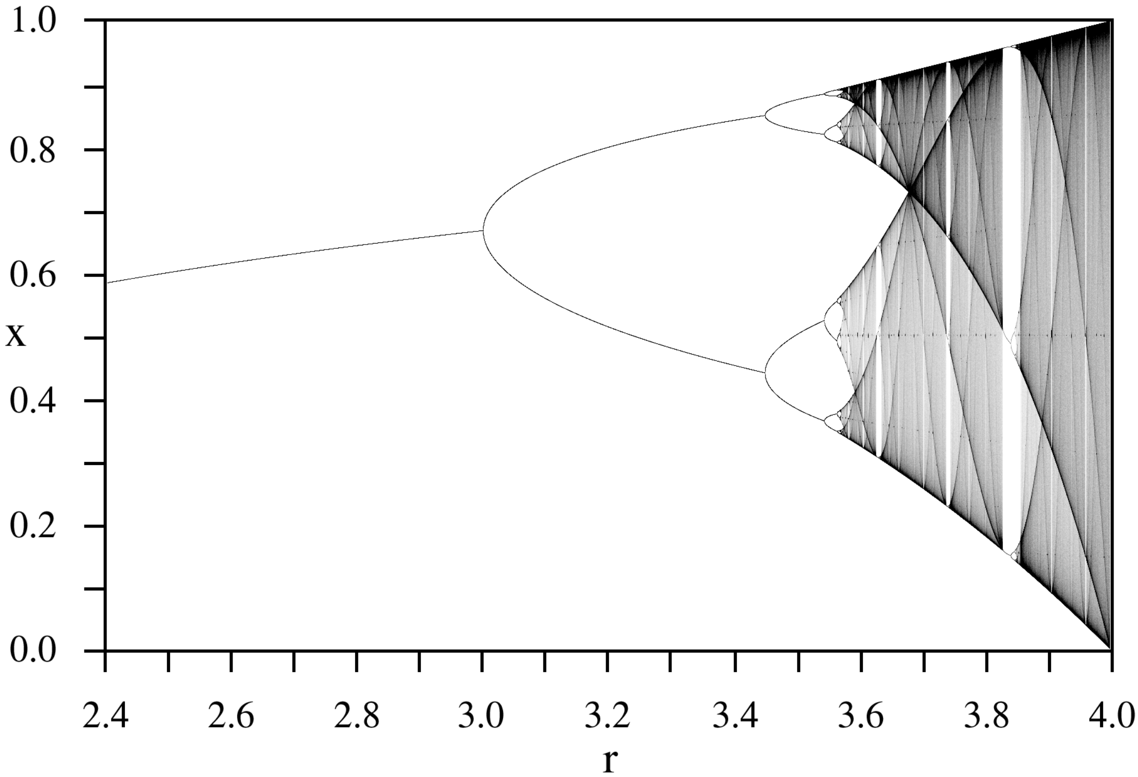
AM108 1:00pm

Class 21: More Maps

Goals for the day:

1. Find the equations of the curves running through the logistic map orbit diagram.
2. Rescale a map.
3. Work with a functional equation.



1. (10.3.13) In the diagram above there are dark tracks of points running through the chaotic regions. The logistic map is given by $x_{n+1} = rx_n(1-x_n)$. The function $f(x;r) = rx(1-x)$ has its maximum at $x = \frac{1}{2}$. This means that many values of x map to close to $f(\frac{1}{2};r) = \frac{r}{4}$.
   (a) One of these dark tracks is the curve $(r, f(\frac{1}{2}, r))$. What are the other curves?
   (b) Set up an equation to find the value of r at the corner where a bunch of the dark track intersect.

2. (10.7.2) Consider a general map
$$y_{n+1} = f(y_n; r).$$
   Let $x_n = \alpha y_n$, so the variables x_n are rescalings of y_n. By rescaling y_{n+1} and y_n show that
$$f^2(y;R) \text{ becomes } \alpha f^2(\frac{x}{\alpha};R).$$
   What does rescaling do to a map?

3. A **functional equation** is an equation whose solution is a function (usually the term is used when the equation can't be reduced to an algebraic equation). Consider the functional equation
$$g(x) = \alpha g^2\left(\frac{x}{\alpha}\right), \quad g(0) = 1.$$
   This functional equation has a boundary condition to help specify the solution.

(a) Use the functional equation to show that $\alpha = 1/g(1)$.

(b) (10.7.3) Show that if g is a solution to the functional equation above, that $h(x) = \mu g(x/\mu)$ is a solution as well (with the same value of α, with a different initial value).

Some answers:

1. Since $f(x;r)$ is often near $r/4$, $f^2(x;r)$ is often near $f(r/4;r)$, etc. These curves are given by $f^m(\frac{1}{2};r)$.

```
Clear[r, x]
f[x_, r_] = r x (1 - x);
Plot[{f[0.5, r], f[f[0.5, r], r],  f[f[f[0.5, r], r], r],
  f[ f[f[f[0.5, r], r], r], r],
  f[f[ f[f[f[0.5, r], r], r], r], r]}, {r, 2.95, 4}]
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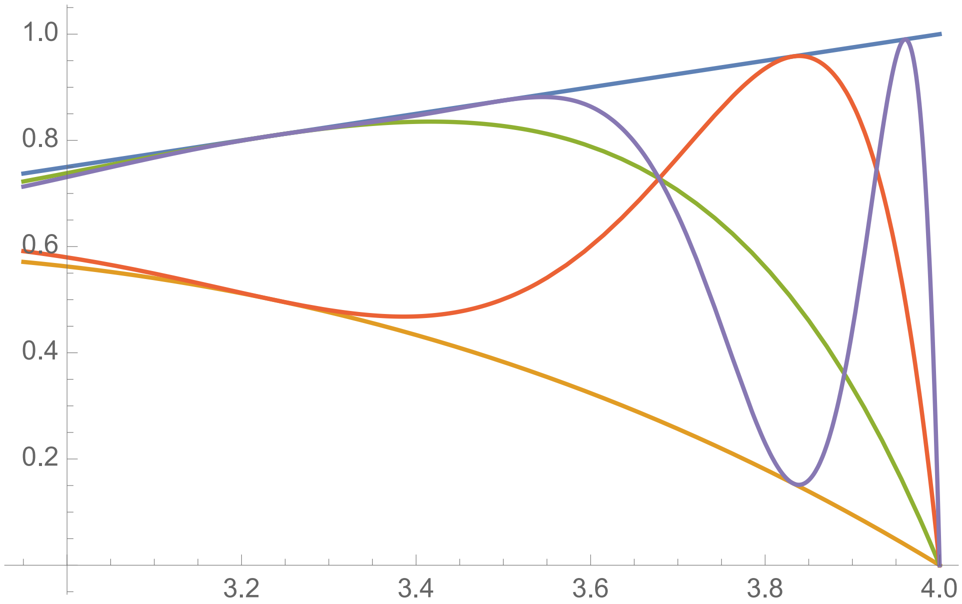

At the intersection, $f^3(1/2;r) = f^4(1/2,r)$. Let $u = f^3(1/2;r)$. Then the intersection happens when $f(u,r) = u$. This is a fixed point of the logistic map, so $u = 0$ or $u = 1 - 1/r$. It must be that $u = 1 - 1/r$. So $f^3(1/2;r) = 1 - 1/r$ is our equation. Expanding:

```
Expand[f[f[f[1/2, r], r], r] - 1 + 1/r]
Solve[Expand[f[f[f[1/2., r], r], r] - 1 + 1/r] == 0, r]
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$$-1 + \frac{1}{r} + r^3/4 - r^4/16 - r^5/16 + r^6/32 - r^7/256 = 0.$$

This factors into $(r-2)^4(r+2)(r^3 - 2r^2 - 4r - 8) = 0$ so $r \approx 3.67857$.

2. $x_{n+1}/\alpha = f(x_n/\alpha;R) \Rightarrow x_{n+1} = \alpha f(x_n/\alpha;R)$. $y_{n+2} = x_{n+2}/\alpha = f(x_{n+1}/\alpha;R)$ so

$$x_{n+2} = \alpha f(x_{n+1}/\alpha;R) = \alpha f(f(x_n/\alpha;R);R) = \alpha f^2(x_n/\alpha;R)$$

so $f^2(y;R)$ becomes $\alpha f^2(x/\alpha;R)$ after rescaling. Rescaling expands or contracts our coordinate. Thinking of the map in terms of a graph from y_n to y_{n+1}, the rescaling expands both axes. We can think of it as just changing the numbers on the axes without changing the shape of the graph.

3. $1 = g(0) = \alpha g(g(0)) = \alpha g(1)$ so $\alpha = 1/g(1)$. Let g satisfy $g = \alpha g^2(x/\alpha)$. Consider $h(x)$.

$$g(x) = \alpha g^2(x/\alpha) \text{ so } g(x/\mu) = \alpha g^2(x/(\mu\alpha)) = h(x)/\mu.$$

$h(0) = \mu g(0) = \mu$ (different initial value).

$h(x) = \mu g(x/\mu) = \mu\alpha g(g(x/(\mu\alpha))) = \alpha\mu g(g(x/(\mu\alpha))) = \alpha\mu g((\mu g(x/(\alpha\mu)))/\mu) = \alpha h(\mu g(x/(\alpha\mu)) = \alpha h^2(x/\alpha)$.