AM108 1:00pm

Class 22: Universality

Goals for the day:

1. Explore the renormalization process and identify limiting functions.
2. Work with the functional equation $g(x) = \alpha g^2\left(\frac{x}{\alpha}\right)$

Problems:

1. Consider the map $f(x) = r(x + 1/2) - r(x + 1/2)^2 - 1/2$. Show that this map is a shifted logistic map with maximum at $x_m = 0$. Make the case that the $x_m = 0$ fixed point is superstable when $f(0) = 0$.

2. (Exploring the renormalization process) *For the map above, argue that superstable 2^k-cycles occur when $f^{2^k}(0; Rk) = 0$ and k is the smallest integer where this expression holds at a given Rk.*

   I find these values in Mathematica:

```
R0 = r /. FindRoot[Nest[f, 0, 1] == 0, {r, 2}]
R1 = r /. FindRoot[Nest[f, 0, 2] == 0, {r, 3}]
R2 = r /. FindRoot[Nest[f, 0, 4] == 0, {r, 3.49}]
R3 = r /. FindRoot[Nest[f, 0, 8] == 0, {r, 3.55}]
R4 = r /. FindRoot[Nest[f, 0, 16] == 0, {r, 3.566}]
(R3 - R2)/(R4 - R3)
4.66296
```

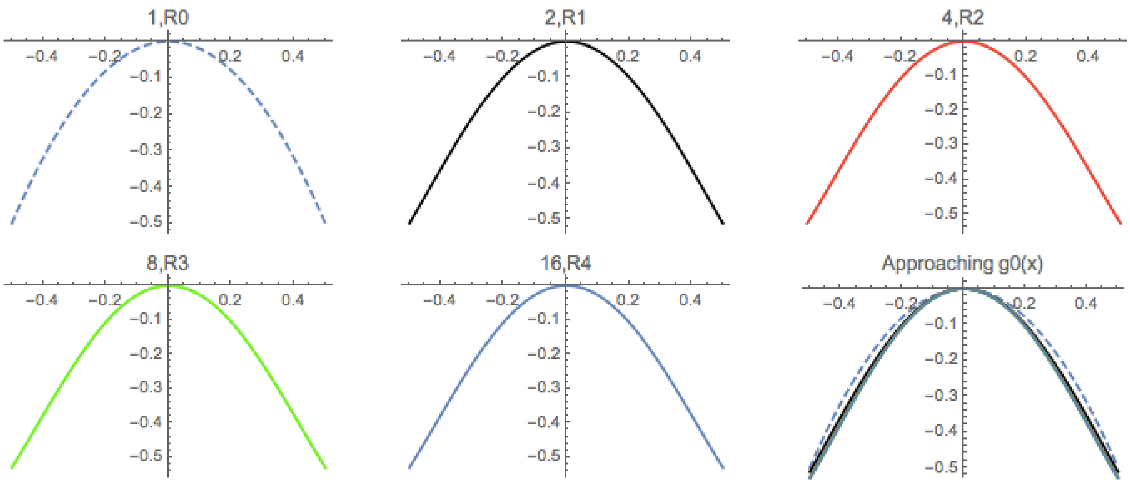
   In the period-doubling regions of the logistic map, we are seeing a phenomenon that is repeating at smaller and smaller length scales. The period-doubling diagram is approximately self similar, meaning that if we zoom in on a small part, it has the same topology (same stable orbit structure occurring at the same transition points) as the whole diagram.

   The renormalization-group algorithm defines a transformation to do this zooming.

   Define the following maps:

```
a = -2.502907875;
f0[x_] = r (x + 1/2) - r (x + 1/2)^2 - 0.5;
f1[x_] = a f0[f0[x/a]];
f2[x_] = a f1[f1[x/a]];
f3[x_] = a f2[f2[x/a]];
f4[x_] = a f3[f3[x/a]] /. r -> R4;
```

   These maps are encoding the renormalization-group algorithm. The results for the value of r corresponding to each superstable orbit are show below (so $f0$ is plotted for $r = R0$, $f1$ for $r = R1$, etc).



What is happening to the function as the map is iterated?

Plotting code:

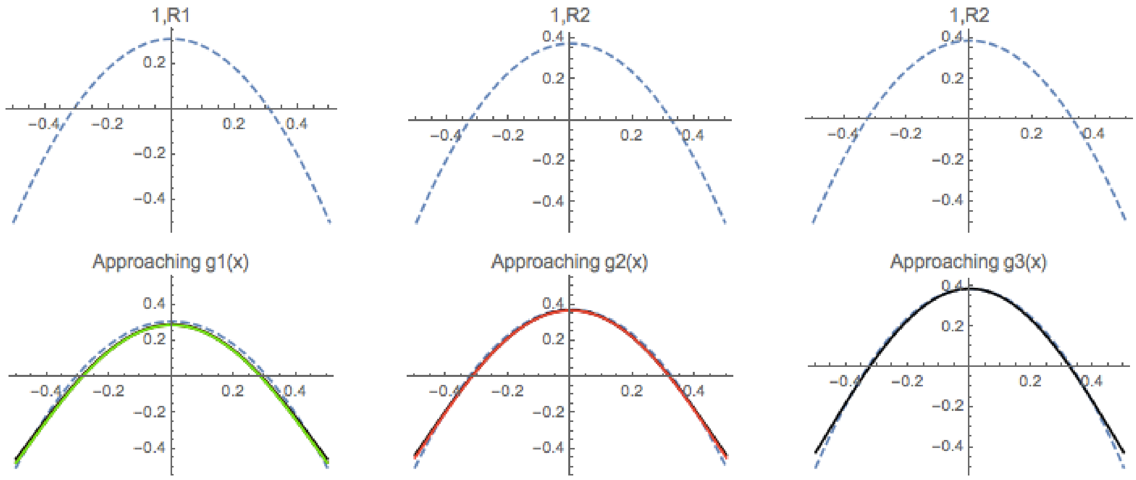
```
p0 = Plot[{(f0[x] /. r -> R0)}, {x, -0.5, 0.5}, PlotLabel -> "1,R0",
   PlotStyle -> {Dashed}];
p1 = Plot[{(f1[x] /. r -> R1)}, {x, -0.5, 0.5}, PlotLabel -> "2,R1",
   PlotStyle -> {Black}];
p2 = Plot[{(f2[x] /. r -> R2)}, {x, -0.5, 0.5}, PlotLabel -> "4,R2",
   PlotStyle -> {Red}];
p3 = Plot[{(f3[x] /. r -> R3)}, {x, -0.5, 0.5}, PlotLabel -> "8,R3",
   PlotStyle -> {Green}];
   p4 = Plot[{(f4[x])}, {x, -0.5, 0.5}, PlotLabel -> "16,R4"];
   GraphicsGrid[{{p0, p1, p2}, {p3, p4,
   Show[p0, p1, p2, p3, p4, PlotLabel -> All]}}]
```

In the limit of this process, $g_0(x) = \lim_{n\to\infty} \alpha^n f^{(2^n)}(\frac{x}{\alpha^n}; R_n)$.

3. How do you guess the set of plots might be different if we plotted $f(x, R_1)$ instead of $f(x, R_0)$, and $\alpha f^2(\frac{x}{\alpha}; R_2)$ instead of $\alpha f^2(\frac{x}{\alpha}; R_1)$, etc?

   In the limit of this process, $g_1(x) = \lim_{n\to\infty} \alpha^n f^{(2^n)}(\frac{x}{\alpha^n}; R_{n+1})$.

4. Let $g_i(x) = \lim_{n\to\infty} \alpha^n f^{(2^n)}(\frac{x}{\alpha^n}; R_{n+i})$.



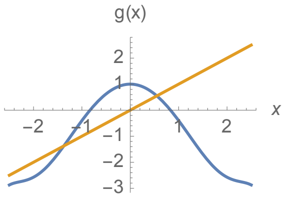
   Consider $g_\infty(x)$. Since R_n is no longer changes, $g_\infty(x) = \alpha g_\infty^2(\frac{x}{\alpha}; R_\infty)$.

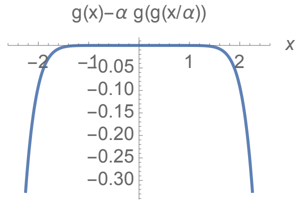
   Let $g(x)$ satisfy $g(x) = \alpha g^2(\frac{x}{\alpha})$. This is a functional equation, and its solution is the function $g(x)$. We can get a sense of this function. We know $g(0) \neq 0$ because the functions were plotted at values of r that didn't coincide with their superstable fixed point. If $g(x)$ is a solution to $g(x) = \alpha g^2(\frac{x}{\alpha})$ then $\mu g(x/\mu)$ is as well. (Optional: *show this*).

   (10.7.1a) *Approximate* $g(x)$. Start by assuming it is even with a quadratic maximum of 1 at $x = 0$. So suppose $g(x) \approx 1 + c_2 x^2$ for small x. Neglect $\mathcal{O}(x^4)$ terms and solve for α and c_2.

   Note that $\sqrt{3} \approx 1.732$. *Compare your values to* $\alpha \approx -2.5029...$, $c_2 \approx -1.527...$

5. (10.7.4) Near the origin, $g(x)$ has a parabolic approximation but it is actually a wiggly function over the real line. To see this, show that if x^* is a fixed point of $g(x)$ then so is αx^*. This means that the function has infinitely many crossings of the line $y = x$. Because it is an even function, it also has infinitely many crossings of the line $y = -x$. To be convinced that it has one crossing, see the figure below.





On the left is a 10th order polynomial approximation to $g(x)$. On the right is the error term, $g(x) - \alpha g^2(\frac{x}{\alpha})$.

Some Answers:

1.

2. Superstable cycles occur when $x = 0$ is one of the points involved in the cycle, so $f^{(2^k)}(0; Rk) = 0$ at a superstable cycle. We need an extra condition so that it is actually a 2^k cycle and not also a 2^{k-1} cycle.
   The function is approaching some limiting distribution.

3. We would no longer be at the parameter value with the superstable fixed point, $g_1(0) \neq 0$.

4. Substitute into the equation so $1 + c_2 x^2 = \alpha g(g(\frac{x}{\alpha}))$. We have $g(\frac{x}{\alpha}) = 1 + c_2 \frac{x^2}{\alpha^2}$. This means $g(g(\frac{x}{\alpha})) = 1 + c_2 \left(1 + c_2 \frac{x^2}{\alpha^2}\right)^2$. Expanding, we find $g(g(\frac{x}{\alpha})) = 1 + c_2 + 2\frac{c_2^2}{\alpha^2} x^2 + \mathcal{O}(x^4)$. Substituting this into our first equation and truncating, $1 + c_2 x^2 = \alpha(1 + c_2) + \frac{2c_x^2}{\alpha} x^2$. Matching terms, $c_2 = \frac{2c_2^2}{\alpha} \Rightarrow c_2 = \frac{\alpha}{2}$ and $1 = \alpha(1 + c_2)$, so $1 = \alpha + \alpha^2/2$. We can solve this quadratic equation to find $\alpha \approx -1 - \sqrt{3} \approx -2.732$. Dividing by 2, $c_2 \approx -1.336$. These are off by about 0.2 or 10%, which is not terrible!

5. We have x^* such that $g(x^*) = x^*$. Now consider $g(\alpha x^*)$. We have $g(\alpha x^*) = \alpha g(g(\frac{\alpha x^*}{x^*}))$. $\Rightarrow$ $g(\alpha x^*) = \alpha g(g(x^*)) = \alpha g(x^*) = \alpha x^*$, so αx^* is a fixed point of $g(x)$!