AM147 11:30am

Class 15: Bifurcations

Goals for the day:

1. How do the bifurcations that are familiar from 1D systems manifest in 2D systems?
2. What is a Hopf bifurcation?
3. Why does subcritical/supercritical matter (what makes subcritical bifurcations "dangerous")?

Team Problems:

1. (8.1.6) Consider the system

$$\dot{x} = y - 2x$$
$$\dot{y} = \mu + x^2 - y.$$

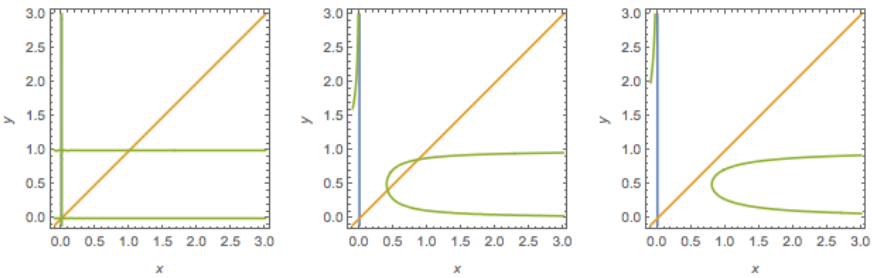
   (a) Sketch the nullclines.
   (b) Find and classify the bifurcations that occur as μ varies.
   (c) Show that at points of bifurcation the nullclines are tangent.

2. (8.1.10) We are back to the spruce budworm model where the budworms are eating a balsam forest. To model the forest, let $S(t)$ be the average size of the trees at time t, and $E(t)$ be an "energy reserve" that is considered a measure of forest health. In the presence of a constant budworm population, B,

$$\dot{S} = r_s S\left(1 - \frac{S}{K_s}\frac{K_E}{E}\right)$$
$$\dot{E} = r_E E\left(1 - \frac{E}{K_E}\right) - P\frac{B}{S}.$$

   Note that all constants are ≥ 0.

   (a) We will nondimensionalize this system. We have four different dimensions present in this problem: tree size, energy, budworm population, and time. Set $x = \frac{S}{S_0}$, $y = \frac{E}{E_0}$, $\tau = \frac{t}{T}$ and $\beta = \frac{B}{B_0}$ where S_0, E_0, T, and B_0 are all characteristic scales that will be chosen to simplify the system. Substitute these into the system and simplify.
   (b) Identify five different nondimensional groups *(one of them is $\frac{S_0}{K_S}$, another is $r_E T$. Find the other three).*
   (c) Choose T, S_0, E_0, B_0 so that all parameters that remain are in the second equation.
   (d) For $\beta = 0$, analyze the system. Find the fixed points, use the Jacobian to classify their type, and sketch the phase portrait. *Note that* $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ *is an eigenvector of the Jacobian for both points.*
   (e) The figures below show the nullclines as β increases from 0 to 0.2. What happens to the fixed points and what kind of bifurcation occurs?



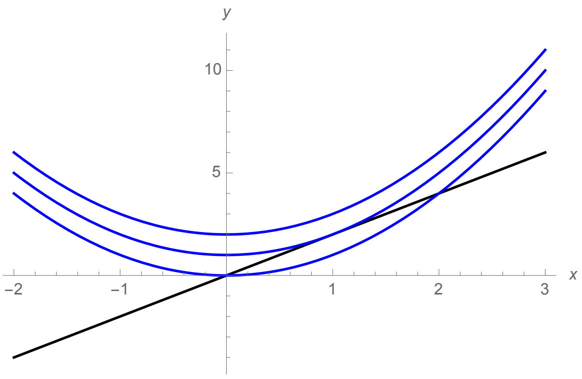
3. (8.2.8) Consider the predator-prey system ($x \geq 0$ is the dimensionless prey, $y \geq 0$ is the dimensionless predators)

$$\dot{x} = x(x(1-x) - y)$$
$$\dot{y} = y(x - a).$$

   (a) Show that the fixed points are $(0, 0)$, $(1, 0)$, and $(a, a - a^2)$.
   (b) Classify these fixed points.
   (c) Show that a Hopf bifurcation (pure imaginary eigenvalues) occurs for $a_c = \frac{1}{2}$.
   (d) Is the bifurcation supercritical or subcritical?

Some Answers:

1. (a) The $\dot{x} = 0$ nullcline is in black and the $\dot{y} = 0$ nullcline is in blue for three different values of μ. These values are below, at, and above the bifurcation point.



   (b) To find and classify the bifurcations we need to identify the fixed points and find where the number of fixed points changes. We can see from the nullclines that the black and blue lines intersect in two places for some values of μ, and as μ increases they intersect at a single point and then no points. This means there's a saddle-node bifurcation. So I've classified the bifurcation just from looking at the nullcline picture. To find the bifurcation, I suppose I need to find the point where there's just one fixed point, or, I can find one of the fixed points and identify where it changes stability. Either option would work.

   At the fixed points $y = 2x$ and $y = \mu + x^2$ because $y - 2x = 0$ and $\mu + x^2 - y = 0$. This means $2x = \mu + x^2$ so $x^2 - 2x + \mu = 0$. Using the quadratic formula,

$$x_{\pm} = 1 \pm \frac{1}{2}\sqrt{4 - 4\mu} = 1 \pm \sqrt{1 - \mu}.$$

   So there is just a single fixed point when $\mu = 1$ and for $\mu > 1$ there are no fixed points. Thus the bifurcation occurs when $\mu = 1$ at the point $(1, 2)$.

   (c) The slope of the $\dot{x} = 0$ nullcline of $y = 2x$ is 2. We want to show the other nullcline has the same slope at the point of bifurcation. The other nullcline is $y = x^2 + \mu$ so its slope is given by $2x$ and at the bifurcation point, $x = 1$, so its slope is also 2. The two lines intersect at $(1, 2)$ for $\mu = 1$ and they have the same slope, so they are tangent.

2. We are studying the forest while the budworms are constant...