# Single Image Spatially Variant Out-of-focus Blur Removal
Supplementary Material

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## 1 Derivation of $\psi(i,j)$

In this section, we explain the intuition of how to derive the equation

$$\psi(i,j) = \frac{1}{|\mathcal{A}|}\sum_{(p,q)\in\mathcal{A}} \left(1+\frac{1}{k}\right)\widetilde{\mathbf{g}}(i+p,i+q) - \frac{1}{k}\widetilde{\mathbf{g}}(i+2p,i+2q). \tag{1}$$

### 1.1 1D Intuition

To start with, we consider the one-dimension case. Filling the missing foreground $\Omega_F$ is equivalent to extrapolate a discrete-time signal $g[n]$ for $n \geq 0$, with known values of $g[n]$ for $n < 0$. There are various ways of extrapolation. Here, we consider the method that enforces the smoothness across the boundary. More precisely, we want

$$g[n] - g[n-1] = g[n-1] - g[n-2], \tag{2}$$

where $g[n]-g[n-1]$ is the finite difference approximation to the derivative at $n$, and $g[n-1]-g[n-2]$ is the finite difference approximation to the derivative at $n-1$. Thus, the condition means that the slope at $g[n]$ should be the same as the slope at $g[n-1]$. Determining $g[n]$ from (2) is straight-forward, because $g[n-1]$ and $g[n-2]$ are known. Thus,

$$g[n] = 2g[n-1] - g[n-2].$$

### 1.2 2D Intuition

Extending the idea to the two-dimensional setting, we want the gradient of a two-dimensional signal $g[i,j]$ at pixel $(i,j)$ to be similar to the gradients of its neighborhood. Since the two-dimensional gradient is directional, there are multiple equations for predicting $g[i,j]$:

$$g[i,j] - g[i+p,j+q] = g[i+p,j+q] - g[+2p,j+2q], \tag{3}$$

where $p = q = \{-1, 0, 1\}$. Determining $g[i,j]$ is not as easy, because there are multiple equations in (3). Unless for some specific situations, in general $g[i,j]$ needs to solved by fitting the neighborhood. To this end, we consider the set of valid neighborhoods

$$\mathcal{A} = \{(p,q) \,|\, g[i+p,i+q] \neq 0, |p| \leq 1, |q| \leq 1\}.$$

Here, the set $\mathcal{A}$ denotes the set of pixels that are neighbors of $g[i,j]$ and they are known. Then, finding $g[i,j]$ from the pixels in $\mathcal{A}$ becomes the minimization problem

$$g[i,j] = \underset{g[i,j]}{\operatorname{argmin}} \sum_{(p,q)\in\mathcal{A}} (g[i,j] - 2g[i+p,i+q] + g[i+2p,i+2q])^2,$$

of which the solution can be found by considering the first order optimality, yielding

$$g[i,j] = \frac{1}{|\mathcal{A}|}\sum_{(p,q)\in\mathcal{A}} 2g[i+p,i+q] - g[i+2p,i+2q].$$

### 1.3 Stability Condition

The condition $g'[n] = g'[n-1]$ has a problem that it leads to unbounded prediction, because if $g'[n-1] > 0$, then $g[n] \to \infty$ as $n \to \infty$. To ensure boundedness, instead of using $g'[n] = g'[n-1]$, we require $g'[n] = \frac{1}{n}g'[n-1]$ for $n > 0$, and $g'[n] = g'[n-1]$ for $n = 0$. Consequently, the recursion is defined as

$$g[n] = \left(1 + \frac{1}{n}\right) g[n-1] - \frac{1}{n} g[n-2] \quad \text{for } n > 0, \tag{4}$$

with the initial condition $g[0] = 2g[-1] - g[-2]$. Intuitively, this recursion forces the slope at every extrapolation location to be reduce by a factor depending on the physical distance from the object boundary. The following proposition shows the boundedness of this method.

**Proposition 1.** *Suppose that $g[-1]$ and $g[-2]$ are bounded, and hence $g[0] = 2g[-1] - g[-2]$ is also bounded. $g[n]$ satisfying the condition $g'[n] = \frac{1}{n}g'[n-1]$ has the recursion*

$$g[n] = \left(1 + \frac{1}{n}\right) g[n-1] - \frac{1}{n} g[n-2], \quad \textit{for } n > 0, \tag{5}$$

*and $g[n]$ is bounded for all $n$.*

*Proof.* Since $g'[n] = g[n] - g[n-1]$, $g'[n] = \frac{1}{n}g'[n-1]$ implies $g[n] - g[n-1] = \frac{1}{n}(g[n-1] - g[n-2])$. By rearranging the terms we have (5). The boundedness can be proved by induction: $g[1]$ and $g[2]$ are bounded, because $g[0]$, $g[-1]$ are bounded. Assume that $g[k]$ and $g[k+1]$ are bounded, then by triangle inequality $|g[k+2]| \leq \left(1 + \frac{1}{k+2}\right)|g[k+1]| + \frac{1}{k+2}|g[k]|$ is also bounded. □

Incorporating the idea of diminishing gradient so that $g[i,j]$ is bounded, we have

$$g[i,j] = \frac{1}{|\mathcal{A}|} \sum_{(p,q)\in\mathcal{A}} \left(1 + \frac{1}{k}\right) g[i+p, i+q] - \frac{1}{k} g[i+2p, i+2q],$$

where $k$ is the shortest distance from the unknown pixel $(i,j)$ to the known set $\Omega_B$. Replace the $g[i,j]$ by $\psi(i,j)$ and $g[i+p, j+p]$ by $\widetilde{\mathbf{g}}(i+p, j+p)$, we have

$$\psi(i,j) = \frac{1}{|\mathcal{A}|} \sum_{(p,q)\in\mathcal{A}} \left(1 + \frac{1}{k}\right) \widetilde{\mathbf{g}}(i+p, i+q) - \frac{1}{k} \widetilde{\mathbf{g}}(i+2p, i+2q).$$

## 2 Boundedness of $\Delta\widehat{\mathbf{f}}_B$

In this section, we show the following statement.

**Proposition 2.**

$$\|(\alpha * \mathbf{h}_F) \cdot (1 - \alpha * \mathbf{h}_F) \cdot \Delta\widehat{\mathbf{f}}_B\| \leq \|(1 - \alpha * \mathbf{h}_F) \cdot \Delta\widehat{\mathbf{f}}_B\|,$$

*where the norm $\|\cdot\|$ is Frobenius-norm.*

*Proof.* Note that for each pixel $\Delta\widehat{\mathbf{f}}_B(i,j)$, $|(\alpha * \mathbf{h}_F)(i,j) \cdot (1 - \alpha * \mathbf{h}_F)(i,j) \cdot \Delta\widehat{\mathbf{f}}_B(i,j)| \leq (1 - \alpha * \mathbf{h}_F)(i,j) \cdot \Delta\widehat{\mathbf{f}}_B(i,j)|$ because $0 \leq (\alpha * \mathbf{h}_F)(i,j) \leq 1$. Summing the squares of individual elements completes the proof. □

# 3 Shock Filter

In this section we briefly describe the shock filter. Given an input image $\mathbf{f}$, shock filter first applies a smoothing blur kernel, typically a Gaussian blur kernel of size $9 \times 9$ and variance $\sigma = 1$. The purpose of applying the smoothing kernel to the input image is to remove textures and noise.

In the $k$-th iteration of the shock filter, the $k+1$-th solution is given by

$$\mathbf{f}^{k+1} = \mathbf{f}^{k} - \beta \, \mathrm{sign}(\Delta \mathbf{f}^{k}) \|\nabla \mathbf{f}^{k}\|_1.$$

Here $\nabla \mathbf{f} = [\mathbf{f}_x; \mathbf{f}_y]$ is the gradient of $\mathbf{f}$ and $\Delta \mathbf{f} = \mathbf{f}_x^2 \mathbf{f}_{xx} + 2\mathbf{f}_x \mathbf{f}_y \mathbf{f}_{xy} + \mathbf{f}_y^2 \mathbf{f}_{yy}$ is the Laplacian of $\mathbf{f}$. $\beta (= 1)$ is the step size.

# 4 Edge Selection

Finally we provide some brief discussion on the edge selection mask $\mathbf{M}$. First, given a blurred image $\mathbf{g}$ we define a metric

$$\mathbf{R} = \frac{\sqrt{|\mathbf{h}_A * \mathbf{g}_x|^2 + |\mathbf{h}_A * \mathbf{g}_y|^2}}{\mathbf{h}_A * \sqrt{|\mathbf{g}_x|^2 + |\mathbf{g}_y|^2} + 0.5},$$

where $\mathbf{h}_A$ is a $5 \times 5$ a $5 \times 5$ uniform average kernel. The numerator $\mathbf{h}_A * \mathbf{g}_x$ is the average of the horizontal gradient within a $5 \times 5$ window. Therefore, if there are small objects/textures/noise, positive and negative gradients will cancel out each other. On the other hand, the denominator $\mathbf{h}_A * \sqrt{|\mathbf{g}_x|^2 + |\mathbf{g}_y|^2}$ is the average of the absolute gradient, which is always positive. As a result, $\mathbf{R}$ differentiates the large objects versus small texture in the window.

To rule out small values of $\mathbf{R}$, one can set a threshold as

$$\widetilde{\mathbf{R}} = \max \left\{ \mathbf{R} - \tau_r, 0 \right\},$$

where $\tau_r$ is a threshold. Finally, we define the edge selection mask as

$$\mathbf{M} = \max \left\{ \widetilde{\mathbf{R}} \cdot \sqrt{|\mathbf{f}_x^s|^2 + |\mathbf{f}_y^s|^2} - \tau_s, 0 \right\},$$

where $\tau_s$ is a threshold, $\mathbf{f}^s$ is the shock filtered image, $\mathbf{f}_x^s$ and $\mathbf{f}_y^s$ are gradients of $\mathbf{f}^s$.