Class 25: Smale Horseshoe + more Hénon

Goals for the day:

1. Find sensitive dependence on initial conditions in the Baker's map.
2. Explore some properties of the Hénon map.

Problems:

1. For the area preserving Baker's map, a sequence $...b_3b_2b_1.a_1a_2a_3...$ was associated with a point in the unit square. The action of the map takes

$$...b_3b_2b_1.a_1a_2a_3... \mapsto ...b_3b_2b_1a_1.a_2a_3...$$

Let $\lambda \approx \frac{1}{n}\ln\left|\frac{\delta_n}{\delta_0}\right|$ be the Liapunov exponent for a map. This is

$$\frac{1}{n}\ln\left|\frac{\delta_n}{\delta_0}\right| = \frac{1}{n}\ln\left|\frac{f^n(x_0+\delta_0)-f^n(x_0)}{\delta_0}\right|.$$

In the limit where $\delta_0 \to 0$ this is $\lambda \approx \frac{1}{n}\ln|(f^n)'(x_0)|$.

Using this notion of a Liapunov exponent, and thinking of the map above as multiplying a binary number by 2 at each step, find the Liapunov exponent for this map. Is there sensitive dependence on initial conditions in this system?

2. (12.1.7) The Smale horseshoe map is illustrated in the figure below. In this map, some of the points that start in the unit square are mapped outside the square after an iteration of the map. In a

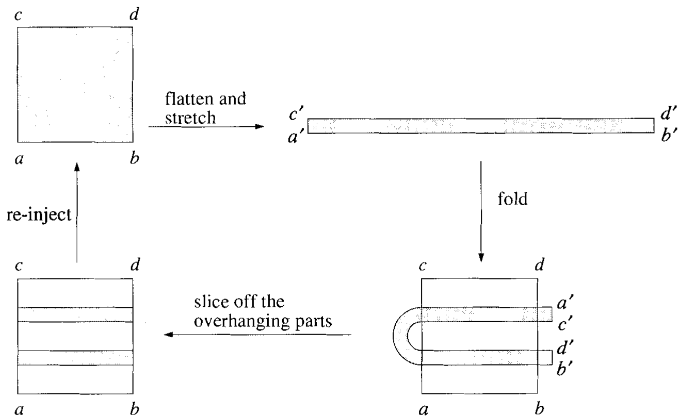

Figure 1: The Smale horseshoe map (from Strogatz)

phenomenon called **transient chaos**, a trajectory of a dynamical system can undergo stretching and folding, with sensitive dependence on initial conditions and a positive Lyapunov exponent, but then eventually leave the region where this is occurring. In the Lorenz map, with $\sigma = 10, b = 8/3$ and $r = 23$ instead of $r = 28$ there is no chaotic attractor, but there are trajectories that show transient chaos before approaching an equilibrium.

Horseshoe maps like this one are used to approximate the Poincaré map of systems with transient chaos. The Hénon map also has a bit of a horseshoe shape.

(a) Consider the unit square. Which regions of the unit square are mapped back into the unit square under one iteration of this map? Mark these regions A and B.

(b) In the manner of the diagram above, sketch the effect of a second iteration of the map. Identify the points in the original unit square that are mapped back into the unit square after two iterations. Mark these regions AA, AB, BA, BB with the first letter indicating whether it was in A or B above and the second letter indicating whether it is in A or B after an iterate of the map.

(c) Work to identify the set of points in the original unit square that will survive forever under forward iterations of the map.

(d) Now consider a backward iterate of the map. Which points stay in the unit square under a single backward iteration? Mark these regions A and B

(e) What about under two backward iterations? Mark these regions AA, AB, BA, BB, similarly to above.

(f) Attempt to construct the set of points that is in the unit square for all time (both forward and backward).

(g) You can use a binary number to describe points in this system, too. For a point v we define its itinerary to be $...S_{-3}S_{-2}S_{-1}S_0.S_1S_2....$ If $f^i(x)$ lies in A set $S_i = A$. If $f^i(x)$ lies in B set $S_i = B$. We can again define a shift map and have chaotic orbits.

Some answers:

1. $\lambda = \ln 2$. Any sequence that does not eventually repeat a finite sequence on the right is a chaotic orbit.

2. (a) In the map, the pieces that return to the unit square correspond to a chunk towards the left and a chunk towards the right of the thin flattened, stretched, bar. Stepping back to the initial square, these chunks correspond to vertical stripes.

   (b) For the second iterate, we stretch the segmented unit square and the 2 horizontal lines stretch to the entire length of the stretched and flattened intermediate step. These are then bent around and put into the square, so we have four thin vertical regions, split in two pairs.

   (c) The points that will stay in for all forward time are a Cantor set (not quite middle thirds, but keeping two segments each time) crossed with a vertical line.

   (d)