

Do We Really Need Gaussian Filters for Feature Detection? (Supplementary Material)

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Lemma 1. *The derivatives of the Gaussian functions*

$$g(t; \sigma) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{t^2}{2\sigma^2}}$$

are

$$g'(t; \sigma) = \frac{1}{\sqrt{2\pi\sigma^2}} \left(-\frac{t}{\sigma^2} e^{-\frac{t^2}{2\sigma^2}} \right) \quad \text{and} \quad g''(t; \sigma) = \frac{1}{\sqrt{2\pi\sigma^2}} \left(\frac{t^2}{\sigma^4} - \frac{1}{\sigma^2} \right) e^{-\frac{t^2}{2\sigma^2}}.$$

Proof. Direct calculation. □

Normalized Derivative.

Scale-space axiom requires that there is no enhancement to the local maximum and local minimum [1]. Therefore, the derivative must be *normalized*. Defining the m th order γ normalized derivative operator as

$$\partial_{\sigma^m, \gamma} = (\sigma^2)^{\frac{m\gamma}{2}} \partial_{t^m},$$

we take $\gamma = 1$ and $m = 2$, and apply the operator to $g(t; \sigma)$ to yield

$$\begin{aligned} \partial_{\sigma^2, 1} [g(t; \sigma)] &= (\sigma^2) \partial_{t^2} [g(t; \sigma)] \\ &= \sigma^2 g''(t; \sigma). \end{aligned}$$

Therefore, in the paper, instead of using $g(t; \sigma)$ we use $\sigma^2 g(t; \sigma)$. In this case,

$$\sigma^2 g''(t; \sigma) = \frac{1}{\sqrt{2\pi}} \left(\frac{t^2}{\sigma^3} - \frac{1}{\sigma} \right) e^{-\frac{t^2}{2\sigma^2}}.$$

Definition 1. Let $T > 0$ be a constant. The signal $I(t)$ is defined as

$$I(t) = \begin{cases} 1, & -T \leq t \leq T \\ 0, & \text{otherwise.} \end{cases} \quad (1)$$

Definition 2. The noise $n(t)$ is i.i.d. Gaussian with zero mean and variance σ_N^2 , denoted by $n(t) \sim \mathcal{N}(0, \sigma_N^2)$.

Proposition 1. Letting

$$\begin{aligned} v(t, \sigma) &= \sigma^2 g''(t; \sigma) * I(t), \text{ and} \\ w(t, \sigma) &= \sigma^2 g''(t; \sigma) * n(t), \end{aligned}$$

it holds that

$$v(t, \sigma) = \frac{1}{\sqrt{2\pi}} \left[-\frac{t+T}{\sigma} e^{-\frac{(t+T)^2}{2\sigma^2}} + \frac{t-T}{\sigma} e^{-\frac{(t-T)^2}{2\sigma^2}} \right], \quad (2)$$

$$\mathbb{E}[w(t, \sigma)] = 0, \quad (3)$$

$$\mathbb{E}[w(t, \sigma)w(t+\tau, \sigma)] = \sigma^4 \sigma_N^2 g''(-\tau, \sigma^2) * g''(\tau, \sigma^2). \quad (4)$$

Hence,

$$\text{Var}[w(t, \sigma)] = \mathbb{E}[w(t, \sigma)^2] = \frac{3\sigma_N^2}{8\sqrt{\pi}\sigma}. \quad (5)$$

Proof. For simplicity we drop the variable σ . First, note that $I(t) = u(t+T) - u(t-T)$, where $u(t)$ is the unit step function. Consequently,

$$\begin{aligned} v(t; \sigma) &= \sigma^2 g''(t) * I(t) = \int_{-\infty}^{\infty} \sigma^2 g''(\tau) I(t-\tau) d\tau \\ &= \int_{-\infty}^{\infty} \left(\frac{\tau^2}{\sigma^4} - \frac{1}{\sigma^2} \right) \frac{\sigma^2}{\sqrt{2\pi\sigma^2}} e^{-\frac{\tau^2}{2\sigma^2}} [u(t+T-\tau) - u(t-T-\tau)] d\tau \\ &= \frac{1}{\sqrt{2\pi}} \int_{t-T}^{t+T} \left[\frac{\tau^2}{\sigma^3} - \frac{1}{\sigma} \right] e^{-\frac{\tau^2}{2\sigma^2}} d\tau \\ &= \frac{1}{\sqrt{2\pi}} \int_{t-T}^{t+T} \left[\frac{\tau^2}{\sigma^3} \right] e^{-\frac{\tau^2}{2\sigma^2}} d\tau - \frac{1}{\sqrt{2\pi}} \int_{t-T}^{t+T} \frac{1}{\sigma} e^{-\frac{\tau^2}{2\sigma^2}} d\tau \\ &= \frac{1}{\sqrt{2\pi}} \left[-\frac{t+T}{\sigma} e^{-\frac{(t+T)^2}{2\sigma^2}} + \frac{t-T}{\sigma} e^{-\frac{(t-T)^2}{2\sigma^2}} \right]. \end{aligned}$$

That the mean is zero is due to linearity of convolution:

$$\mathbb{E}[w(t)] = \mathbb{E} \left[\int_{-\infty}^{\infty} \sigma^2 g''(\tau) n(t-\tau) d\tau \right] = \int_{-\infty}^{\infty} \sigma^2 g''(\tau) \mathbb{E}[n(t-\tau)] d\tau = 0.$$

The autocorrelation of the output is

$$\begin{aligned}
\mathbb{E}[w(t)w(t+\gamma)] &= \mathbb{E}\left[\int_{-\infty}^{\infty} \sigma^2 g''(\tau_1) n(t-\tau_1) d\tau_1 \int_{-\infty}^{\infty} \sigma^2 g''(\tau_2) n(t+\gamma-\tau_2) d\tau_2\right] \\
&= \int \int \sigma^4 g''(\tau_1) g''(\tau_2) \mathbb{E}[n(t-\tau_1)n(t+\gamma-\tau_2)] d\tau_1 d\tau_2 \\
&= \int \int \sigma^4 g''(\tau_1) g''(\tau_2) \sigma_N^2 \delta(-\tau_1 - \gamma + \tau_2) d\tau_1 d\tau_2 \\
&= \int_{-\infty}^{\infty} \sigma^4 \sigma_N^2 g''(\tau_1) \underbrace{\int_{-\infty}^{\infty} g''(\tau_2) \delta(\tau_2 - (\tau_1 + \gamma)) d\tau_2}_{d\tau_1} d\tau_1 \\
&= \sigma^4 \sigma_N^2 \int_{-\infty}^{\infty} g''(\tau_1) g''(\tau_1 + \gamma) d\tau_1 \\
&= \sigma^4 \sigma_N^2 g''(-\gamma) * g''(\gamma).
\end{aligned}$$

Since $\mathbb{E}[w(t)] = 0$, we have $\text{Var}[w(t)] = \mathbb{E}[w(t)^2] = \mathbb{E}[w(t)w(t+\gamma)]|_{\gamma=0}$. Therefore, $\text{Var}[w(t)] = \sigma^4 \sigma_N^2 g''(-\tau) * g''(\tau)|_{\tau=0}$. The convolution is

$$\begin{aligned}
g''(\tau) * g''(-\tau) &= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi\sigma^2}} \left(\frac{\gamma^2}{\sigma^4} - \frac{1}{\sigma^2} \right) e^{-\frac{\gamma^2}{2\sigma^2}} \frac{1}{\sqrt{2\pi\sigma^2}} \left(\frac{(\gamma+\tau)^2}{\sigma^4} - \frac{1}{\sigma^2} \right) e^{-\frac{(\gamma+\tau)^2}{2\sigma^2}} d\gamma \\
&= \frac{1}{2\pi\sigma^2} \int_{-\infty}^{\infty} \left(\frac{\gamma^2}{\sigma^4} - \frac{1}{\sigma^2} \right) \left(\frac{(\gamma+\tau)^2}{\sigma^4} - \frac{1}{\sigma^2} \right) e^{-\frac{\gamma^2+\gamma^2+2\gamma\tau+\tau^2}{2\sigma^2}} d\gamma \\
&= \frac{1}{2\pi\sigma^2} e^{-\frac{\tau^2}{2\sigma^2}} \int_{-\infty}^{\infty} \left(\frac{(\gamma^2+\gamma\tau)^2}{\sigma^8} - \frac{\gamma^2}{\sigma^6} - \frac{(\gamma+\tau)^2}{\sigma^6} + \frac{1}{\sigma^4} \right) e^{-\frac{\gamma^2+\gamma\tau}{\sigma^2}} d\gamma.
\end{aligned}$$

Putting $\tau = 0$ yields

$$\begin{aligned}
[g''(\tau) * g''(-\tau)]_{\tau=0} &= \frac{1}{2\pi} \int_{-\infty}^{\infty} \left(\frac{\gamma^4}{\sigma^8} - \frac{2\gamma^2}{\sigma^6} + \frac{1}{\sigma^4} \right) e^{-\frac{\gamma^2}{\sigma^2}} d\gamma \\
&= \frac{1}{2\pi\sigma^2} \int_{-\infty}^{\infty} \frac{\gamma^4}{\sigma^8} e^{-\frac{\gamma^2}{\sigma^2}} d\gamma - \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{2\gamma^2}{\sigma^6} e^{-\frac{\gamma^2}{\sigma^2}} d\gamma + \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{1}{\sigma^4} e^{-\frac{\gamma^2}{\sigma^2}} d\gamma \\
&= \frac{2}{2\pi\sigma^2} \int_0^{\infty} \frac{\gamma^4}{\sigma^8} e^{-\frac{\gamma^2}{\sigma^2}} d\gamma - \frac{2}{2\pi} \int_{-\infty}^{\infty} \frac{\gamma^2}{\sigma^6} e^{-\frac{\gamma^2}{\sigma^2}} d\gamma + \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{1}{\sigma^4} e^{-\frac{\gamma^2}{\sigma^2}} d\gamma \\
&= \frac{1}{2\pi\sigma^2} \left[\frac{2}{\sigma^8} \frac{(2*2-1)!!}{2^{(2+1)}(\frac{1}{\sigma^2})^2} \sqrt{\pi\sigma^2} - \frac{2}{\sigma^6} \frac{1}{2} \sqrt{\pi\sigma^6} + \frac{1}{\sigma^4} \sqrt{\pi\sigma^2} \right] \\
&= \frac{3}{8\sqrt{\pi}\sigma^5}.
\end{aligned}$$

Therefore,

$$\text{Var}[w(t)] = \sigma^4 \sigma_N^2 g''(-\tau) * g''(\tau)|_{\tau=0} = \frac{3\sigma_N^2}{8\sqrt{\pi}\sigma}.$$

□

Proposition 2. Given T , $v(0, \sigma)$ is a local minimum of $v(t, \sigma)$ if and only if $\sigma > T/\sqrt{3}$. Furthermore, the optimal scaling parameter σ^* for which $\partial v(0, \sigma)/\partial \sigma = 0$ is $\sigma^* = T$.

Proof. By KarushKuhnTucker optimality condition, $v(0, \sigma)$ is a local minimum of $v(t, \sigma)$ if and only if $v'(0) = 0$ and $v''(0) > 0$. The first order derivative is

$$\begin{aligned} \frac{\partial v(t, \sigma)}{\partial t} \Big|_{t=0} &= \frac{1}{\sqrt{2\pi}} \left[-\frac{1}{\sigma} e^{-\frac{(t+T)^2}{2\sigma^2}} - \left(-\frac{2(t+T)^2}{2\sigma^3} \right) e^{-\frac{(t+T)^2}{2\sigma^2}} + \frac{1}{\sigma} e^{-\frac{(t-T)^2}{2\sigma^2}} + \left(-\frac{2(t-T)^2}{2\sigma^3} \right) e^{-\frac{(t-T)^2}{2\sigma^2}} \right]_{t=0} \\ &= \frac{1}{\sqrt{2\pi}} \left[-\frac{1}{\sigma} e^{-\frac{(T)^2}{2\sigma^2}} - \left(-\frac{2(T)^2}{2\sigma^3} \right) e^{-\frac{(T)^2}{2\sigma^2}} + \frac{1}{\sigma} e^{-\frac{(-T)^2}{2\sigma^2}} + \left(-\frac{2(-T)^2}{2\sigma^3} \right) e^{-\frac{(-T)^2}{2\sigma^2}} \right] = 0, \end{aligned}$$

thus the first order criteria is satisfied. The second order derivative is

$$\begin{aligned} \frac{\partial^2 v(t, \sigma)}{\partial t^2} \Big|_{t=0} &= \frac{1}{\sqrt{2\pi}} \left[\left(\frac{t+T}{\sigma^3} + \frac{2(t+T)}{\sigma^3} - \frac{(t+T)^3}{\sigma^5} \right) e^{-\frac{(t+T)^2}{2\sigma^2}} \right]_{t=0} \\ &\quad - \frac{1}{\sqrt{2\pi}} \left[\left(\frac{t-T}{\sigma^3} + \frac{2(t-T)}{\sigma^3} - \frac{(t-T)^3}{\sigma^5} \right) e^{-\frac{(t-T)^2}{2\sigma^2}} \right]_{t=0} \\ &= \frac{1}{\sqrt{2\pi}\sigma^3} \left[\left(3T - \frac{T^3}{\sigma^2} \right) e^{-\frac{T^2}{2\sigma^2}} + \left(3T - \frac{T^3}{\sigma^2} \right) e^{-\frac{T^2}{2\sigma^2}} \right] \\ &= \frac{2}{\sqrt{2\pi}\sigma^3} \left[3T - \frac{T^3}{\sigma^2} \right] e^{-\frac{T^2}{2\sigma^2}} \end{aligned}$$

Therefore, $v''(0) > 0$ if and only if $3T - \frac{T^3}{\sigma^2} > 0$, which is

$$\frac{T}{\sigma} < \sqrt{3}.$$

The second statement can be derived by considering the first order optimality condition of $v(0, \sigma)$ over σ , i.e., find σ such that $\frac{\partial v(0, \sigma)}{\partial \sigma} = 0$. Since

$$v(0, \sigma) = \frac{1}{\sqrt{2\pi}} \left[-\frac{2T}{\sigma} e^{-\frac{T^2}{2\sigma^2}} \right],$$

we have

$$\begin{aligned} \frac{\partial v(0, \sigma)}{\partial \sigma} &= -\frac{2T}{\sqrt{2\pi}} \left[(-1)\sigma^{-2} e^{-\frac{T^2}{2\sigma^2}} + (\sigma^{-1})(-2)\frac{-T^2}{2\sigma^3} e^{-\frac{T^2}{2\sigma^2}} \right] \\ &= -\frac{2T}{\sqrt{2\pi}} \left[\frac{T^2}{\sigma^4} - \frac{2}{\sigma^2} \right] e^{-\frac{T^2}{2\sigma^2}} = -\frac{2T}{\sqrt{2\pi}\sigma^2} \left[\frac{T^2}{\sigma^2} - 1 \right] e^{-\frac{T^2}{2\sigma^2}}. \end{aligned}$$

Setting $\frac{\partial v(0, \sigma)}{\partial \sigma} = 0$ yields $\sigma^* = T$. \square

Proposition 3. Given σ , and let

$$C^* = \underset{C}{\operatorname{argmin}} \int_{-\infty}^{\infty} [\sigma^2 g''(t; \sigma) - h''(t; \sigma)]^2 dt,$$

then $C^* = \alpha\sigma$ where $\alpha \approx 0.91$ is a constant.

Proof. We let

$$\begin{aligned} f(t) &\stackrel{\text{def}}{=} \sigma^2 g''(t) = \frac{1}{\sqrt{2\pi}} \left(\frac{t^2}{\sigma^3} - \frac{1}{\sigma} \right) e^{-\frac{t^2}{2\sigma^2}}, \text{ and} \\ \hat{f}(t) &\stackrel{\text{def}}{=} h''(t) \\ &= \frac{1}{6C}[u(t+3C) - u(t+C)] - \frac{2}{6C}[u(t+C) - u(t-C)] + \frac{1}{6C}[u(t-C) - u(t-3C)], \end{aligned}$$

where $u(t)$ is the unit step function. Our goal is to find C such that it minimizes the residue

$$C^* = \underset{C}{\operatorname{argmin}} \left\| f(t) - \hat{f}(t) \right\|_{L_2}^2, \quad (6)$$

where

$$\left\| f(t) - \hat{f}(t) \right\|_{L_2}^2 = \int_{-\infty}^{\infty} [f(t) - \hat{f}(t)]^2 dt.$$

Observe that

$$\left\| f(t) - \hat{f}(t) \right\|_{L_2}^2 = \|f(t)\|^2 - 2\langle f(t), \hat{f}(t) \rangle + \|\hat{f}(t)\|^2.$$

We now investigate each term individually.

$$\begin{aligned} \|\hat{f}(t)\|^2 &= \int_{-\infty}^{\infty} \frac{1}{(6C)^2} [U(t+3C) - U(t-3C)] dt \\ &\quad - 2 \int_{-\infty}^{\infty} \frac{2^2}{(6C)^2} [U(t+C) - U(t-C)] dt + \int_{-\infty}^{\infty} \frac{1}{(6C)^2} [U(t-C) - U(t-3C)] dt \\ &= \int_{-3C}^{-C} \frac{1}{36C^2} dt + \int_{-C}^C \frac{4}{36C^2} dt + \int_C^{3C} \frac{1}{36C^2} dt \\ &= \frac{-C+3C}{36C^2} + \frac{4(C+C)}{36C^2} + \frac{3C-C}{36C^2} \\ &= \frac{12C}{36C^2} = \frac{1}{3C}. \end{aligned}$$

Before we calculate $\langle f(t), \hat{f}(t) \rangle$, we calculate $\int_a^b f(t) dt$ in advance. Due to the unit step function,

the integral will be in a certain interval $[a, b]$.

$$\begin{aligned}
\int_a^b f(t)dt &= \int_a^b \frac{1}{\sqrt{2\pi}} \left(\frac{t^2}{\sigma^3} - \frac{1}{\sigma} \right) e^{-\frac{t^2}{2\sigma^2}} dt \\
&= \frac{1}{\sqrt{2\pi}} \left[\int_a^b \frac{t^2}{\sigma^3} e^{-\frac{t^2}{2\sigma^2}} dt - \int_a^b \frac{1}{\sigma} e^{-\frac{t^2}{2\sigma^2}} dt \right] \\
&= \frac{1}{\sqrt{2\pi}} \left[-\frac{t}{\sigma} e^{-\frac{t^2}{2\sigma^2}} \right]_a^b + \frac{1}{\sqrt{2\pi}} \left[\int_a^b \frac{1}{\sigma} e^{-\frac{t^2}{2\sigma^2}} dt \right] - \frac{1}{\sqrt{2\pi}} \left[\int_a^b \frac{1}{\sigma} e^{-\frac{t^2}{2\sigma^2}} dt \right] \\
&= \frac{1}{\sqrt{2\pi}} \left[-\frac{t}{\sigma} e^{-\frac{t^2}{2\sigma^2}} \right]_a^b = \frac{1}{\sqrt{2\pi}} \left[-\frac{b}{\sigma} e^{-\frac{b^2}{2\sigma^2}} + \frac{a}{\sigma} e^{-\frac{a^2}{2\sigma^2}} \right]
\end{aligned}$$

Using previous result, we can simplified the following equation.

$$\begin{aligned}
\langle f(t), \hat{f}(t) \rangle &= \int_{-\infty}^{\infty} f(t) \hat{f}(t) dt = \frac{1}{6C} \int_{-3C}^{-C} f(t) dt - \frac{2}{6C} \int_{-C}^C f(t) dt + \frac{1}{6C} \int_C^{3C} f(t) dt \\
&= \frac{1}{6C\sqrt{2\pi}} \left[\frac{C}{\sigma} e^{-\frac{C^2}{2\sigma^2}} - \frac{3C}{\sigma} e^{-\frac{9C^2}{2\sigma^2}} \right] - \frac{2}{6C\sqrt{2\pi}} \left[\frac{-C}{\sigma} e^{-\frac{C^2}{2\sigma^2}} - \frac{C}{\sigma} e^{-\frac{9C^2}{2\sigma^2}} \right] \\
&\quad + \frac{1}{6C\sqrt{2\pi}} \left[-\frac{3C}{\sigma} e^{-\frac{9C^2}{2\sigma^2}} + \frac{C}{\sigma} e^{-\frac{C^2}{2\sigma^2}} \right] \\
&= \frac{1}{6C\sqrt{2\pi}} \left[\frac{6C}{\sigma} e^{-\frac{C^2}{2\sigma^2}} - \frac{6C}{\sigma} e^{-\frac{9C^2}{2\sigma^2}} \right] = \frac{1}{\sqrt{2\pi}\sigma} \left[e^{-\frac{C^2}{2\sigma^2}} - e^{-\frac{9C^2}{2\sigma^2}} \right].
\end{aligned}$$

Finally, we have

$$\begin{aligned}
\|f(t)\|^2 &= \int_{-\infty}^{\infty} f^2(t) dt = \int_{-\infty}^{\infty} \frac{1}{2\pi} \left(\frac{t^2}{\sigma^3} - \frac{1}{\sigma} \right)^2 e^{-\frac{t^2}{\sigma^2}} dt = \int_{-\infty}^{\infty} \frac{1}{2\pi} \left(\frac{t^4}{\sigma^6} - \frac{2t^2}{\sigma^4} + \frac{1}{\sigma^2} \right) e^{-\frac{t^2}{\sigma^2}} dt \\
&= \int_{-\infty}^{\infty} \frac{1}{2\pi\sigma^2} \left(\frac{t^4}{\sigma^4} - \frac{2t^2}{\sigma^2} + 1 \right) e^{-\frac{t^2}{\sigma^2}} dt \\
&= \frac{1}{2\pi\sigma^2} \left[\int_{-\infty}^{\infty} \frac{t^4}{\sigma^4} e^{-\frac{t^2}{\sigma^2}} dt - \int_{-\infty}^{\infty} \frac{2t^2}{\sigma^2} e^{-\frac{t^2}{\sigma^2}} dt + \int_{-\infty}^{\infty} e^{-\frac{t^2}{\sigma^2}} dt \right] \\
&= \frac{1}{2\pi\sigma^2} \left[\frac{2}{\sigma^4} \int_0^{\infty} t^4 e^{-\frac{t^2}{\sigma^2}} dt - \frac{4}{\sigma^2} \int_0^{\infty} t^2 e^{-\frac{t^2}{\sigma^2}} dt + \int_{-\infty}^{\infty} e^{-\frac{t^2}{\sigma^2}} dt \right] \\
&= \frac{1}{2\pi\sigma^2} \left[\frac{2}{\sigma^4} \frac{(2*2-1)!!}{2^{2+1}} \sqrt{\pi\sigma^{10}} - \frac{4}{\sigma^2} \frac{1}{4} \sqrt{\pi\sigma^6} + \sqrt{\pi\sigma^2} \right] \\
&= \frac{1}{2\pi\sigma^2} \left[\frac{2}{\sigma^4} \frac{3}{8} \sigma^5 \sqrt{\pi} - \frac{\sigma^3}{\sigma^2} \sqrt{\pi} + \sqrt{\pi}\sigma \right] = \frac{3}{8\sqrt{\pi}\sigma}
\end{aligned}$$

Therefore,

$$\|f(t) - \hat{f}(t)\|^2 = \frac{1}{3C} - \frac{2}{\sqrt{2\pi}\sigma} \left[e^{-\frac{C^2}{2\sigma^2}} - e^{-\frac{9C^2}{2\sigma^2}} \right] + \frac{3}{8\sqrt{\pi}\sigma}.$$

Given a value C , we can find the local minima by considering the first derivative of the cost function $\varepsilon(\sigma) = \|(f(t) - \hat{f}(t))\|^2$ with respect to σ .

$$\begin{aligned}\frac{\partial \varepsilon(\sigma)}{\partial \sigma} &= \frac{\partial}{\partial \sigma} \left[\frac{1}{3C} - \frac{2}{\sqrt{2\pi}\sigma} \left(e^{-\frac{C^2}{2\sigma^2}} - e^{-\frac{9C^2}{2\sigma^2}} \right) + \frac{3}{8\sqrt{\pi}\sigma} \right] \\ &= 0 - \frac{2}{\sqrt{2\pi}} \left[-\sigma^{-2} e^{-\frac{C^2}{2\sigma^2}} + \sigma^{-1} \left(-\frac{C^2}{2} \right) (-2)\sigma^{-3} e^{-\frac{9C^2}{2\sigma^2}} \right] \\ &\quad + \frac{2}{\sqrt{2\pi}} \left[-\sigma^{-2} e^{-\frac{9C^2}{2\sigma^2}} + \sigma^{-1} \left(-\frac{9C^2}{2} \right) (-2)\sigma^{-3} e^{-\frac{9C^2}{2\sigma^2}} \right] + (-\sigma^{-2} \frac{3}{8\sqrt{\pi}}) \\ &= -\frac{2}{\sqrt{2\pi}} \left[(C^2\sigma^{-4} - \sigma^{-2}) e^{-\frac{C^2}{2\sigma^2}} - (9C^2\sigma^{-4} - \sigma^{-2}) e^{-\frac{9C^2}{2\sigma^2}} \right] - \frac{3\sigma^{-2}}{8\sqrt{\pi}}.\end{aligned}$$

No analytical solution exists for solving $\frac{\partial \varepsilon(\sigma)}{\partial \sigma} = 0$. However, numerical results suggest that there exists a linear relationship between C and σ , shown in Fig. 1. \square

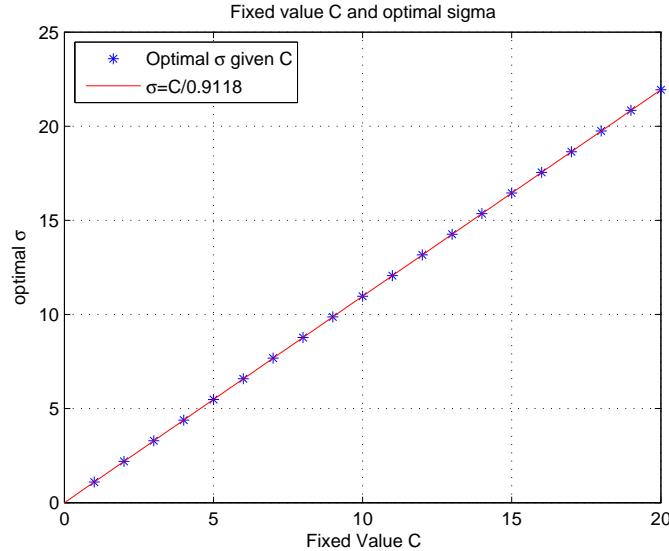


Figure 1: Linear relationship between C and σ is found with the ratio $\sigma \approx \frac{C}{0.9118}$.

Proposition 4. *Letting*

$$\begin{aligned} v(t) &= h''(t; \sigma) * I(t), \text{ and} \\ w(t) &= h''(t; \sigma) * n(t), \end{aligned}$$

then

$$v(t) = \begin{cases} -\frac{2}{3}, & t = 0, \\ -\frac{2}{3} - \frac{t}{2C}, & 0 < |t| \leq 2C, \\ \frac{1}{3} + \frac{t}{6C}, & 2C < |t| \leq 4C, \end{cases} \quad \text{and} \quad \text{Var}[w(t)] = \frac{\sigma_N^2}{3C}.$$

Proof. Since $\mathbb{E}[w(t)] = 0$, we have $\text{Var}[w(t)] = \mathbb{E}[w(t)^2] = \mathbb{E}[w(t)w(t+\gamma)]_{\gamma=0}$. Therefore, $\text{Var}[w(t)] = \sigma_N^2 h(-\tau) * h(\tau)|_{\tau=0}$. The convolution is the summation of square of amplitude times the interval

$$h(-\tau) * h(\tau)|_{\tau=0} = \frac{2C}{(6C)^2} + \frac{2*4C}{(6C)^2} + \frac{2C}{(6C)^2} \quad (7)$$

$$= \frac{12C}{36C^2} = \frac{1}{3C}. \quad (8)$$

Therefore, the $\text{Var}[w(t)] = \frac{\sigma_N^2}{3C}$. □

References

- [1] Tony Lindeberg. Feature detection with automatic scale selection. *International Journal of Computer Vision*, 30:79–116, 1998.