Class 14: small perturbations to simple harmonic oscillation

Goals for the day:

1. Review and understand the van der Pol oscillator in the large μ (strongly nonlinear) case.
2. What is a weakly nonlinear oscillator?
3. How does averaging theory allow us to determine the long time evolution of oscillations?
4. What does it mean to have multiple timescales in a system?

In class:

1. Review the van der Pol problem from last time.
2. Summarize the method of averaging.

Team problems:

1. Consider a differential equation

$$\ddot{x} + x + \epsilon h(x, \dot{x}) = 0, \quad 0 \le \epsilon \ll 1.$$

We define $r(t)$ and $\phi(t)$ such that $x(t) \sim \bar{r}\cos(t + \bar{\phi})$ and $\dot{x} \sim -\bar{r}\sin(t + \bar{\phi})$. Here $\sim$ means "is asymptotically approximated by", so $\lim_{\epsilon\downarrow 0} \frac{x(t)}{\bar{r}\cos(t+\bar{\phi})} = 1$. Specifically $x(t) = \bar{r}\cos(t + \bar{\phi}) + \mathcal{O}(\epsilon^2)$.

From the method of averaging, we have

$$\frac{d\bar{r}}{dt} = \epsilon \left\langle h \sin(t + \bar{\phi}) \right\rangle + \mathcal{O}(\epsilon^2)$$
$$\bar{r}\frac{d\bar{\phi}}{dt} = \epsilon \left\langle h \cos(t + \bar{\phi}) \right\rangle + \mathcal{O}(\epsilon^2).$$

(a) Consider the equation $\ddot{x} + x - \epsilon x\dot{x} = 0$. Show that $\dot{\bar{r}} = 0 + \mathcal{O}(\epsilon^2)$ and $\bar{r}\dot{\bar{\phi}} = 0 + \mathcal{O}(\epsilon^2)$.
Recall that $\langle \cos^n t \sin t\rangle = 0$ *for the time average over a single period of* $\sin t$. *And similarly for* $\langle \sin^n t \cos t\rangle$.

(b) Consider the equation $\ddot{x} + x + \epsilon x^2 = 0$. Show that $\dot{\bar{r}} = 0 + \mathcal{O}(\epsilon^2)$ and $\bar{r}\dot{\bar{\phi}} = 0 + \mathcal{O}(\epsilon^2)$.
Note that $\langle \cos^n t\rangle = 0$ *for* n *an odd integer.*

(c) Consider the equation $\ddot{x} + x + \epsilon\dot{x}^3 = 0$. Find $\bar{r}(t)$ and then $x(t, \epsilon)$ to first order in ϵ.
Note that $\sin^4 t = \left(\frac{e^{it} - e^{-it}}{2i}\right)^4$
$= \frac{1}{16}\left(e^{4it} - 4e^{3it}e^{-it} + 6e^{2it}e^{-2it} - 4e^{it}e^{-3it} + e^{-4it}\right)$
$= \frac{1}{16}\left(e^{4it} + e^{-4it} - 4(e^{2it} + e^{-2it}) + 6\right)$
$= \frac{1}{16}\cos 4t - \frac{1}{4}\cos 2t + \frac{3}{8}$.

(d) Consider the equation $\ddot{x} + x + \epsilon\dot{x}^n$, $n \in \mathbb{Z}_+$. Write a conjecture for when you think the correction term at order ϵ will be nonzero for either the $\dot{\bar{r}}$ equation or the $\bar{r}\dot{\bar{\phi}}$.

2. In the method of multiple time scales, we create new time variables: $t \to \tau + T$ where τ is t, the short time scale, and $T = \epsilon t$, the long time scale. We could define more timescales, such as $T_1 = \epsilon^2 t$, an even longer time scale. We assume two time scales is enough, however.

$$\partial_t \to \frac{d\tau}{dt}\partial_\tau + \frac{dT}{dt}\partial_T = \partial_\tau + \epsilon\partial_T.$$

Consider the differential equation

$$\ddot{x} + x + \epsilon h(x, \dot{x}) = 0.$$

Using our new multiple time scales, this equation becomes

$$x_{\tau\tau} + 2\epsilon x_{\tau T} + \epsilon^2 x_{TT} + x + \epsilon h(x, x_\tau + \epsilon x_T) = 0.$$

Sorting by scales, we have

$$x_{\tau\tau} + x + 2\epsilon x_{\tau T} + \epsilon(h(x, x_\tau + \epsilon x_T) + \epsilon^2 x_{TT} = 0.$$

Assume our solution is of the form $x(\tau, T, \epsilon) = x_0(\tau, T) + \epsilon x_1(\tau, T) + ...$

Substituting this in, we have

$$\begin{aligned}
x_{0\tau\tau} &+ \epsilon x_{1\tau\tau} + \epsilon^2 x_{2\tau\tau} \\
&+ x_0 + \epsilon x_1 + \epsilon^2 x_2 \\
&+ 2\epsilon x_{0\tau T} + 2\epsilon^2 x_{1\tau T} \\
&+ \epsilon h(x_0, x_{0\tau}) + \epsilon^2 \text{from h term} \\
&+ \epsilon^2 x_{0TT} \\
&+ \mathcal{O}(\epsilon^3) = 0.
\end{aligned}$$

Sorting by order, our equations are

$$x_{0\tau\tau} + x_0 = 0$$

$$\epsilon(x_{1\tau\tau} + x_1 + 2x_{0\tau T} + h(x_0, x_{0\tau})) = 0$$

Solving the order 1 equations, we find $x_0(\tau, T) = R(T)\cos(\tau + \phi(T))$. This can also be written $x_0(\tau, T) = A(T)\cos\tau + B(T)\sin\tau$).

We have $x_{0\tau} = -R(T)\sin(\tau + \phi(T))$ and $x_{0\tau T} = -R_T \sin(\tau + \phi(T)) - R(T)\sin(\tau + \phi(T))\phi_T$. Plugging in to the order ϵ equations, we find

$$x_{1\tau\tau} + x_1 = -2x_{0\tau T} - h(x_0, x_{0\tau}))$$

$$\Rightarrow x_{1\tau\tau} + x_1 = 2R_T \sin(\tau + \phi) + 2R\sin(\tau + \phi)\phi_T - h(R\cos(\tau + \phi), -R\sin(\tau + \phi))).$$

We choose R and ϕ to avoid resonant forcing on the right hand side, so we want all $\sin\tau$ and $\cos\tau$ terms to disappear.

Use this method to find $x(t, T, \epsilon)$ for $\ddot{x} + x + \epsilon\dot{x}^3 = 0$.

Note: $\cos^3 t = \frac{3}{4}\cos t + \frac{1}{4}\cos 3t$.
$\cos^2 t \sin t = \frac{1}{4}\sin t + \frac{1}{4}\sin 3t$.
$\cos t \sin^2 t = \frac{1}{4}\cos t - \frac{1}{4}\cos 3t$.
$\sin^3 t = \frac{3}{4}\sin t - \frac{1}{4}\sin 3t$.

Answers to 1:

1. (a) We have $\dot{\bar{r}} = \epsilon\left\langle h\sin(t + \bar{\phi})\right\rangle + \mathcal{O}(\epsilon^2)$. In the equation $\ddot{x} + x - \epsilon x\dot{x} = 0$, the function $h(x, \dot{x})$ is given by $-x\dot{x}$. We have defined r and ϕ such that $x \approx \bar{r}\cos(t + \bar{\phi})$ and $y \approx -\bar{r}\sin(t + \bar{\phi})$, so h is $-\bar{r}^2\cos(t + \bar{\phi})\sin(t + \bar{\phi}) + \mathcal{O}(\epsilon)$. Substituting in,

$$\dot{\bar{r}} = \epsilon\left\langle \bar{r}^2\sin^2(t + \bar{\phi})\cos(t + \bar{\phi})\right\rangle + \mathcal{O}(\epsilon^2) = \epsilon\bar{r}^2\left\langle \sin^2(t + \bar{\phi})\cos(t + \bar{\phi})\right\rangle + \mathcal{O}(\epsilon^2).$$

Because $\left\langle \sin^2(t + \bar{\phi})\cos(t + \bar{\phi})\right\rangle = \frac{1}{2\pi}\int_t^{t+2\pi}\sin^2(\tau + \bar{\phi})\cos(\tau + \bar{\phi})d\tau = 0$, we are left with $\dot{\bar{r}} = 0 + \mathcal{O}(\epsilon^2)$.

In this case things work out very similarly for $\bar{r}\dot{\bar{\phi}}$.

(b) Now $h = x^2$. This gives us

$$\dot{\bar{r}} = \epsilon\left\langle \bar{r}^2\sin(t + \bar{\phi})\cos^2(t + \bar{\phi})\right\rangle + \mathcal{O}(\epsilon^2),$$

$$\bar{r}\dot{\bar{\phi}} = \epsilon\left\langle \bar{r}^2\cos^3(t + \bar{\phi})\right\rangle + \mathcal{O}(\epsilon^2).$$

Since $\left\langle \cos^3(t + \bar{\phi})\right\rangle = \frac{3}{4}\left\langle \cos(t + \bar{\phi})\right\rangle + \frac{1}{4}\left\langle \cos 3(t + \bar{\phi})\right\rangle$, this is 0. Similarly for $\dot{\bar{r}}$.

(c) Now $h = \dot{x}^3$.

$$\dot{\bar{r}} = -\epsilon \left\langle \bar{r}^3 \sin^4(t+\bar{\phi}) \right\rangle + \mathcal{O}(\epsilon^2),$$

which gives us something that will be nonzero in the time average at order ϵ. Exciting!! We have $\left\langle \sin^4(t+\bar{\phi}) \right\rangle = \frac{1}{16}\left\langle \cos 4(t+\bar{\phi}) \right\rangle - \frac{1}{4}\left\langle \cos 2(t+\bar{\phi}) \right\rangle + \frac{3}{8} = \frac{3}{8}$. This gives us

$$\dot{\bar{r}} = -\frac{3}{8}\epsilon \bar{r}^3 + \mathcal{O}(\epsilon^2).$$

The origin is now an attracting fixed point (but the timescale is set by $\frac{3}{8}\epsilon$ so it is slowly attracting). We can actually solve this differential equation to find $\bar{r}(t)$.

$$-\bar{r}^{-3}\dot{\bar{r}} = \frac{3}{8}\epsilon$$

$$\Rightarrow \frac{1}{2}\bar{r}^{-2} = \frac{3}{8}\epsilon t + C.$$

To set C, let $\bar{r}(0) = r_0$. Then $\frac{1}{2}r_0^{-2} = C$. Solving for $\bar{r}$, we have

$$\bar{r}(t) = \left(\frac{1}{\frac{3}{4}\epsilon t + r_0^{-2}} \right)^{\frac{1}{2}}.$$

As t gets large (note that ϵt won't matter much until t is of order $\frac{1}{\epsilon}$ for r_0 of order 1, so t isn't large until $\frac{1}{\epsilon} \ll t$), this will approach 0, as it should.

(d) Thinking for generally, let $h = \dot{x}^n$. This puts a $\sin^n(t+\bar{\phi})\sin(t+\bar{\phi})$ in the time average, so if $n+1$ is even this will have a nonzero time average. Thus for n odd there should be a nonzero order ϵ evolution of the $\bar{r}$ equation.