Class 23: Fractals

Goals for the day:

1. Use similarity dimension to determine fractal dimensions.
2. Find a basin of attraction in a map.
3. Identify a condition for stability of a fixed point in 2D maps.

Problems:

1. (11.1.1) Consider the diagonal argument used to show that $S = \{x : 0 \leq x < 1\}$ is an uncountable set. Why doesn't this argument also show that the rational numbers are uncountable?

2. (11.2.1) We want to find the total length of the points in the Cantor set. To do this, consider the lengths of the intervals that we removed to construct the set. First we removed an interval of length $\frac{1}{3}$. Then we removed two intervals, each of length $\frac{1}{9}$, etc. Show that the total length of all of the removed intervals is 1 and this the leftover points making up the Cantor set have length 0.

3. (Example 4.9, Alligood et al) Consider the tent map $x_{n+1} = T_3(x_n)$, defined as

$$x_{n+1} = \begin{cases} 3x_n, & 0 \leq x_n \leq \frac{1}{2} \\ 3(1 - x_n), & \frac{1}{2} \leq x_n \leq 1. \end{cases}$$

   For the slope-3 tent map, the basin of attraction of infinity is related to the Cantor set. The basin of attraction is the set of initial conditions that will converge to ∞ or $-\infty$ with iteration of T_3.

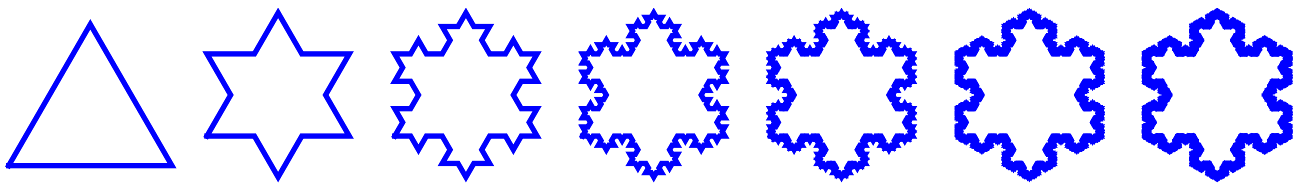
   (a) Sketch the map.

   (b) Convince your team that initial conditions outside of $[0, 1]$ converge to $-\infty$ under iteration of T_3. In addition, note that initial conditions in $(1/3, 2/3)$ do the same.

   (c) Convince your team that the basin of infinity of T_3 is the complement in $\mathbb{R}^1$ of the middle-third Cantor set, meaning that the basin is the set of points in $\mathbb{R}^1$ that are left if we remove the Cantor set.

   (d) What is the long term behavior if your initial condition is $1/3^k$ for $k = 1, 2, ...$? What if it is $2/3^k$? Guess the basin of attraction for 0 in this system.

4. (11.3.7) The von Koch snowflake curve is shown below. It is made by starting with an equilateral triangle and then replacing each side using the von Koch procedure.

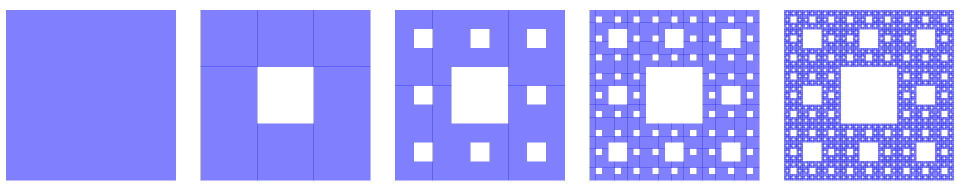


   (a) Show that the snowflake curve has infinite arc length.

   (b) Assuming that the initial side lengths are 1, find the area of the region enclosed by the snowflake.

   (c) Find the similarity dimension of the snowflake. *Recall that # copies = (scale)^d.*

5. (11.3.8) The Sierpinski carpet is formed by dividing a closed unit box into nine equal boxes, removing the central (open) box, and repeating.



   1. Find the similarity dimension.
   2. Show the Sierpinski carpet has zero area.

6. We will be using 2D maps to make connections between fractal structure and chaos, so we will begin to work with 2D maps today.
(12.1.1) Consider the 2D linear map

$$x_{n+1} = ax_n$$
$$y_{n+1} = by_n,$$

$a, b \in \mathbb{R}$. Identify the possible patterns of orbits near the origin depending on the signs and sizes of a and b. Draw the possible patterns for orbits that converge to the origin. *Note that the map is linear and uncoupled.*

7. (12.1.2) Now consider the 2D linear map

$$x_{n+1} = ax_n + by_n, \quad y_{n+1} = cx_n + dy_n,$$

$a, b, c, d \in \mathbb{R}$. Find conditions on the parameters so that the origin is globally attracting (globally asymptotically stable).

A linear algebra interlude: this equation can be written $\mathbf{x}_{n+1} = A\mathbf{x}_n$. *Any matrix* A *can be transformed in the following way:* $A = P^{-1}DP$ *where* D *is diagonal or* $A = P^{-1}CP$ *where* C *is an upper triangular matrix and the diagonal entries of* C *or* D *are the eigenvalues of* A. *Here,* C *or* D *is known as the* ***Jordan normal form*** *of the matrix, and* P, D, C *may be over the complex numbers. We have* $P^{-1}P = PP^{-1} = \mathbb{I}$; *if* $D = \begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix}$ *then* $D^k = \begin{pmatrix} \alpha^k & 0 \\ 0 & \beta^k \end{pmatrix}$; *if* $C = \begin{pmatrix} \alpha & 1 \\ 0 & \beta \end{pmatrix}$ *then* $C^k = \begin{pmatrix} \alpha^k & \gamma_k \\ 0 & \beta^k \end{pmatrix}$ *for some* $\gamma_k \in \mathbb{C}$.

8. (11.3.10) A fat fractal has nonzero measure (length for $d < 1$, area for $1 < d < 2$). Consider the set created by starting with $[0, 1]$, deleting the open middle $\frac{1}{2}$, then the open middle $\frac{1}{4}$ of the two remaining subintervals, then the open middle $\frac{1}{8}$, etc.
   (a) Show that the length of the limiting set is greater than zero.
   (b) Show that the set is a topological Cantor set. First show it is "totally disconnected". To do this, pick any two points in the set, and argue that they are in different "halves" of the set at some point. Then show that it has no "isolated points". To do this, show that for any point in the set, p, and any neighborhood length, ϵ, there is a second point within distance ϵ of p.

Some Answers:

1. We make a list of all of the rational numbers with their decimal expansions. As we construct a new number that is not on the list (and this is definitely something we can do), it is not clear that the new number is rational - it definitely does not have a finite decimal expansion and there is no reason the decimals would be repeating.

2. We remove $L = \frac{1}{3} + 2\frac{1}{9} + 4\frac{1}{27} + ... = \sum_1^\infty \frac{2^{n-1}}{3^n}$. So $\frac{2}{3}L = \frac{2}{9} + \frac{4}{27} + ...$ Thus $L - \frac{2}{3}L = \frac{1}{3}$. This means $\frac{1}{3}L = \frac{1}{3}$, so $L = 1$.

3. Points $(-\infty, 0) \cup (1/3, 2/3) \cup (1, \infty)$ will go off to infinity. This is the complement of S_1 in $\mathcal{R}^1$. Which points map to there? $(1/9, 2/9)$ (and its mirror image interval, $(7/9, 8/9)$, across $x = 1/2$). Assume that an interval $(\frac{a}{3^k}, \frac{a+1}{3^k})$ will go off to infinity. Clearly $(\frac{a}{3^{k+1}}, \frac{a+1}{3^{k+1}}) \in [0, \frac{1}{2})$ will, too. As will its mirror image across $x = 1/2$. So each interval of length $1/3^k$ corresponds to two intervals of length $1/3^{k+1}$ that also diverge. These intervals are all distinct and make up the set of intervals removed in constructing the Cantor set.

4. The number of line segments is a sequence $3, 4*3, 4*4*3, ..., 4^k 3, ...$ so $n_k = 4^k 3,\ k = 0, 1, 2, ...$ Length of a line segment at step k gives a sequence $1, \frac{1}{3}, \frac{1}{9}, ..., \frac{1}{3^k}, ...,$ so $l_k = \frac{1}{3^k},\ k = 0, 1, 2, ...$ Total length at each iterate is $n_k l_k$, so $3, 4, 4\frac{4}{3}, ..., 4\frac{4^{k-1}}{3^{k-1}}, ...$ In the limit $k \to \infty$, this is unbounded.

   For the area, we start with area A_0 in the triangle. Then the next structure adds n_0 triangles each with area $\frac{1}{9}$ the original area. Area sequence is

   $$A_0, A_0 + 3\frac{1}{9}A_0, A_0 + \frac{1}{3}A_0 + 4*3\frac{1}{3^4}A_0, ..., A_0 + \sum_{r=1}^{k} n_{r-1}{l_r}^2 A_0, ...$$

   So $a_k = A_0(1 + \sum_{r=1}^k 4^{r-1}3\frac{1}{3^{2r}}) = A_0(1 + \frac{1}{3}\sum_{r=0}^{k-1}(\frac{4}{9})^r)$. The area of the fractal is $A = A_0(1 + \frac{1}{3}\sum_{r=0}^\infty(\frac{4}{9})^r)$. $\frac{4}{9}A = A_0(\frac{4}{9} + \frac{1}{3}\sum_{r=1}^\infty(\frac{4}{9})^r)$, so $A - \frac{4}{9}A = A_0(\frac{5}{9} + \frac{1}{3})$. Thus $A = \frac{9}{5}\frac{8}{9}A_0 = \frac{8}{5}A_0$.

   Still scaling down by 3 and making 4 copies, so $d = \frac{\ln 4}{\ln 3} \approx 1.26$.

5.

6. The map is uncoupled, so we can tackle x and y separately. For x, if $|a| > 1$ then orbits diverge. If $|a| < 1$ then orbits converge. For $a > 0$, the change in x is monotonic. For $a < 0$, we alternate sides of the origin. All of the same is true for y, so there are sixteen possibilities. There are four possibilities that lead to convergence: $\{(a, b) : |a| < 1, |b| < 1\}$ is the interior of the unit square and the qualitative possibilities correspond to points in each of the four quadrants.

7. P and P^{-1} serve as an invertible pair of transformations. We can think about the system $P\mathbf{x}_{n+1} = CP\mathbf{x}_n$. If $P\mathbf{x_n}$ approaches the origin then $\mathbf{x}_n$ does as well. Let $\mathbf{z}_n = P\mathbf{x}_n$. $\mathbf{z}_0 = \begin{pmatrix} r \\ s \end{pmatrix}$. $\mathbf{z}_k = D^k\mathbf{z}_0 = \begin{pmatrix} \alpha^k r \\ \beta^k s \end{pmatrix}$. In this case it is clear that we need $|\alpha| < 1$ and $|\beta| < 1$ so we need both eigenvalues of the matrix to be less than 1.

   The nondiagonal case is slightly trickier. $\mathbf{z}_k = C^k\mathbf{z}_0 = \begin{pmatrix} \alpha^k r + c_k s \\ \beta^k s \end{pmatrix}$. It is not obvious how to deal with the $c_k s$ term. We can see it is not a problem by iterating another k times. $\mathbf{z}_{2k} = C^k C^k \mathbf{z}_0 = C^k \begin{pmatrix} \alpha^k r + c_k s \\ \beta^k s \end{pmatrix} = \begin{pmatrix} \alpha^k(\alpha^k r + c_k s) + c_k\beta^k s \\ \beta^{2k} s \end{pmatrix}$. In this case, in the limit as $k \to \infty$, we still need $|\alpha| < 1$ and $|\beta| < 1$ to approach the origin.