Class 10: 2D conservative systems

Goals for the day:

1. How do we find a conserved quantity?
2. What are the implications of having a conserved quantity?

Team problems:

1. (6.5.1) Consider the system $\ddot{x} = -\mu x + x^3$.
   (a) Rewrite this as a first order system.
   (b) Find an expression for $\frac{dy}{dx}$ by taking $\frac{dy}{dt}/\frac{dx}{dt}$. When this results in a separable differential equation, you can solve the equation to find a conserved quantity. Use this method to find a conserved quantity for this system.
   (c) For systems of the form $m\ddot{x} + \frac{dV}{dx} = 0$, $\frac{1}{2}m\dot{x}^2 + V(x)$ is a conserved quantity. Use this to check your result.
   (d) Find the equilibrium points of the system and classify them as saddles or centers.
   (e) For $\mu = 1$, the eigenvalues of the system $\begin{pmatrix} 0 & 1 \\ 2 & 0 \end{pmatrix}$ are $\lambda_\pm = \pm\sqrt{2}$, $\mathbf{v}_\pm = \begin{pmatrix} \mp\sqrt{2}/2 \\ 1 \end{pmatrix}$. Use this information to sketch the phase portrait for this case.
   (f) Identify any homoclinic or heteroclinic orbits, and find an equation for them.
   (g) Identify any regions of the phase space where oscilliations will occur.
   (h) This system is a model of a nonlinear spring. Add a small amount of drag, so $\dot{y} = -x + x^3 - \epsilon y$ with $\epsilon > 0$. What happens to the system? Attempt to sketch a new phase portrait.

2. Consider the system

$$\begin{aligned} \dot{x} &= -\mu y + xy \\ \dot{y} &= \mu x + \frac{1}{2}(x^2 - y^2). \end{aligned}$$

   Assume $\mu > 0$.
   (a) Find a conserved quantity for this system.
   (b) Find the fixed points of the system.