AM147 11:30am

Class 13: post quiz activity

1. After a change of variables the van der Pol system becomes

$$\dot{x} = \mu(y - \frac{1}{3}x^3 + x)$$
$$\dot{y} = -\frac{1}{\mu}x.$$

   (a) In this relaxation oscillation, the trajectory moves very quickly when it jumps between the two parts of the $\dot{x} = 0$ nullcline. Convince yourself that it is moving at a velocity of order $\mathcal{O}(\mu)$ during these jumps. Since it is traversing a distance of about 3 (this is an order one number), estimate the order of magnitude of the time that it spends jumping.

   (b) How close does the trajectory need to be to the $\dot{x} = 0$ nullcline for both $\dot{x}$ and $\dot{y}$ to be the same order of magnitude?

   (c) While on the nullcline, the trajectory moves about 1 in x and a bit less than 2 in y. It is basically moving on the curve $y = \frac{1}{3}x^3 - x$ (not quite, but it is very close to that curve the whole time). The time it spends traversing the curve is

$$\mathcal{O}(\mu^k)$$

   for some integer k. To estimate time (just as we did for the oscillators in chapter 4), we set up an integral of the form

$$\int_{x_1}^{x_2} \frac{dt}{dx} dx$$

   or something like this, and then see how it depends on μ. The suggestion I just made is not a great one because it puts

$$y - (\frac{1}{3}x^3 - x)$$

   in the denominator and that quantity is basically zero, and how far it is from zero probably depends on μ, so it isn't a great choice. Try again with

$$\int_{y_1}^{y_2} \frac{dt}{dy} dy.$$

   This is going to have some problems, too, because it will be a function of x in the integral. So actually, we want

$$\int_{x_1}^{x_2} \frac{dt}{dy}\frac{dy}{dx} dx.$$

   (This is an example of persisting until something works, and luckily getting something to work before we run out of options). Use the setup of this final integral to identify k. Note that for $\frac{dy}{dx}$ we're thinking of ourselves as basically being stuck on the nullcline, so you can compute this directly.

   (d) Compare the amount of time spent jumping to the amount of time spend moving along the curve. (We have the timescales of these processes, and not the exact amounts of time, so compare those).

Answer:

1. (a) When the trajectory is flowing across, $\dot{y}$ is small: $|x| < 3$ or so, and μ is big, so $|\frac{x}{\mu}| < \frac{3}{\mu}$, which is small (specifically order of $\frac{1}{\mu}$. This means the motion is basically horizontal. And it is moving at a speed of $\mu(y - x^3/3 + x)$ in the horizontal direction. Away from the nullcline itself, $y - x^3/3 + x$ is order 1, so μ times it is order μ. It jumps across a distance of maybe 3 as it moves horizontally. This is an order 1 distance. Since $distance/time = velocity$ the time is $distance/velocity$ which is order $\frac{1}{\mu}$. This is a small number. That means it jumps across pretty quickly.

(b) For $\dot{x}$ and $\dot{y}$ to be the same order we need them to both be order $\frac{1}{\mu}$ so we need $(y - x^3/3 + x)$ to be order $\frac{1}{\mu^2}$ so that when it is multiplied by μ it is order $\frac{1}{\mu}$. So we need to be within order $\frac{1}{\mu^2}$ of the nullcline.

(c) $\frac{dt}{dy} = -\mu\frac{1}{x}$ and $\frac{dy}{dx} = x^2 - 1$ on the nullcline. So $T = \int_{x_1}^{x_2} -\mu\frac{1}{x}(x^2-1)dx$. This is μ multiplied by a number, and the number will be order 1. So this time is order of μ.

(d) The jump is order of $\frac{1}{\mu}$ and the motion along the curve is order of μ. This means we spend a ton more time on the curve compared to doing the jump.