Class 11: index theory in 2D

Goals for the day:

1. What is the value of index theory?
2. How can we use index theory to restrict possibilities?

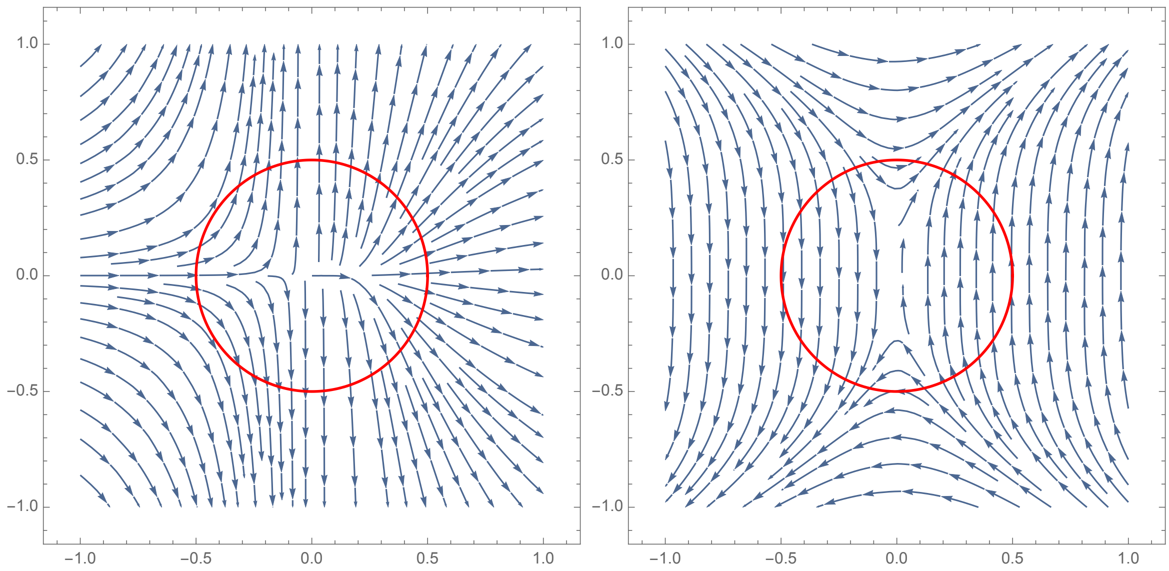
Team problems:

1. (6.8.8) A smooth vector field on the phase plane is known to have exactly three closed orbits. Two of the cycles, C_1 and C_2, lie inside the third cycle, C_3. However, C_1 does not lie inside C_2 or vice versa.
   (a) Sketch the arrangement of the three cycles.
   (b) Show there there must be at least one fixed point in the region bounded by C_1, C_2, and C_3. What can you say about its type?
   (c) How does this change, if at all, if C_1 now lies inside of C_2?
2. For each of the following systems, locate the fixed points and calculate the index
   (a) (6.8.2) $\dot{x} = x^2, \dot{y} = y$.
   (b) (6.8.4) $\dot{x} = y^3, \dot{y} = x$.
3. Consider a system with a saddle point at the origin and a stable spiral at $(1, 0)$, and no other fixed points. Based on index theory, where could closed trajectories potentially exist in this system (can they enclose both fixed points? one? none?)
4. (More 2D practice: 6.3.10) Consider the system

$$\dot{x} = xy$$
$$\dot{y} = x^2 - y$$

   (a) Show that the linearization predicts that the origin is a non-isolated fixed point (non-hyperbolic).
   (b) There is a center subspace for this system. Find the associated eigenvector.
   (c) To determine the stability of the fixed point we will need to do an analysis that includes the nonlinear information. A common method to use is to find the quadratic approximation to the center manifold (recall that this manifold is tangent to the center subspace). Then we study the dynamics on this manifold.
      i. Assume the center manifold can be represented by the function $y = V(x)$. It is tangent to the center subspace so the quadratic approximation is $y = 0 + 0x + \frac{1}{2}wx^2 + \ldots$.
      If we want to study the dynamics along the center manifold direction, then we want to know what is happening along $\dot{x}$ given that we are restricted to the manifold. We replace $y = \frac{1}{2}wx^2$ in the $\dot{x}$ equation to find $\dot{x} = \frac{1}{2}wx^3$ along the manifold. Using a 1D perspective, we can see that the sign of w will determine the stability of the $x = 0$ point on this manifold.
      ii. To start the process of finding w, differentiate the expression $y = V(x)$ with respect to time to find $\dot{y} = \frac{\partial V}{\partial x}\dot{x}$.
      iii. We know $\dot{y}$ from the original definition of the dynamical system, and we know $\dot{x}$ as well. Substitute in this information. Also substitute $y = \frac{1}{2}wx^2$ in place of y. Making the assumption that $|x| \ll 1$, so $x^4 \ll x^2$, keep the leading terms and set w so that the equation is true (up to its leading terms).
      iv. The center manifold has the quadratic approximation $V(x) = \frac{1}{2}wx^2$. Now that you know w, and combining this information with the nonzero eigenvalue you found above, is the origin attracting, repelling, or a saddle?

   The type of analysis is called *center manifold theory*.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