AM108 1:00pm

Class 24: 2D Maps

Goals for the day:

1. Show that the Hénon map is not area preserving via composition of transformations.
2. Explore the Baker's map.

Two-dimensional maps can represent flow in a three-dimensional space. For example, a Poincaré section in in 3D flow would lead to a 2D map (since we would find the map on a 2D region transverse to the flow). Invertible 2D maps can be useful models of chaotic systems (see notes of Michael Cross, Caltech).

1. The Baker's map is given by

$$B(x_n, y_n) = (x_{n+1}, y_{n+1}) = \begin{cases} (2x_n, ay_n) & \text{for } 0 \leq x_n \leq \frac{1}{2} \\ (2x_n - 1, ay_n + \frac{1}{2}) & \text{for } \frac{1}{2} \leq x_n \leq 1 \end{cases}.$$

It is illustrated by Figure 12.1.4 of the text, shown below.

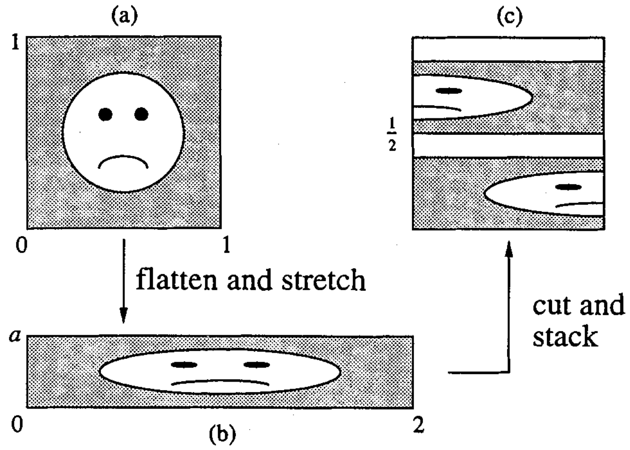

Figure 12.1.4

The Baker's map is a simple model with chaotic dynamics. It can be expressed via **symbolic dynamics**, or dynamics that are represented via shifts on sequences of numbers.

(a) Explain why the map is equivalent to the procedure of stretching by 2 and flattening by a, then cutting and stacking, that is shown in the figure.

(b) Sketch what will happen after one more iterate of the map shown in the figure. (Include the face!)

(c) This process should remind you of forming the Cantor set. Consider covering the n^{th} iterate of the map with square boxes of side length a^n. Note that the first iterate has 2 stripes and the second has 4. The box dimension is given by $d = \lim_{\epsilon \to 0} \frac{\ln N}{\ln \frac{1}{\epsilon}}$ where N is the number of boxes needed to cover the set and ϵ is the side length of the boxes. Compute the box dimension for the limiting set of the Baker's map.

(d) In the case $a = \frac{1}{2}$, your box dimension should be 2 because the map is area preserving. Check that this is the case.

(e) (12.1.5) Symbolic dynamics involve understanding the map via sequences of numbers. For the area preserving Baker's map, consider a binary representation of a point in the unit square:

$$(x, y)_2 = (0.a_1a_2a_3..., \ 0.b_1b_2b_3...)$$

where $a_1 = 0$ indicates the point has $0 \leq x < \frac{1}{2}$ and $a_1 = 1$ indicates the point has $\frac{1}{2} \leq x < 1$. Find the binary representation of $B(x, y)$.

Multiplying a coordinate by 2 has the effect of shifting the decimal place once to the right.

(f) Represent the point (x, y) as $...b_3b_2b_1.a_1a_2a_3....$ In this notation, what is $B(x, y)$?

(g) Use the binary version of the map to show that B has a period-2 orbit. Plot the locations of the two points involved in the orbit in the unit square.

2. The Hénon map is given by $x_{n+1} = 1 + y_n - ax_n^2$ and $y_{n+1} = bx_n$. Consider the series of transformations (from the video) $T' : x' = x, y' = 1 + y - ax^2$, $T'' : x'' = bx', y'' = y'$, $T''' : x''' = y'', y''' = x''$.

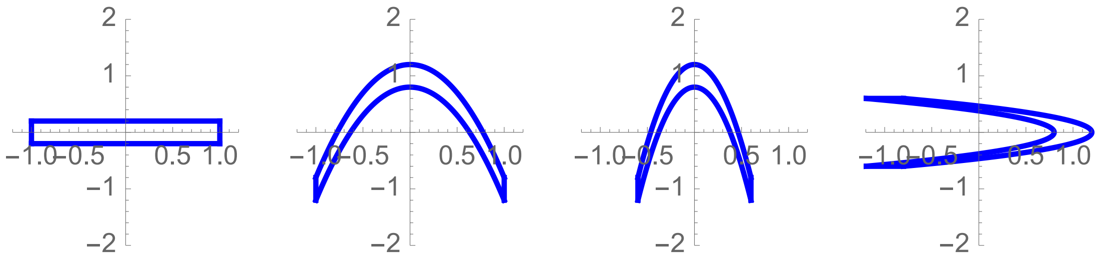

Figure 1: The transformations T', T'' and T''' are composed from left to right, with T' operating on the rectangle on the far left.

(a) (12.2.1) Show that composing this series ($T'''T''T'$) of transformations yields the Hénon map.

(b) (12.2.2) Show that the transformations T' and T'' are area preserving but T'' is not.
A vector calculus interlude: think of the map T' as a coordinate transformation from coordinates xy to coordinates $x'y'$. We are interested in the area of a region of the xy plane after it undergoes the coordinate transformation. Recall: $\iint_R dx\ dy = \iint_S \left|\frac{\partial(x,y)}{\partial(x',y')}\right| dx'dy'$ where
$\frac{\partial(x,y)}{\partial(x',y')} = \begin{vmatrix} \frac{\partial x}{\partial x'} & \frac{\partial x}{\partial y'} \\ \frac{\partial y}{\partial x'} & \frac{\partial y}{\partial y'} \end{vmatrix}$.

(c) (12.2.4) Find the fixed points of the Hénon map, and show that they exist only if $a > a_0$.

(d) (12.2.5) Determine the stability of the fixed points as a function of a and b by finding the Jacobian and determining whether $|\lambda| < 1$. *This seems pretty ugly, so just make a contour plot of the eigenvalues to figure out the region.*

Some Answers:

1. (a) For the unit square, consider the sets $S_0 = \{(x, y) : 0 \le x < \frac{1}{2}, 0 \le y < 1\}$ and $S_1 = \{(x, y) : \frac{1}{2} \le x < 1, 0 \le y < 1\}$. Under the action of the map, in the x direction, S_0 is stretched by a factor of two to take up the whole range $0 \le x < 1$ and y is squished by a factor of a. It is clear that this is the same thing as what happens to S_0 under a stretching by 2 and a flattening by a and then cutting, as S_0 is not impacted by the cutting and stacking procedure. For S_1, it is also stretched and flattened. Then $(\frac{1}{2}, 0)$ corner of S_1 is placed at $(0, \frac{1}{2})$, setting the placement of the whole stretched/flattened set. This is equivalent to what happens to the set under flattening/stretching and cutting/stacking.

(b)

(c) We have 2^n stripes and need $\frac{1}{a^n}$ boxes to cover a single stripe (stripes are of width a^n), so there are $\left(\frac{2}{a}\right)^n$ boxes being used and the box size is a^n. $d = \lim_{n\to\infty} \frac{\left(\frac{2}{a}\right)^n}{\ln \frac{1}{a^n}} = 1 - \frac{\ln 2}{\ln a}$

(d) If we plug in $a = \frac{1}{2}$ we have $d = 1 - \frac{\ln 2}{-\ln 2} = 2$.

(e) The x coordinate should be right shifted by the stretch, so it becomes $a_1.a_2a_3a_4....$ Cutting and stacking turns it into $0.a_2a_3a_4....$ For the y coordinate, it depends on the x coordinate. If $a_1 = 0$ then y becomes $0.0b_1b_2...$ while if $a_1 = 1$ then y becomes $0.1b_1b_2....$ So $(0.a_1a_2a_3, 0.b_1b_2b_3) \mapsto (0.a_2a_3..., 0.a_1b_1b_2...)$.

(f) $...b_3b_2b_1.a_1a_2a_3... \mapsto ...b_2b_1a_1.a_2a_3a_4...$ so the map acts as a shift map on this representation.

(g) For a period-2 orbit, we are looking for a binary number that returns to itself after two shifts. These are the repeating fractions ...101010.101010... and ...010101.010101.... Their coordinates are given by $x = \frac{1}{2} + \frac{1}{8} + ..., y = \frac{1}{4} + \frac{1}{16} + ...$ and vice versa. Thus $x - \frac{1}{4}x = \frac{1}{2} \Rightarrow x_1 = \frac{2}{3}$ and $y - \frac{1}{4}y = \frac{1}{4} \Rightarrow y = \frac{1}{3}$. The points are $(\frac{2}{3}, \frac{1}{3})$ and $(\frac{1}{3}, \frac{2}{3})$.

2. (a) $T'(x, y) = x, 1 + y - ax^2$. $T''(T'(x, y)) = (bx, 1 + y - ax^2)$. $T'''(T''(T'(x, y))) = T'''(bx, 1 + y - ax^2) = (1 + y - ax^2, bx)$.

(b)

(c) No fixed points when $a = 0, b = 1$. Otherwise, $x^* = \frac{b-1}{2a} \pm \sqrt{\frac{(1-b)^2}{4a} + 1}, \quad y^* = bx^*$. Need $\frac{(1-b)^2}{4a} + 1 \leq 0$ so $-(1-b)^2/4 \leq a$.