Class 16: Bifurcations

Goals for the day:

1. How do chemical oscillators relate to bifurcations?
2. How do the bifurcations of cycles (saddle node of cycles, saddle node infinite period, homoclinic bifurcation) differ from each other?

Team Problems:

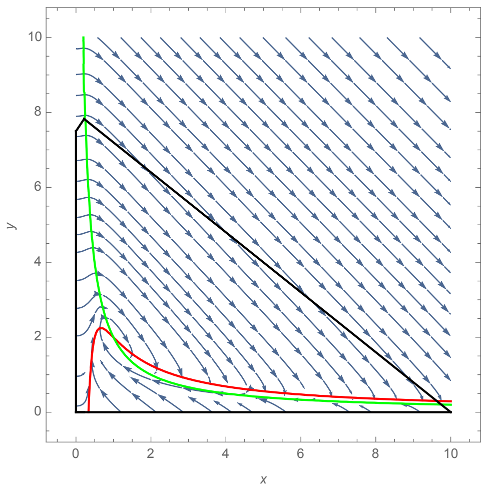
1. (8.3.1, The Brusselator). This is a simple chemical oscillator model. Its kinetics are

$$\dot{x} = 1 - (b+1)x + ax^2 y$$
$$\dot{y} = bx - ax^2 y$$

with $a, b > 0$ parameters of the system and $x, y \geq 0$ dimensionless concentration.

   (a) Find the fixed point of this system and use its Jacobian to classify it.

   (b) Argue that the region shown below is a trapping region, and that a similar region can be constructed for any choice of $a, b > 0$.



   (c) Show that a Hopf bifurcation occurs at some value of b and determine b_c, the value where the bifurcation occurs. *b_c will be a function of a.*

   (d) Use the Poincaré Bendixson theorem to determine whether the limit cycle exists for $b > b_c$ or for $b < b_c$.

   (e) The frequency of the limit cycle at the bifurcation is given by the imaginary part of the pure imaginary eigenvalues. Find the corresponding period, which is the approximate period of the limit cycle for $b \approx b_c$.

2. (8.4.1) Consider the system given by

$$\dot{r} = r(1 - r^2)$$
$$\dot{\theta} = \mu - \sin\theta$$

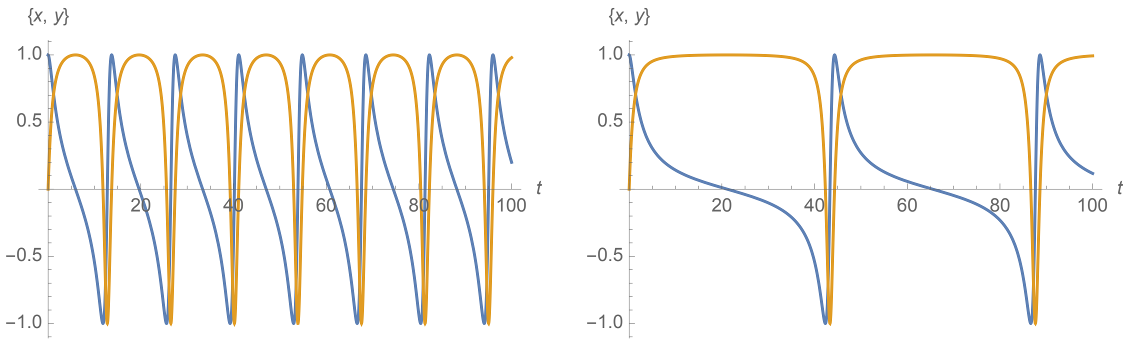
with mu slightly greater than 1, so we are on the verge of an infinite period bifurcation. In lecture we saw the approximate waveform for $x(t)$. Reproduce this argument to approximate the $x(t)$ waveform, and also find the waveform for $y(t)$.

3. (8.4.2) Discuss the bifurcations of the system $\dot{r} = r(\mu - \sin r)$, $\dot{\theta} = 1$ as μ varies.

Some Answers:

1. Fixed point is $(1, \frac{b}{a})$, trace is $b - a - 1$, determinant is $a > 0$ so stability changes with trace at $b_c = a + 1$. At $b_c = a + 1$ the Jacobian at the fixed point is $\begin{pmatrix} a & a \\ -a-1 & -a \end{pmatrix}$ with trace $= 0$ and $\det = a$, so $a = -(i\omega)^2 = \omega^2$ meaning frequency is $\sqrt{a}$ and period is about $\frac{2\pi}{\sqrt{a}}$.

2. Waveforms for $\mu = 1.1$ and $\mu = 1.01$.



3. The fixed points of r in the μr plane for $r \geq 0$ are show in the figure.

   There is always a fixed point at the origin and its stability is given by μ so for $\mu < 0$ it is stable and for $\mu > 0$ it is unstable. At $\mu = 0$ there must be a bifurcation because $r = 0$ changes stability. Specifically, a (stable) limit cycle appears at the origin and grows as μ increases, so this is a supercritical Hopf bifurcation.

   For $\mu < -1$ the origin is the only fixed point in the μr plane so no limit cycles exist. At $\mu = -1$ an infinite number of limit cycles are born in saddle-node of cycles bifurcations. These occur at $r = 2\pi n$ for $n = 1, 2, \ldots.$ These cycles are stable on their upper branch and unstable on their lower (we can reason this out by knowing that the Hopf produces a stable cycle and the rest of the stability must alternate).

   At $\mu = 1$ they disappear in another set of saddle node bifurcations, and these occur at $r = \pi + 2\pi n$ for $n = 0, 1, 2, \ldots.$