Class 17: Coupled oscillators

Goals for the day:

1. How might coupled oscillators behave if they are interacting?
2. How do we use a Poincaré map?
3. What is quasiperiodicity?

Team Problems:

1. (8.6.1: "Oscillator death" and bifurcations on a torus), from Ermentrout and Kopell (1990). Consider the following model:

$$\dot{\theta}_1 = \omega_1 + \sin\theta_1 \cos\theta_2$$
$$\dot{\theta}_2 = \omega_2 + \sin\theta_2 \cos\theta_1.$$

   The oscillators have a natural frequency, but they also are responding to each other. We'd like to determine the different behaviors that manifest in this system.

   (a) Consider $\phi = \theta_1 - \theta_2$. Set up a differential equation to describe the evolution of ϕ, and identify any bifurcations or fixed points.

   (b) Also consider $\psi = \theta_1 + \theta_2$ and do a similar analysis.

   (c) Classify the different behaviors of the system in the $\omega_1\omega_2$ plane.
   Note: If $\theta_1 - \theta_2 = C_1$ and $\theta_1 + \theta_2 = C_2$ then the system has a fixed point.

2. (8.7.2) Consider the vector field on the cylinder given by $\dot{y} = ay$ and $\dot{\theta} = 1$. This system is not in polar coordinates because y in not a polar angle. Instead the system evolves on a cylinder so that θ can change periodically why $y \in \mathbb{R}$.

   Let $\Sigma = \{(y, 0) : y \in [-1, 1]\}$ be a line segment on the cylinder. Define a Poincaré map, $P : \Sigma \to \Sigma$. To do this, note that it takes time 2π for θ to evolve by 2π and thus for the trajectory to return to Σ.

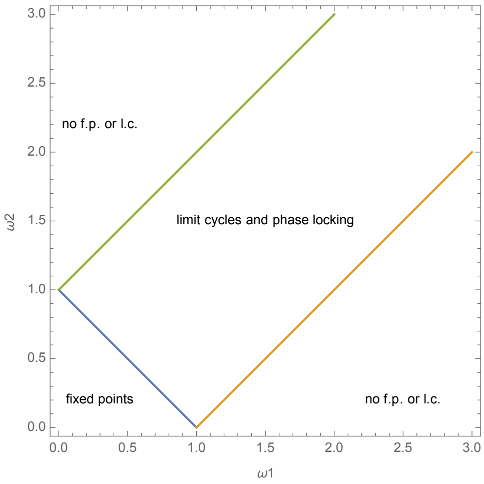
$$2\pi = \int_0^{2\pi} dt = \int_y^{P(y)} \frac{dt}{dy} dy.$$

   After finding the map, show that the system has a periodic orbit (meaning that P has a fixed point), and use its Floquet multiplier (*eigenvalue*) to identify the stability.

   When you examine the Floquet multiplier, it is important to remember that we are wondering whether perturbations grow in length or shrink near the fixed point, so we are not comparing to 0 in this case.

Some Answers:

1. The chart shown for $\omega_1 > 0$ and $\omega_2 > 0$.



2. We have $2\pi = \int_y^P \frac{1}{a}\frac{1}{y}dy$. So $2\pi a = \ln\frac{P(y)}{y}$. $\Rightarrow P(y) = ye^{2\pi a}$. This has a fixed point when $P(y) = y$ so when $y = ye^{2\pi a}$. This means $y^* = 0$ is a fixed point. The Floquet multiplier is $e^{2\pi a}$ and it has magnitude less than 1 when $a < 0$. This is all consistent of our understanding of the system based on examining the flow.