Names: ______________________________

1. (3.7.4) Let $\dot{N} = RN(1 - N/K) - HN/(A + N)$. This model is describing harvesting of fish.

   (a) If your group completed this on Tuesday then skip these first steps - you may not have chosen this nondimensionalization, though, and please redo it if you didn't.
   Nondimensionalize, choosing N_0 and T so that the nondimensional equation is
   $$\frac{dx}{d\tau} = x(1 - x) - h\frac{x}{a + x}.$$
   Note: In the last class we learned that $\frac{x}{1+x}$ *is called a Monod function. Similar functions* $\frac{x^n}{1+x^n}$ *are called Hill functions. They switch more sharply with larger* n*. This Monod function is a switching function but the switching won't be too sharp.*

   (b) This is a harvesting model for a fish population. Assume the harvesting is being done by humans. In this context, why do we retain h and a as the nondimensional parameters? We could have chosen r and k instead, but did not.

   (c) As a note, the Monod function is a switching function, so you should be alert to the possibility of bistability in this model.

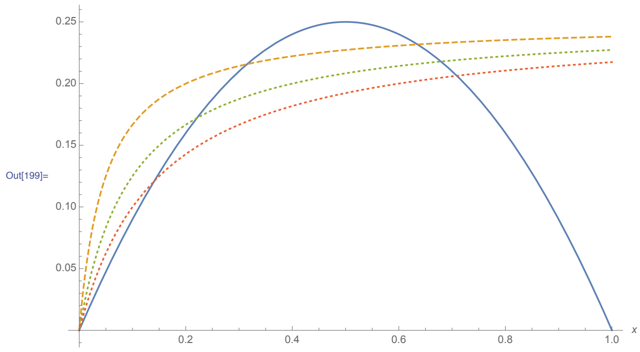


Figure 1: The solid line is a plot of $x(1-x)$. The dashed line (top) is a plot of $\frac{1}{4}\frac{x}{0.05+x}$. The upper dotted line (middle) is a plot of $\frac{1}{4}\frac{x}{0.1+x}$. The lower dotted line (bottom) is a plot of $\frac{1}{4}\frac{x}{0.15+x}$. The parameter a is changing while h stays fixed.

   Based on these plots it should be clear that the system can have three fixed points. It can also have one or two. By solving for the fixed points algebraically, show that this is the case.

   (d) Use linear stability analysis (so compute $\frac{df}{dx}$) to find the stability of the $x = 0$ fixed point.

   (e) A bifurcation occurs when $h = a$. What type of bifurcation is it?

   (f) Classify the stability of the nonzero fixed points (the stability of the zero fixed point should allow you to determine the stability of other fixed points when we are at a particular parameter set because stability alternates).

   (g) Show that another bifurcation occurs when $h = \frac{1}{4}(a + 1)^2 = \frac{1}{4}(1 - a)^2 + a$. Classify this bifurcation.

   Which of these bifurcation diagrams show bistability? Identify the bistable region.

   (i) Plot the stability diagram of the system in the ah-plane. To do this, draw the $h = a$ and $h = \frac{1}{4}(a + 1)^2$ bifurcation lines. This divides the plane into regions. Label each region with the number of fixed points. Which regions exhibit bistability?

   (j) Hysteresis occurs when two steady state solutions exist and the one that we access depends on our initial conditions. Imagine we start with $a = 0.25$ and set harvesting at $h = 0.1$. See the bifurcation diagrams above to find the state of the system. Now slowly ramp up harvesting, h. What eventually happens to the number of fish in the system, and is the change in population gradual or abrupt? What will it take for the population to recover?

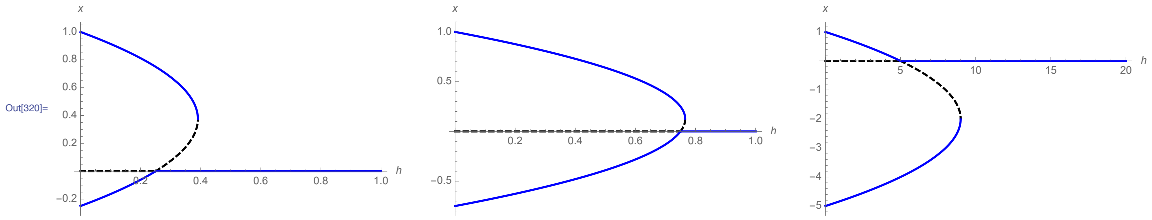
(h) 

Figure 2: This is the bifurcation diagram as h varies for three different values of a. From left to right, $a = 0.25$, $a = 0.75$, $a = 5$. Note that when $a = 1$ the two bifurcations coincide.

2. This problem is from Margo and is an introduction to singular perturbation theory

In the video lecture we were introduced to the idea that a first order system can sometimes be used to approximate a second order system if the second derivative term is small relative to the first derivative term. Unfortunately neglecting this term means we must solve a first-order equation subject to two initial or boundary conditions, which isn't possible. However, the first order equation solution can still be quite useful. This problem shows us how to address the problem of two conditions in the boundary value case, by constructing an approximate solution.

Consider the system

$$\begin{aligned} \epsilon\ddot{x} + 2\dot{x} + 2x &= 0 \\ x(0) &= 0 \\ x(1) &= 1 \end{aligned}$$

where $0 < t < 1$ and $\epsilon << 1$. This is a *boundary value problem*, meaning the value of x is specified at two points (the initial time and the final time) instead of having its position and velocity specified at an initial time.

Assume the solution $x(t)$ can be expanded in powers of ϵ

$$x(t) \sim x_0(t) + \epsilon x_1(t) + \dots.$$

This expansion is called an *asymptotic expansion*, and a Taylor series is one example of such an expansion. In this method, we are splitting the solution into some function $x_0(t)$ that captures most of what is happening, plus some small correction to that function. These expansions are almost always truncated after two terms, as above.

(a) By substituting this expansion into the equation and matching terms based on how many ϵ they have in front of them, derive a first order equation for $x_0(t)$. This is a first order approximation to our second order equation.

This is much like what we do when equate two polynomials, matching terms of the same power.

Solve the first order, $\mathcal{O}(1)$, equation for $x_0(t)$. We use the notation $\mathcal{O}(1)$ because ϵ is not present. Your solution $x_0(t)$ should contain only one arbitrary constant, a.

(b) Now we run into the conundrum of how to deal with the two conditions on the solution. We don't know which boundary condition, if any, we should require $y_0(t)$ to satisfy. Here's what we'll do. We'll assume that $x_0(t)$ describes the solution over most of the interval, but that there is a small region at either $t = 0$ or $t = 1$ where we must use a different approximation. This region is called a *boundary layer*.

(c) Let's assume that there exists a boundary layer at $t = 0$. The boundary layer is a solution that applies just for a short time very close to $t = 0$. We introduce a boundary layer coordinate given by $\tau = t/\epsilon$. Now $\tau = 1$ when $t = \epsilon$ so we have zoomed in on the region in time that we care about, by rescaling our time coordinate.

Change variables in our original differential equation, substituting $\tau = t/\epsilon$ for t.

(d) Assume there is a boundary layer solution of the form $X(\tau) \sim X_0(\tau) + \epsilon X_1(\tau) + \ldots$ to this new equation. Show that the $\mathcal{O}(1/\epsilon)$ problem is

$$\begin{aligned} \ddot{X}_0 + 2\dot{X}_0 &= 0, \quad 0 < \tau < \infty \\ X_0(0) &= 0 \end{aligned} \tag{1}$$

(e) If time permits (if not, just read on), solve for the general solution $X_0(\tau)$. After satisfying the initial condition, your solution $X_0(\tau)$ should contain one arbitrary constant, A.

(f) The solution you just computed describes the solution of the original differential equation in the vicinity of $t = 0$. It is reasonable to assume that the first solution should satisfy the boundary condition at $t = 1$. Use the boundary condition $x(1) = 1$ to solve for the constant a above.

(g) We still have one more constant to solve for, A. We use A to match the boundary layer solution up with the solution $x_0(t)$ that applies for most of the range of t. To do this matching, we assume that as $t \to 0$ $x_0(t)$ is approaching the boundary layer. The boundary layer is using a rescaled time where $\tau \to \infty$ corresponds to leaving the layer. So we will choose A so that $X_0(\tau)$ as $\tau \to \infty$ matches up with $x_0(t)$ as $t \to 0$.

If time permits (otherwise just read on), impose this matching condition and show that

$$X_0(\tau) = e - e^{1-2\tau}$$

(h) We're almost done! Now we combine these two expansions into a composition expansion. We do this by adding the expansions and then subtracting the part that is common to both. Do this by writing $x \sim x_0(t) + X_0(t/\epsilon) - x_0(0)$.

(i) You could go back and solve the boundary value problem analytically for $x(t)$, and then plot the approximate solution together with the analytical solution. Here is the comparison plot for four values of ϵ.

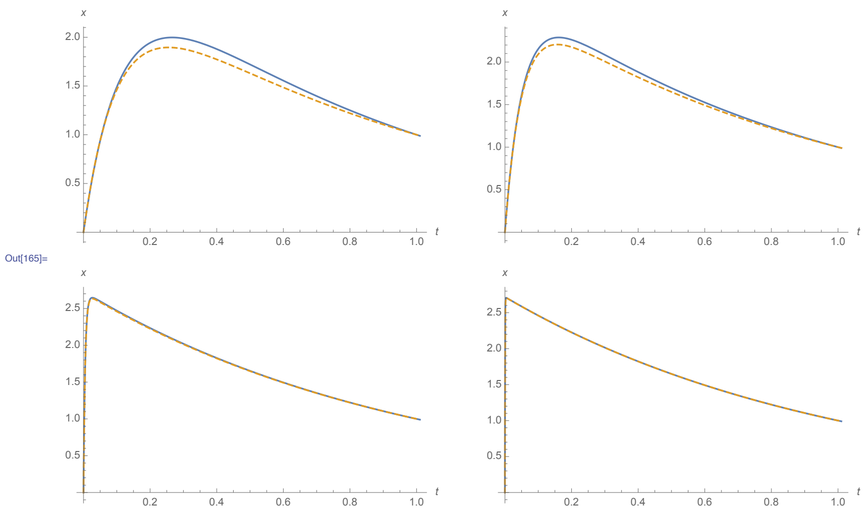


Figure 3: ϵ decreases from left to right. Top row: $\epsilon = 0.2$, $\epsilon = 0.1$. Bottom row: $\epsilon = 0.01$, $\epsilon = 0.001$. The exact solution is the solid blue line and our approximation is the dashed orange line.

Find the boundary layer on each of these plots. How good is the approximation? How does the effectiveness of the approximation change as ϵ changes?

Congratulations, you've now completed a crash course in singular perturbation theory!