The Value of Information: Why You Should Add the Second Order Conditions

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Abstract

When conducting estimation based on agent optimization, we show that one can improve the performance of the estimator when information such as the second order condition is appropriately incorporated as moment inequality restrictions, especially when there are weak instruments. We run a simulation study to demonstrate the effectiveness of this approach in both continuous and discrete choice problems, and illustrate to empirical researchers how to include the additional moment inequalities in practice.

Keywords: Second order condition, moment inequality, weak instrument, discrete choice.

1 Introduction

We study the estimation of an optimal choice problem inspired by the empirical example in Pakes, Porter, Ho, and Ishii (2015), where agents make investment decisions based on their private information on productivity. In such settings, we worry that some of the instruments used might be weak, which motivates the analysis here.

We first conduct estimation for a continuous choice problem with varying instrument strengths, where we show the effects of explicitly including the second order condition. Then, we analyze the corresponding discrete choice problem using a revealed preference set-up, as in Ciliberto and Tamer (2009), and show how the second order condition can be incorporated there.

Given the inequality nature of both the second order condition and the revealed preferences, we follow Chernozhukov, Hong, and Tamer (2007) for estimation. Meanwhile, although the literature on the inference for partial identification proposes various approaches such as Imbens and Manski (2004) and Andrews and Guggenberger (2009), we do not address the inference problem here, because we think that the Monte-Carlo simulation results are effective in conveying our main point. Furthermore, even though we will be working with a specific model in this paper, the key issues raised here can be relevant in a wide variety of settings as long as the estimation relies on optimality conditions combined with instruments, such as Berry, Levinsohn, and Pakes (1995).

2 The Model

Consider the optimal investment choice of a firm d_i given the investments d_{-i} already made by its competitors in the same market, analogous to the set-up in Pakes, Porter, Ho, and Ishii (2015).

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Suppose the revenue of the firm is as follows:

$$r(d_i, d_{-i}) = A \times \frac{d_i}{d_i + d_{-i}} \tag{1}$$

where the constant A is known.

The cost of installing d_i units of the investments is quadratic:

$$c_i(d_i) = (\beta_1 + \nu_i)d_i + \beta_2 d_i^2 \tag{2}$$

$$\mathbb{E}[\nu_i] = 0 \tag{3}$$

where ν_i represents the firm's independent draw of its idiosyncratic productivity shock that is known to the firm but unobservable to the econometrician. Thus, the firm makes its investment decision based on ν_i and d_{-i} to maximize its profit $\Pi_i(d_i, d_{-i}) = r(d_i, d_{-i}) - c_i(d_i)$.

Lastly, we assume that d_{-i} is drawn from a Poisson distribution whose mean negatively depends on $\nu_i + u_i$, where u_i represents an additional independent cost shock that is only relevant for the competitors.

2.1 The Continuous Optimal Choice Problem

Suppose the firm's optimal choice is continuous, that is $d_i \in \mathbb{R}$, we want to obtain an estimate of β_1 and β_2 based on the relevant moment conditions.

2.1.1 Moment Conditions Based on First Order Conditions (FOC)

For ease of notation, denote $c(d_i) = \beta_1 d_i + \beta_2 d_i^2$ and $\Pi(d_i, d_{-i}) = r(d_i, d_{-i}) - c(d_i)$. We form the following moment conditions based on the first order condition of each optimizing firm:

$$\mathbb{E}\left[\Pi'(d_i, d_{-i}) z_i\right] = \mathbb{E}\left[\left(A\frac{d_{-i}}{(d_i + d_{-i})^2} - (\beta_1 + 2\beta_2 d_i)\right) z_i\right] = \mathbb{E}\left[\nu_i z_i\right] = 0 \tag{4}$$

where z_i is any positive instrument that satisfies $\mathbb{E}[\nu_i z_i] = 0$. We have two valid instruments:

(1) $z_i^1 = 1$

(2) $z_i^2 = u_i$

Note that d_i and d_{-i} are both endogenous. Here, u_i is a valid instrument for d_i because u_i affects d_i through the number of competitors in the market d_{-i}, but is independent from ν_i. This problem is just identified and we can use the standard IV estimator.

However, in practical settings, one may not observe the cost shock u_i precisely and could suffer weak instrument problems. We model this through scaling u_i by $\pi > 0$ and adding a positive random noise, following Staiger and Stock (1997):

$$z_i^2 = \pi u_i + \epsilon_i$$
$$\epsilon_i \sim \text{Uniform}[0,1)$$

2.1.2 Moment Conditions Based on Second Order Conditions (SOC)

Given that the firm profit is *maximized*, we also know that $\Pi_i'' \leq 0$. Since the instruments are positive, we can form the following inequality moments based on this second order condition:[2]

$$\mathbb{E}\left[\Pi''(d_i, d_{-i}) z_i\right] = \mathbb{E}\left[\left(-A\frac{2d_{-i}}{(d_i + d_{-i})^3} - 2\beta_2\right) z_i\right] \leq 0 \tag{5}$$

Combining with Eq (4), we obtain a lower bound $\underline{\beta}_2^j$ and an upper bound $\bar{\beta}_1^j$ from each instrument:

$$\beta_2 \geq \underline{\beta}_2^j := \frac{\mathbb{E}\left[\left(-A\frac{d_{-i}}{(d_i+d_{-i})^3}\right) z_i^j\right]}{\mathbb{E}\left[z_i^j\right]}$$
$$\beta_1 \leq \bar{\beta}_1^j := \frac{\mathbb{E}\left[\left(A\frac{d_{-i}}{(d_i+d_{-i})^2} - 2\underline{\beta}_2 d_i\right) z_i^j\right]}{\mathbb{E}\left[z_i^j\right]}$$

Geometrically, Figure 1 shows that each moment equality condition generated by the FOC identifies a line in the space of (β_1, β_2), where their intersection produces the IV estimator. However, the moment inequality condition generated by the SOC further restricts each line to a *ray* starting at $(\bar{\beta}_1^j, \underline{\beta}_2^j)$. If an instrument becomes weak, producing an intersection that is not on the ray, the SOC restriction will become binding.

2.1.3 Simulation Results

We run simulations to illustrate the properties of the estimators.

First, Figure 2 shows that as the instrument weakens, the IV estimator becomes increasingly noisy and biased, exhibiting the classical weak instrument problem.

Next, we add the inequality moments generated by the SOC to the equality moments generated by the FOC, where the estimation is conducted following Chernozhukov, Hong, and Tamer (2007) using an identity weighting matrix. Figure 3 shows that this noticeably "tucks in" one of the tails.

Therefore, even when a problem has enough equality restrictions for identification, incorporating the

[2] Although in this example the inequality can be applied at the observation level, practically, any measurement error in Π_i'' would require the expectation operator.

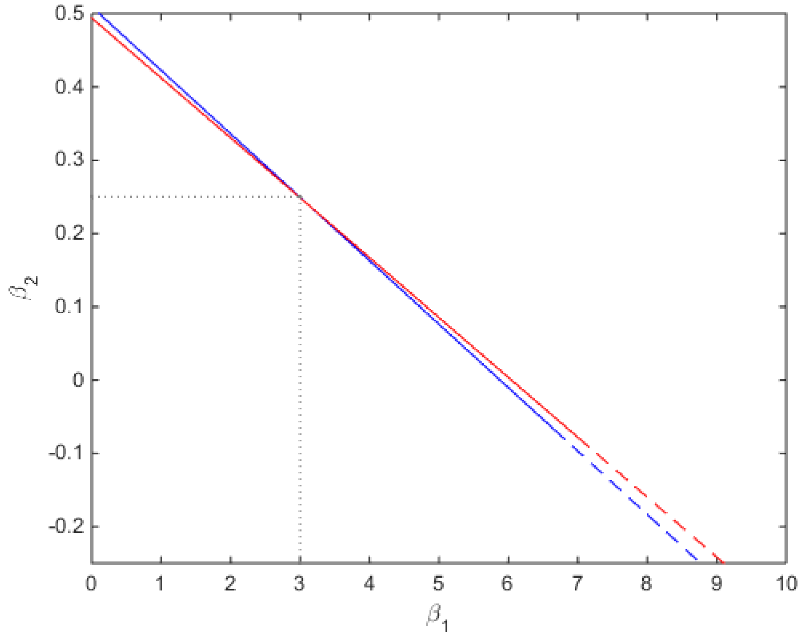

Figure 1: Each FOC produces a line and the intersection produces the IV point estimate, while the dashed part shows the portion ruled out by the SOC. The true parameter value $\beta_1 = 3$ and $\beta_2 = 0.25$.

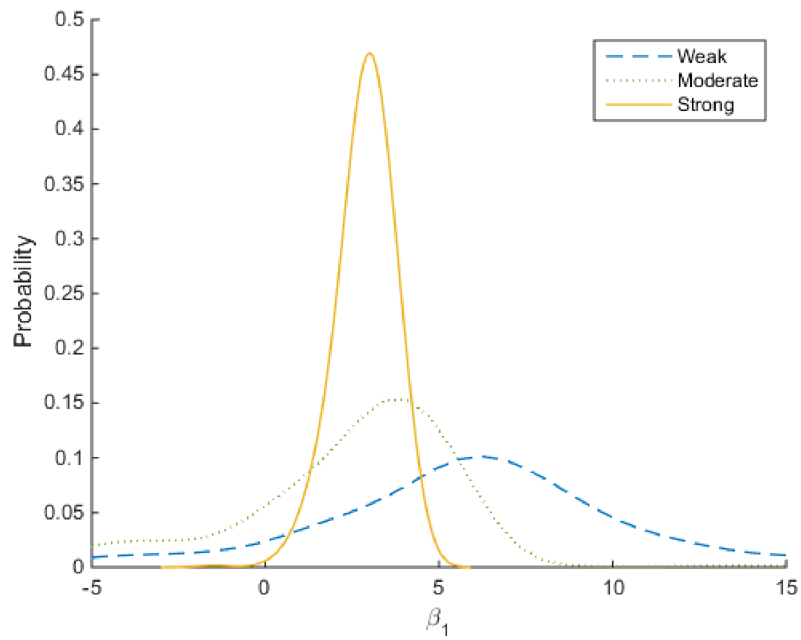

Figure 2: The effects of the instrument strengths. $\pi = 0.02, 0.1, 0.5$ are used for the weak, moderate and strong label respectively.

second order condition could still improve the efficiency of the estimator, especially when some of the instruments are weak.

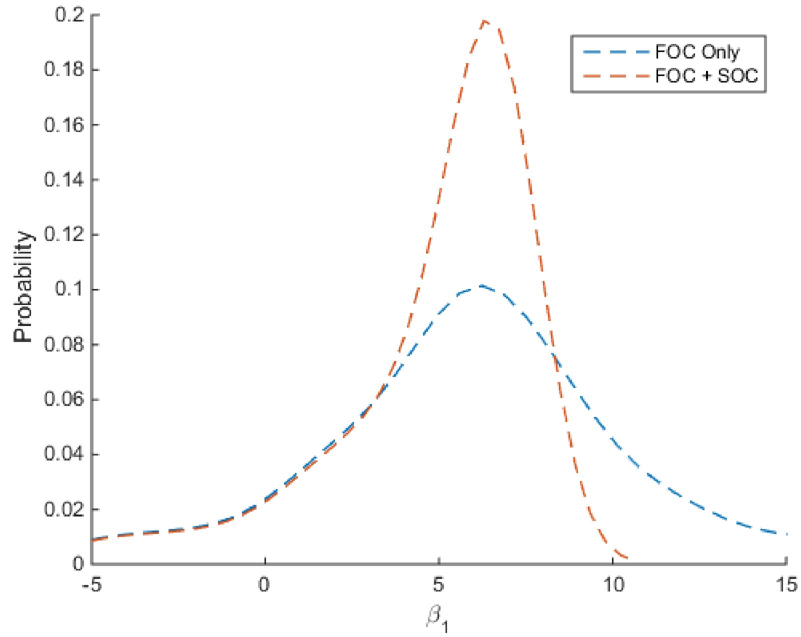

Figure 3: The instrument is weak. Comparing to only using the FOC restrictions, including moments from the second order condition "tucks in" one of the tails.

2.2 The Discrete Optimal Choice Problem

In this section, we study the corresponding discrete choice problem, which is analogous to the previous section except that the firm can no longer choose any investment $d_i \in \mathbb{R}$, but only discrete units with a discretization step of S. Specifically, $S = 1$ implies that $d_i \in \mathbb{Z}$. The revenue function, the cost function and the agent's information set remain the same.

2.2.1 Inequality Moment Conditions Based on Optimality

Based on revealed preferences, namely $\Pi_i(d_i, d_{-i}) \geq \Pi_i(d_i - S, d_{-i})$ and $\Pi_i(d_i, d_{-i}) \geq \Pi_i(d_i + S, d_{-i})$, we can construct the following moment inequality restrictions for the same positive instruments:

$$\mathbb{E}\left[\left(\frac{\Pi(d_i, d_{-i}) - \Pi(d_i - S, d_{-i})}{S}\right) z_i\right] \geq \mathbb{E}\left[\nu_i z_i\right] = 0 \tag{6}$$

$$\mathbb{E}\left[\left(\frac{\Pi(d_i, d_{-i}) - \Pi(d_i + S, d_{-i})}{S}\right) z_i\right] \geq \mathbb{E}\left[-\nu_i z_i\right] = 0 \tag{7}$$

The estimator will find the bounds of the identified set if feasible, and minimizes the deviations otherwise. Meanwhile, combining (6) and (7), we obtain

$$\mathbb{E}\left[\left(\frac{\Pi(d_i + S, d_{-i}) - \Pi(d_i, d_{-i})}{S} - \frac{\Pi(d_i, d_{-i}) - \Pi(d_i - S, d_{-i})}{S}\right) z_i\right] \leq 0 \tag{8}$$

which resembles SOC because it computes the difference of the first derivative $\Pi'(d_i, d_{-i})$ estimated above and below d_i.

2.2.2 Simulation Results

We run simulations using the two inequality moment conditions constructed in (6) and (7).

To build intuition, we show in Figure 4 the identified set using the constant only. The intersection of the two inequalities forms a "wedge", which contains the ray constructed by the FOC and SOC of the corresponding continuous problem up to an approximation term.[3]

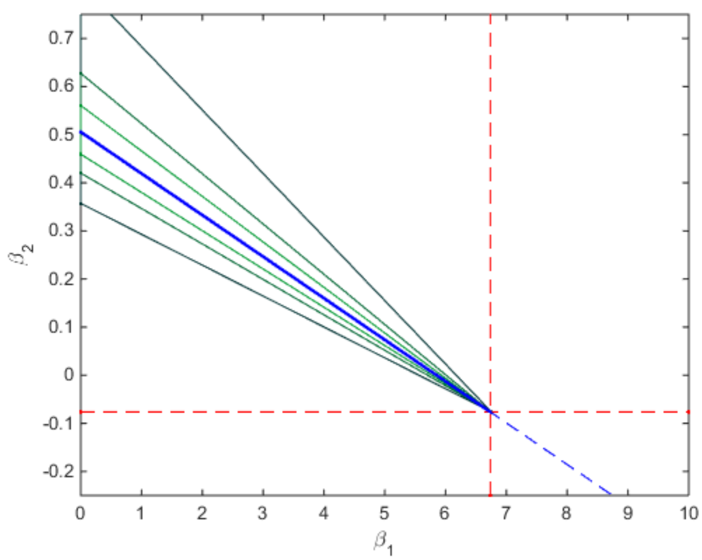

Figure 4: Identification using the constant. The thick blue ray shows the FOC and SOC restrictions of the continuous problem, where the start of the ray is emphasized by the red-dashed lines. The pair of green lines shows the "wedge" identified by the moment inequalities, which becomes "thinner" as S decreases.

Then, we show in Figure 5 the effects of the instrument strengths. With $S = 1$ fixed, the bounds of the identified set is much less sensitive to the weakening of the instrument, compared to the IV estimator of the corresponding continuous problem. This nice behavior is due to the implicit incorporation of the SOC as shown in Eq (8).

Next, Figure 6 shows as the discretization step size decreases, the bounds estimated from the discrete problem starts to resemble the IV estimator of the continuous problem, increasingly breaching the second order condition. To understand this, take the limit of Eq (6) and (7) with $S \to 0$:

$$\mathbb{E}\left[\Pi'_{-}(d_i, d_{-i})z_i\right] \geq 0$$
$$\mathbb{E}\left[\Pi'_{+}(d_i, d_{-i})z_i\right] \leq 0$$

[3]Note that $\mathbb{E}\left[\left(\frac{\Pi(d_i+S;\hat{\beta})}{S} - \frac{\Pi(d_i;\hat{\beta})}{S}\right) z_i\right] = \mathbb{E}\left[\left(\Pi'(d_i;\hat{\beta})\right) z_i\right] + \frac{1}{2}\mathbb{E}\left[\left(\Pi''(d_i;\hat{\beta})\right) z_i\right] S + \mathcal{O}(S^2) \leq 0$, provided $\mathbb{E}\left[\left(\Pi'(d_i;\hat{\beta})\right) z_i\right] = 0$ and $\mathbb{E}\left[\left(\Pi''(d_i;\hat{\beta})\right) z_i\right] \leq 0$ and the third term is not too large.

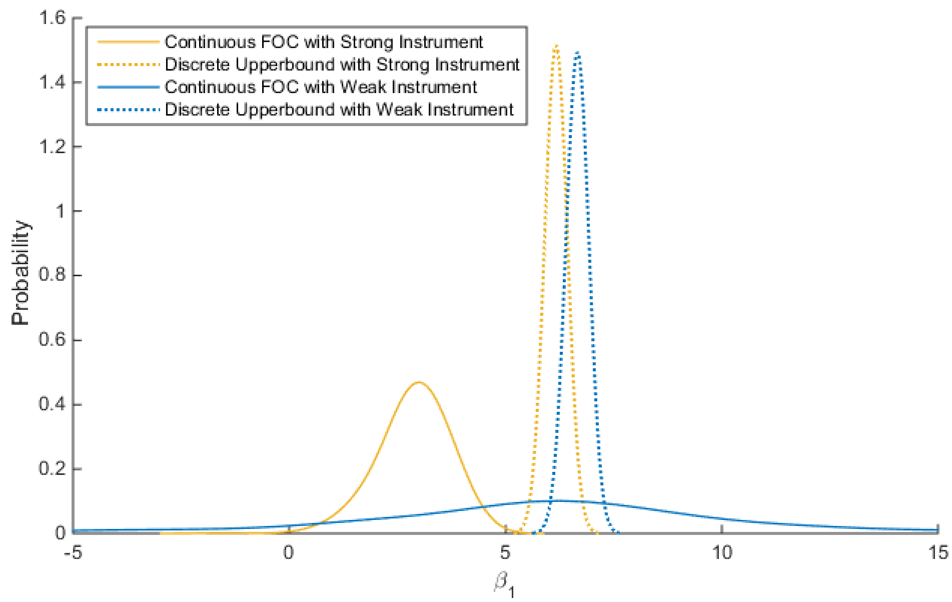

Figure 5: The upperbound of the identified set of the discrete problem (the dotted lines) is much less sensitive to the weak instrument than the corresponding parameter estimates obtained from the FOCs of the continuous problem (the solid lines).

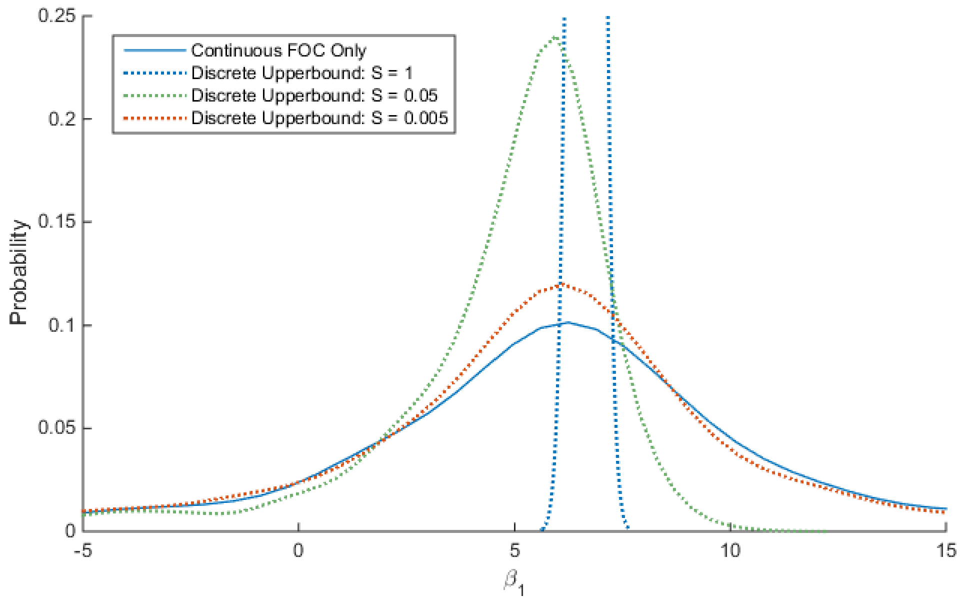

Figure 6: With large S, the upperbound of the identified set is further from the true value, but the distribution of the bound itself is narrow. As S decreases, the distribution starts to resemble the IV estimator of the continuous problem.

Since Π is differentiable, we recover the first order condition:

$$\mathbb{E}\left[\Pi'(d_i, d_{-i}) z_i\right] = 0 \tag{9}$$

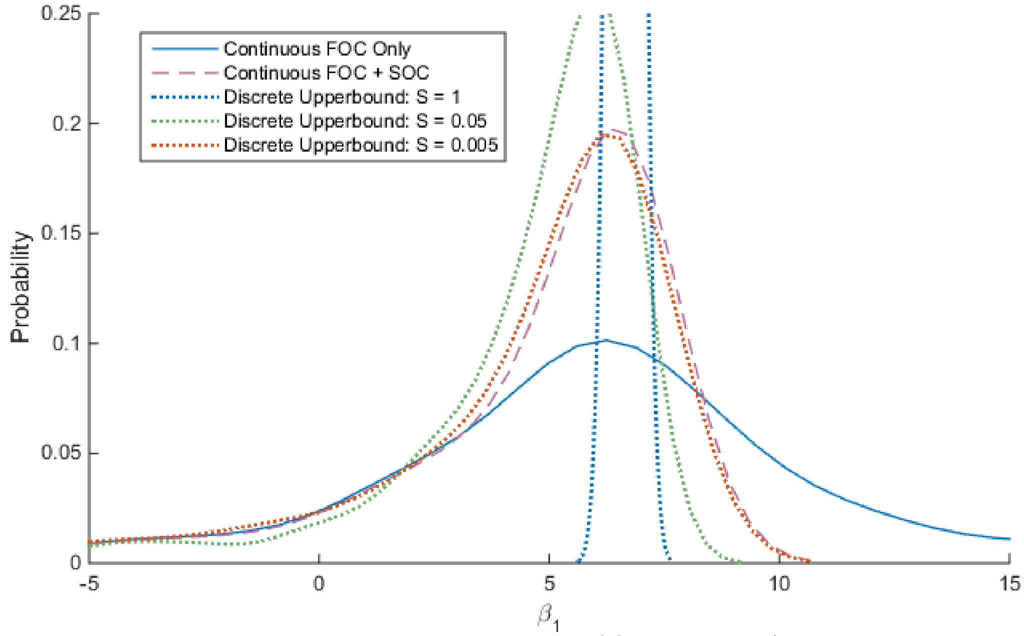

Figure 7: With additional moment inequalities specified by (8) scaled by $1/S$, as S decreases, the distribution of the upperbounds starts to resemble the FOC + SOC estimator of the continuous problem.

However, we can rewrite Eq (8) as

$$\mathbb{E}\left[\left(\Pi''(d_i, d_{-i})S + \mathcal{O}(S^2)\right) z_i\right] \leq 0 \tag{10}$$

Notice that the strength of the second order condition is scaled by S. As $S \to 0$, the revealed preference set-up converges to that of the FOC only and the SOC loses its effect.

To address this perverse behavior, we suggest explicitly constructing the additional moment for the SOC as in Eq (8) but scaled by $1/S$. In the limit when $S \to 0$, this becomes the explicit addition of the SOC moments to the continuous problem, shown in Figure 7.

Relatedly, by adding moment conditions that look beyond the "immediate neighbor" for $N \times S$ steps away, one also improves the relevance of the SOC by a factor of N. However, one needs to trade off these additional moments with potentially larger confidence sets. Indeed, Pakes, Porter, Ho, and Ishii (2015) included larger steps ($d = \pm 2$) and found the estimate of the identified set unchanged.

3 Conclusion

Using a simple optimal choice setting, we showed why it can be useful to include additional moment conditions for both the continuous and the discrete choice problem. Therefore, regardless whether there are already enough moments for identification, we suggest empirical researchers to consider explicitly incorporating moments based on the second order condition, which may be particularly useful when there are weak instruments.

Appendix: Simulation Details

$\beta_1 = 3, \beta_2 = 0.25, A = 500$. ν_i and u_i are drawn independently from Uniform$[-2.5, 2.5)$. d_{-i} is drawn from a Poisson distribution with $\lambda = 50 + 50 \times (1 - 0.2(\nu_i + u_i))$. The number of simulation draws $ns = 500$. The sample size for each draw $N = 500$.

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