AM108 1:00pm

Class 19: Lorenz Attractor

Goals for the day:

1. Work with definitions to gain practice with the term *attractor*.
2. Identify what the largest Liapunov exponent of a system conveys about the system.
3. Find orbits in a map.

1. We defined an *attractor*, A, as a closed set with some specific properties: A is an *invariant set*; A attracts an open set of initial conditions; A is *minimal*.
   (a) Consider the system $\dot{x} = x - x^3$, $\dot{y} = -y$. This is the set that was used as an example in the video. Let $I = \{(x, y) : -1 \leq x \leq 1, y = 0\}$. Argue that I is an invariant set.
   (b) Consider the system $\dot{x} = x - x^3$, $\dot{y} = -y$. Let $I = \{(x, y) : -1 \leq x \leq 1, y = 0\}$. The set I is closed and invariant. It is also attracting for a set of points larger than the set itself. Identify its basin of attraction.
   (c) Is the set I an *attractor*?

2. (9.3.1) Consider the 2D system

$$\dot{\theta_1} = \omega_1$$
$$\dot{\theta_2} = \omega_2,$$

   with $\frac{\omega_1}{\omega_2}$ irrational. The phase space is the torus. Consider two trajectories that start close to each other, so they have similar $(\theta_1(0), \theta_2(0))$. For example, let $\theta_1(0)$ differ by δ between the two trajectories and $\theta_2(0)$ be identical for the two trajectories. Using $\|\delta(t)\| = \sqrt{\Delta\theta_1^2 - \Delta\theta_2^2}$ as the distance between the trajectories at time t, find the largest Liapunov exponent of this system.

   $\|\delta(t)\| \sim \|\delta_0\|e^{\lambda t}$ *where* λ *is the largest Liapunov exponent.*

   This system is quasiperiodic. Why isn't it considered chaotic?

3. Let $\lambda = 1$ year^{-1} for some unidentified chaotic system. Recall that $\|\delta(t)\| \sim \|\delta_0\|e^{\lambda t}$. A team of scientists is trying to make a prediction about the evolution of the system given its initial state. Their initial measurement is within 10^{-3} m of the actual state of the system, and they believe their model equations are exactly correct. They want their estimate of the state of the system to be correct to within 1 m. Over what time scale is their prediction achieving the accuracy they desire?

   You can assume that $\ln 10^3 \approx 7$.

   If they wanted to double the length of time over which your estimate held, how much more accurate would your initial estimate need to be?

4. (9.4.2) The tent map is a simple analytical model that has some properties in common with the Lorenz map. Let

$$x_{n+1} = \begin{cases} 2x_n, & 0 \leq x_n \leq \frac{1}{2} \\ 2 - 2x_n, & \frac{1}{2} \leq x_n \leq 1. \end{cases}$$

   (a) Draw $f(x)$ where $x_{n+1} = f(x_n)$. Why is this map the "tent map"?
   (b) Find the fixed points of this map.
   (c) Classify the stability of the fixed points.
   (d) Show the map has a period-2 orbit. This means that there is an x such that $f(f(x)) = x$.
   (e) Classify the stability of any period-2 orbits.
   (f) Look for a period-3 or period-4 point. If you find one, are such orbits stable or unstable?
   (g) If you want, you can think about this for a period-k orbit...

Some answers on back

Answers:

1. (a) Let $x_0 \in [-1,1], y_0 = 0$. For all points (x_0, y_0), $\dot{y}|_{y_0} = 0$ so we don't leave the set in the y direction. We also need to check the x direction: If $x_0 = \pm 1$ or $x_0 = 0$ then $\dot{x}|_{x_0} = 0$, so if we start at one of those points we definitely stay in the set for all time. If $x_0 \in (-1, 0)$ then $\dot{x}|_{x_0} < 0$ and we flow towards $x = -1$ but we can't pass that point, so we stay in I. Similarly if $x_0 \in (0,1)$ then we flow towards $x = 1$ but we can't pass that point, so we also stay in I. No matter where we start in I we stay in I for all time, so the set is invariant.

   (b) Its basin of attraction is the whole plane, actually (considered an open set).

   (c) Actually, the trajectories are moving towards the two stable points $(-1,0)$ and $(1,0)$, and not towards all of I, so not an attractor.

2. $\theta_1(t) = \omega_1 t + \theta_1(0)$ and $\theta_2(t) = \omega_2 t + \theta_2(0)$, so $\Delta\theta_1 = \delta$ and $\Delta\theta_2 = 0$. $\delta(t) = \delta$, so $\lambda = 0$. There is no sensitive dependence on initial conditions and no exponential divergence of nearby trajectories.

3. $1 \sim 10^{-3} e^t$, so we have $t \sim \ln 10^3 \approx 7$ years. To double the length of time, we want the time to be $2 \ln 10^3$ years and still want $1 \sim \delta_0 e^t$ so $1 \sim \delta_0 e^{2 \ln 10^3}$. $\Rightarrow 1 \sim \delta_0 (10^3)^2$. $\Rightarrow \delta_0 \sim 10^{-6}$. This is 1 μm of initial measurement accuracy instead of 1 mm.

4. (a)

   (b) If $2x = x$ then $x = 0$, so 0 is a fixed point. If $2 - 2x = x$ then $2 - 3x = 0$ so $x = \frac{2}{3}$. $\frac{1}{2} \le \frac{2}{3} \le 1$, so $\frac{2}{3}$ is also a fixed point. Looking at the graph of this below, there are two intersections between $y = f(x)$ and $y = x$ corresponding to two fixed points.

   (c) $\frac{df}{dx} = \pm 2$ so it is greater than 1 in magnitude. This means the fixed points are unstable.

   (d) $x = f(f(x))$. If $0 \le x \le \frac{1}{2}$ then $x \to 2x$. If $0 \le x \le \frac{1}{4}$ then $x \to 2x \to 4x$. This has a fixed point of 0 but that isn't a period 2 point. If $\frac{1}{4} \le x \le \frac{1}{2}$ then $x \to 2x \to 2 - 4x$. This has a fixed point of $x = 2 - 4x$ so $x = \frac{2}{5}$. $f(\frac{2}{5}) = \frac{4}{5}$ so the period-2 orbit is $x_1 = \frac{2}{5}, x_2 = \frac{4}{5}$.
   Looking at the graph of $y = f(f(x))$ below, there are 4 intersection points with $y = x$. These correspond to the two period-1 fixed points and two new fixed points. The two new fixed points form a period-2 orbit.

   (e) For the stability, we created the map explicitly, so it is clear that $x_{n+2} = f(f(x_n))$ has a Floquet multiplier of 4, and is unstable. More generally, thinking about the growth of a perturbation near the period-2 point, let z_n be a point near the period-2 orbit and let η_n be the distance of z_n from the orbit. $\eta_{n+2} \approx f'(z_{n+1})\eta_{n+1} \approx f'(z_{n+1})f'(z_n)\eta_n$. In our case, $|f'(z)| = 2$ for all z, so $|f'(z_{n+1})f'(z_n)| = 4$.

   (f) From above, any such orbit is unstable. Now, can we find one? For the period-3, looking at the $y = f(f(f(x)))$ graph below, there are six intersection points. Two of these correspond to the period-1 points. The other 6 are new and correspond to two different period-3 orbits ($\frac{2}{9}, \ldots$ and $\frac{2}{7} \ldots.$)
   The map for period-4 should have $2^4 = 16$ intersections, of which two are period-1 and two are period-2 but the other 12 should be new, so 3 period-4 orbits.

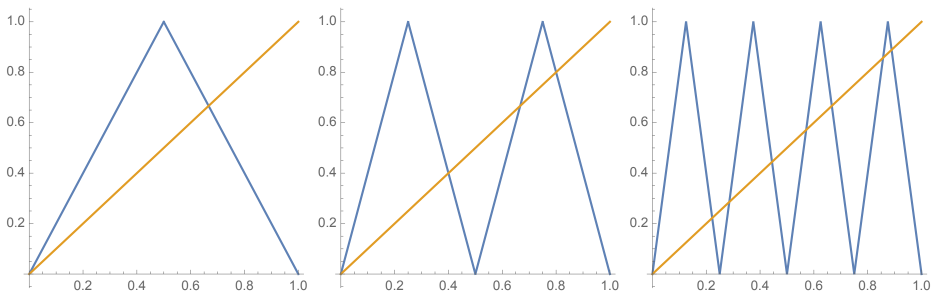

Figure 1: Maps from left: $x_{n+1} = f(x_n)$, $x_{n+2} = f(f(x_n))$ and $x_{n+3} = f(f(f(x_n)))$.