AM108 1:00pm

Class 17: Coupled oscillators and Poincar maps

Goals for the day:

1. Describe how coupled oscillators might behave if they are interacting.
2. Use a Poincaré map to analyze a limit cycle.
3. Distinguish between quasiperiodicity and periodic motion.

Team Problems:

1. (8.6.1: "Oscillator death" and bifurcations on a torus) This model is from Ermentrout and Kopell (1990), where the authors were considering a system of interacting neural oscillators. They developed a simple example with two oscillators that captured many of the interaction properties they wanted. Specifically, they wanted to capture that coupling can suppress oscillation ("oscillator death") and lead to a steady state of the coupled system. Here is their example model:

$$\dot{\theta_1} = \omega_1 + \sin\theta_1 \cos\theta_2$$
$$\dot{\theta_2} = \omega_2 + \sin\theta_2 \cos\theta_1.$$

   The oscillators have a natural frequency, but they also are responding to each other. Find the different behaviors possible in this system, with the aim of identifying bifurcations and plotting a stability diagram in $\omega_1\omega_2$ space.

   (a) In the video, looking for fixed points of $\phi = \theta_1 - \theta_2$ allowed us to identify curves where $\theta_1 = \theta_2 + const$. Here, use both $\phi = \theta_1 - \theta_2$ and $\psi = \theta_1 + \theta_2$ to aid your analysis. *Note: If $\theta_1 - \theta_2 = C_1$ and $\theta_1 + \theta_2 = C_2$ then the system has a fixed point.* Find the curves in $\omega_1\omega_2$ parameter space along with bifurcations occur, and classify the bifurcations.

   (b) Classify the different behaviors of the system in the $\omega_1\omega_2$ plane, sketching phase diagrams corresponding to each region of the plane.

2. (8.7.2) Consider the vector field on the cylinder given by $\dot{y} = ay$ and $\dot{\theta} = 1$. This system is not in polar coordinates because y in not a polar angle. Instead the system evolves on a cylinder so that θ can change periodically why $y \in \mathbb{R}$.

   Let $\Sigma = \{(y, 0) : y \in [-1, 1]\}$ be a line segment on the cylinder. Define a Poincaré map, $P : \Sigma \to \Sigma$. To do this, note that it takes time 2π for θ to evolve by 2π and thus for the trajectory to return to Σ.

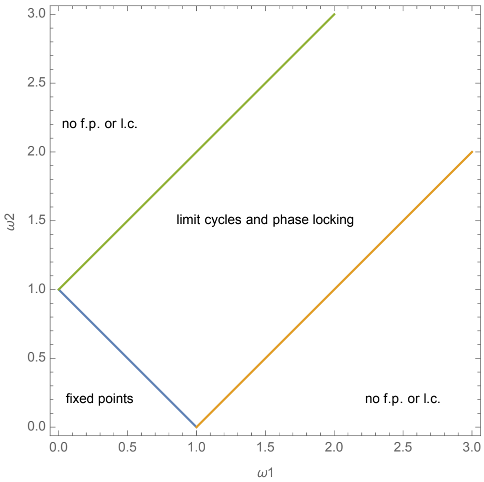
$$2\pi = \int_0^{2\pi} dt = \int_y^{P(y)} \frac{dt}{dy} dy.$$

   After finding the map, show that the system has a periodic orbit (meaning that P has a fixed point), and use its Floquet multiplier (*eigenvalue*) to identify the stability.

   When you examine the Floquet multiplier, it is important to remember that we are wondering whether perturbations grow in length or shrink near the fixed point, so we are not comparing to 0 in this case.

Some Answers:

1. The chart shown for $\omega_1 > 0$ and $\omega_2 > 0$.



The green and orange lines are saddle-node bifurcations of cycles ($|\omega_1 - \omega_2| = 2$). The blue line is a line of infinite period bifurcations ($\omega_1 + \omega_2 = 1$) where the saddle-node bifurcations cross the limit cycles.

2. We have $2\pi = \int_y^P \frac{1}{a}\frac{1}{y}dy$. So $2\pi a = \ln \frac{P(y)}{y}$. $\Rightarrow P(y) = ye^{2\pi a}$. This has a fixed point when $P(y) = y$ so when $y = ye^{2\pi a}$. This means $y^* = 0$ is a fixed point. The Floquet multiplier is $e^{2\pi a}$ and it has magnitude less than 1 when $a < 0$. This is all consistent of our understanding of the system based on examining the flow.