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CONTINUOUS TIME AUTOREGRESSIVE MODELS WITH COMMON STOCHASTIC TRENDS

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A multivariate continuous time model is presented in which a n-dimensional process is represented as the sum of k stochastic trends plus a n-dimensional stationary term, assumed to obey a system of higher-order autoregressive stochastic differential equations. When $k < n$, the variables are cointegrated and can be represented as linear combinations of a reduced number of common trends. An algorithm to estimate the parameters of the model is presented for the case that the trend and stationary disturbances are uncorrelated. This algorithm is used to extract a common (continuous time) stochastic trend from postwar U.S. GNP and consumption.

1. Introduction

Macroeconomic reasoning often assigns different roles to movements in the permanent or 'trend' components of aggregate economic time series and to transitory or 'cyclical' fluctuations. Several recent empirical studies have employed time series models in attempts to distinguish the trend from the cyclical (or 'stationary') components of GNP and to estimate their relative contributions to quarterly changes in output; see, for example, Nelson and Plosser (1982), Harvey (1985), Watson (1986), Campbell and Mankiw (1987a), Clark (1987a), and Cochrane (1986). These studies have used various univariate discrete time models to isolate these two components of GNP. A broad conclusion emerging from these investigations is that the trend in GNP estimated using the different univariate procedures is sensitive to the imposition of restrictions that are at best only partially identified.

This paper takes an alternative perspective on estimating the relative importance of the permanent and transitory components of GNP. Our approach is motivated by two observations. First, the previous studies estimate discrete

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time models using data that refer to a temporal average, total output over the quarter; their results are therefore potentially subject to temporal aggregation bias [e.g., Christiano and Eichenbaum (1987)]. Second, in part because of technical limitations these studies have focused on univariate methods, thereby excluding potentially useful information in additional series that might possess the same stochastic trend as GNP.

Our strategy is therefore to formulate a time series model where a n-dimensional continuous time multivariate process is expressed as the sum of two components: a reduced number k of common stochastic trends and a n-dimensional stationary component. The stationary component is assumed to obey a pth-order linear stochastic differential equation system, and the permanent component (i.e., the trend) is assumed to be multivariate Brownian motion, either with a nonstochastic drift or with a stochastic drift that itself is multivariate Brownian motion. In this 'unobserved components–autoregressive' (UC–AR) formulation, the innovations in the permanent and stationary components are uncorrelated by assumption. This model is set out in section 2.[1]

An algorithm for estimating the parameters of the continuous time UC–AR model by maximum likelihood is presented in section 3. The procedure is based on the Kalman–Bucy (1961) filter, modifying the algorithms in Jones (1981) and Harvey and Stock (1985) to handle the multiple zero roots in the state space transition matrix that arise from the stochastic trends. The algorithm is developed for the case that some of the variables are observed at a point in time (i.e., are stocks) while some of the variables are observed as time averages (i.e., are flows). (This algorithm is generalized in the appendix to allow for data irregularities such as missing observations and mixtures of monthly and quarterly observations.) When there are a reduced number of common stochastic trends in the continuous time model, we show that the discrete time variables will be cointegrated as defined by Engle and Granger (1987) whether or not they are measured a stocks or flows.

This algorithm is then used to estimate two UC–AR models using postwar U.S. aggregate data: first a univariate model with GNP alone and second a bivariate model with GNP and consumption. The results are presented in section 4. Our conclusions are summarized in section 5.

2. The continuous time unobserved components autoregressive model

Let $y(t)$ be a n-dimensional continuous time process with individual elements integrated up to order two. We adopt the convention that y_t refers to

[1] Our procedure exploits the presence of the trend in more than one series to obtain improved estimates of this common permanent component. This can be contrasted with the approaches of Campbell and Mankiw (1987b), Clark (1987b), and Blanchard and Quah (1987) who use unemployment – which is assumed not to contain a stochastic trend – to isolate the stationary component of GNP.

values of the continuous time process $y(t)$ at integer times t and that Y_t refers to the vector of observable discrete time variables. The observations are either taken at a point in time (for stock variables) or are integrals over the interval from $t-1$ to t (i.e., are flows). Let Y_t^s denote the n_s elements of Y_t corresponding to the stock variables and let Y_t^f denote the remaining n_f elements of Y_t measured as flows. Then

$$Y_t^s = Z^s y_t, \tag{1a}$$

$$Y_t^f = Z^f \int_{s=t-1}^{t} y(s)\,\mathrm{d}s, \tag{1b}$$

for $t = 1, \ldots, T$, where Z^s and Z^f are respectively $n_s \times n$ and $n_f \times n$ selection matrices of ones and zeros.

The continuous time process $y(t)$ is assumed to have a representation in terms of k_1 stochastic trends $\mu(t)$ (where $k_1 \leq n$), plus a stationary component $\xi(t)$. The stationary component evolves according to a pth-order continuous time autoregression [see for example Bergstrom (1983, 1985)]. The stochastic trends evolve according to a Browian motion process, which contains a drift which might itself evolve according to a Browian motion (of dimension k_2, with $k_2 \leq k_1$) with a nonrandom drift. Summarizing, the multivariate continuous time UC–AR (p) model can be written as

$$y(t) = \theta_1 \mu(t) + \xi(t), \tag{2}$$

where

$$\mathrm{d}\mu(t) = (\delta_1 + \theta_2 \beta(t))\,\mathrm{d}t + \mathrm{d}\eta(t),$$

$$\mathrm{d}\beta(t) = \delta_2\,\mathrm{d}t + \mathrm{d}\nu(t),$$

$$\mathrm{dD}^{p-1}\xi(t) = \left[\gamma + A_1 \mathrm{D}^{p-1}\xi(t) + \ldots + A_{p-1}\mathrm{D}\xi(t) + A_p \xi(t)\right]\mathrm{d}t + \mathrm{d}\zeta(t),$$

where D is the mean square differential operator as, for example, in Bergstrom (1983); θ_1 and θ_2 respectively are $n \times k_1$ and $k_1 \times k_2$ matrices; γ, δ_1, and δ_2 respectively are $n \times 1$, $k_1 \times 1$, and $k_2 \times 1$ vectors of intercept terms; and $A_1, \ldots, A_p$ are $n \times n$ matrices of unknown coefficients describing the AR process, which is assumed to be stable. The unobserved vector $[\eta(t)'\ \nu(t)'\ \zeta(t)']'$ is assumed to be $(k_1 + k_2 + n)$-dimensional Brownian motion with covariance matrix Σ, where

$$\Sigma = \begin{bmatrix} \Sigma_\eta & 0 & 0 \\ 0 & \Sigma_\nu & 0 \\ 0 & 0 & \Sigma_\zeta \end{bmatrix},$$

so that $[\eta(t)' \quad \nu(t)' \quad \zeta(t)']'$ has uncorrelated Gaussian increments and, for $s > t$,

$$\mathrm{E}\begin{bmatrix} \int_t^s \mathrm{d}\eta(\tau) \\ \int_t^s \mathrm{d}\nu(\tau) \\ \int_t^s \mathrm{d}\zeta(\tau) \end{bmatrix}\left[\int_t^s \mathrm{d}\eta(\tau)' \quad \int_t^s \mathrm{d}\nu(\tau)' \quad \int_t^s \mathrm{d}\zeta(\tau)'\right] = (s-t)\Sigma.$$

The model (2) requires additional normalization conditions for the parameters to be identified. We therefore let Σ_η and Σ_ν be diagonal, so that the disturbances in the stochastic trend and drift are mutually independent but have different variances. The 'factor loading' matrices θ_1 and θ_2 are respectively normalized to be the first k_1 and k_2 columns of a lower triangular matrix with ones on the diagonal. Finally, $\mu(0)$ and $\beta(0)$ are assumed to be fixed and equal to zero vectors. An alternative way of handling the initialization problem is to assume that $\mu(0)$ and $\beta(0)$ have a 'diffuse prior' and to modify the definition of the intercept vectors γ, δ_1 and δ_2 appropriately; see Fernandez-Macho, Harvey and Stock (1987).

Cointegration and common trends in continuous time

The formulation (2) admits various orders of integration and cointegration. If $k_2 = 0$ but $k_1 \neq 0$ the elements of $y(t)$ are integrated of order 1 or 0, depending on whether the stochastic trend enters the particular equation for that element of $y(t)$. When $k_1 > 0$, some elements of $y(t)$ will be integrated of order 2. Moreover, depending on k_1, k_2, θ_1, and θ_2, $y(t)$ can be cointegrated of various orders. Of particular interest is the case that $k_2 = 0$, $0 < k_1 < n$, and θ_1 has at least one nonzero element in every column. Then $y(t)$ is cointegrated [in Engle and Granger's (1987) notation, CI(1,1)] with $n - k_1$ cointegrating vectors; the cointegrating vectors are any set of vectors spanning the subspace of $\mathscr{R}^n$ orthogonal to θ_1. Finally, when $k_2 > 0$ so that there is a random drift, various linear combinations of the elements of $y(t)$ will be integrated of orders 0, 1, or 2. We return to this point in the next section to argue that these results concerning cointegration apply even if the observed variable vector contains both stocks and flows.

3. State space formulation and estimation equations

The parameters of the continuous time UC–AR(p) model (2) can be estimated by using the appropriate state space formulation and applying the Kalman filter. We assume that there are no missing observations and that all variables are observed at the same sampling frequency.

The state space representation is obtained by combining the discrete time transition equations for the unobserved variables $\xi(t)$, $\mu(t)$, and $\beta(t)$ comprising the model for $y(t)$, with eq. (1a) for stocks and the integral (1b) for flows. Setting $Z^{\alpha} = [I_n \quad 0 \quad \dots \quad 0]$ and combining (1) and (2), one can write the discrete time variables in terms of their unobserved components (and integrals of these) as

$$Y_t^s = Z^s\theta_1\mu(t) + Z^sZ^{\alpha}\alpha(t), \tag{3a}$$

$$Y_t^f = Z^f\theta_1\int_{\tau=t-1}^{t}\mu(\tau)\,\mathrm{d}\tau + Z^fZ^{\alpha}\int_{\tau=t-1}^{t}\alpha(\tau)\,\mathrm{d}\tau, \tag{3b}$$

for $t = 1,\dots,T$, where $\alpha(t)$ is defined below. The augmented state vector for the system is composed of $\alpha(t)$, $\mu(t)$, $\beta(t)$, and Y_t^f.[2]

Turning first to the transition equations for the stationary component, let $\alpha(t)$ denote the state vector formed by stacking $\xi(t)$ and its mean square derivatives, let A denote the companion matrix for the differential equation system for $\xi(t)$, and let R be a $np \times n$ selection matrix:

$$\alpha(t) = \begin{bmatrix} \xi(t) \\ \mathrm{D}\xi(t) \\ \vdots \\ \mathrm{D}^{p-1}\xi(t) \end{bmatrix}, \quad A = \left[\begin{array}{c|c} 0 & I_{n(p-1)} \\ \hline A_p & A_{p-1} \cdots A_1 \end{array}\right], \quad R = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ I_n \end{bmatrix},$$

where I_n denotes the $n \times n$ identity matrix. The roots of A are assumed to have negative real parts so that the differential equation system is stable. Using these definitions, the stochastic differential equation system for $\xi(t)$ in (2) can be rewritten as

$$\mathrm{d}\alpha(t) = [R\gamma + A\alpha(t)]\,\mathrm{d}t + R\,\mathrm{d}\zeta(t). \tag{4}$$

For $\tau > t-1$, $\alpha(\tau)$ satisfies

$$\alpha(\tau) = \mathrm{e}^{A(\tau-(t-1))}\alpha_{t-1} + \left[\int_{s=t-1}^{\tau}\mathrm{e}^{A(\tau-s)}\mathrm{d}s\right]R\gamma + \int_{s=t-1}^{\tau}\mathrm{e}^{A(\tau-s)}R\,\mathrm{d}\zeta(s), \tag{5}$$

[2] For further discussion of the device of augmenting the state vector by integrals of its components to handle multiple orders of integration in continuous time variables, see Harvey and Stock (1987).

from which it follows that

$$\alpha_t = e^{A}\alpha_{t-1} + \left[\int_{s=t-1}^{t} e^{A(t-s)}\,ds\right]R\gamma + \int_{s=t-1}^{t} e^{A(t-s)}R\,d\zeta(s), \tag{6a}$$

$$\begin{aligned}\int_{\tau=t-1}^{t}\alpha(\tau)\,d\tau &= \int_{\tau=t-1}^{t} e^{A(\tau-(t-1))}\alpha_{t-1}\,d\tau \\ &\quad+\left[\int_{\tau=t-1}^{t}\int_{s=t-1}^{\tau} e^{A(\tau-s)}\,ds\,d\tau\right]R\gamma \\ &\quad+\int_{\tau=t-1}^{t}\int_{s=t-1}^{\tau} e^{A(\tau-s)}R\,d\zeta(s)\,d\tau.\end{aligned} \tag{6b}$$

Evaluating the integrals, it follows from (6) that

$$\alpha_t = e^{A}\alpha_{t-1} + W^{[1]}R\gamma + \int_{s=t-1}^{t} e^{A(t-s)}R\,d\zeta(s), \tag{7a}$$

$$\begin{aligned}\int_{\tau=t-1}^{t}\alpha(\tau)\,d\tau &= W^{[1]}\alpha_{t-1} + W^{[2]}R\gamma \\ &\quad+\int_{\tau=t-1}^{t}\int_{s=t-1}^{t} e^{A(\tau-s)}R\,d\zeta(s)\,d\tau,\end{aligned} \tag{7b}$$

where

$$W^{[1]} = A^{-1}\left(e^{A} - I_{np}\right) \quad\text{and}\quad W^{[2]} = A^{-1}\left(W^{[1]} - I_{np}\right).$$

Turning to the transition equations for $\beta(t)$ and its unit integrals,

$$\beta(\tau) = \int_{s=t-1}^{\tau}\delta_2\,ds + \beta_{t-1} + \int_{s=t-1}^{\tau} d\nu(s), \tag{8}$$

so

$$\beta_t = \delta_2 + \beta_{t-1} + \int_{s=t-1}^{t} d\nu(s), \tag{9a}$$

$$\int_{\tau=t-1}^{t}\beta(\tau)\,d\tau = \tfrac{1}{2}\delta_2 + \beta_{t-1} + \int_{\tau=t-1}^{t}\int_{s=t-1}^{\tau} d\nu(s)\,d\tau, \tag{9b}$$

$$\begin{aligned}\int_{\tau=t-1}^{t}\int_{s=t-1}^{\tau}\beta(s)\,ds\,d\tau &= \delta_2/6 + \tfrac{1}{2}\beta_{t-1} \\ &\quad+\int_{\tau=t-1}^{t}\int_{s=t-1}^{\tau}\int_{r=t-1}^{s} d\nu(r)\,ds\,d\tau.\end{aligned} \tag{9c}$$

Similarly,

$$\mu(\tau) = \int_{s=t-1}^{\tau} ds\,\delta_1 + \theta_2\int_{s=t-1}^{\tau}\beta(s)\,ds + \mu_{t-1} + \int_{s=t-1}^{\tau} d\eta(s). \tag{10}$$

Integrating (10) and using (9), the transition equations for $\mu(t)$ and its unit integrals are

$$\mu_t = \left(\delta_1 + \tfrac{1}{2}\theta_2\delta_2\right) + \mu_{t-1} + \theta_2\beta_{t-1} + \int_{s=t-1}^{t} \mathrm{d}\eta(s) + \theta_2\int_{\tau=t-1}^{t}\int_{s=t-1}^{\tau} \mathrm{d}\nu(s)\,\mathrm{d}\tau, \tag{11a}$$

$$\int_{\tau=t-1}^{t}\mu(\tau)\,\mathrm{d}\tau = \left(\tfrac{1}{2}\delta_1 + \theta_2\delta_2/6\right) + \mu_{t-1} + \tfrac{1}{2}\theta_2\beta_{t-1} + \int_{\tau=t-1}^{t}\int_{s=t-1}^{\tau}\mathrm{d}\eta(s)\,\mathrm{d}\tau + \theta_2\int_{\tau=t-1}^{t}\int_{s=t-1}^{r}\int_{r=t-1}^{s}\mathrm{d}\nu(r)\,\mathrm{d}s\,\mathrm{d}\tau. \tag{11b}$$

The transition equation for y_t^f obtains by combining the expressions in (3b), (7b), and (11b):

$$\begin{aligned} Y_t^f = {} & \left(\tfrac{1}{2}Z^f\theta_1\delta_1 + Z^f\theta_1\theta_2\delta_2/6 + Z^fZ^\alpha W^{[2]}R\gamma\right) \\ & + Z^fZ^\alpha W^{[1]}\alpha_{t-1} + Z^f\theta_1\mu_{t-1} + \tfrac{1}{2}Z^f\theta_1\theta_2\beta_{t-1} \\ & + Z^fZ^\alpha\int_{\tau=t-1}^{t}\int_{s=t-1}^{\tau}\mathrm{e}^{A(\tau-s)}R\,\mathrm{d}\zeta(s)\,\mathrm{d}\tau \\ & + Z^f\theta_1\int_{\tau=t-1}^{t}\int_{s=t-1}^{\tau}\mathrm{d}\eta(s)\,\mathrm{d}\tau \\ & + Z^f\theta_1\theta_2\int_{\tau=t-1}^{t}\int_{s=t-1}^{\tau}\int_{r=t-1}^{s}\mathrm{d}\nu(r)\,\mathrm{d}s\,\mathrm{d}\tau. \end{aligned} \tag{12}$$

Let $\alpha_t^\dagger$ denote the augmented discrete time state vector comprised of the various unobserved components and Y_t^f, i.e., $\alpha_t^\dagger = [\alpha_t' \quad \mu_t' \quad \beta_t' \quad Y_t^{f\prime}]'$. Combining (7a), (9a), (11a), and (12), the transition equation for $\alpha_t^\dagger$ is thus:

$$\alpha_t^\dagger = \gamma^\dagger + T^\dagger\alpha_{t-1}^\dagger + \varepsilon_t^\dagger, \tag{13}$$

where

$$\gamma^\dagger = \begin{bmatrix} W^{[1]}R\gamma \\ \delta_1 + \tfrac{1}{2}\theta_2\delta_2 \\ \delta_2 \\ \tfrac{1}{2}Z^f\theta_1\delta_1 + \tfrac{1}{2}Z^f\theta_1\theta_2\delta_2 + Z^fZ^\alpha W^{[2]}R\gamma \end{bmatrix},$$

$$T^\dagger = \begin{bmatrix} e^A & 0 & 0 & 0 \\ 0 & I & \theta_2 & 0 \\ 0 & 0 & I & 0 \\ Z^fZ^\alpha W^{[1]} & Z^f\theta_1 & \tfrac{1}{2}Z^f\theta_1\theta_2 & 0 \end{bmatrix},$$

and where $\varepsilon_t^\dagger = [\varepsilon_{1t}' \quad \varepsilon_{2t}' \quad \varepsilon_{3t}' \quad \varepsilon_{4t}']'$, where

$$\varepsilon_{1t} = \int_{s=t-1}^{t} e^{A(t-s)} R \, d\zeta(s),$$

$$\varepsilon_{2t} = \int_{s=t-1}^{t} d\eta(s) + \theta_2 \int_{s=t-1}^{t} \int_{\tau=t-1}^{s} d\nu(\tau) ds,$$

$$\varepsilon_{3t} = \int_{\tau=t-1}^{t} d\nu(\tau),$$

$$\begin{aligned}\varepsilon_{4t} = {} & Z^f Z^\alpha \int_{\tau=t-1}^{t} \int_{s=t-1}^{\tau} e^{A(\tau-s)} R \, d\zeta(s) \, d\tau \\ & + Z^f \theta_1 \int_{\tau=t-1}^{t} \int_{s=t-1}^{\tau} d\eta(s) \, d\tau \\ & + Z^f \theta_1 \theta_2 \int_{\tau=t-1}^{t} \int_{s=t-1}^{\tau} \int_{r=t-1}^{s} d\nu(r) \, ds \, d\tau.\end{aligned}$$

3.1. Disturbance covariance matrix

The disturbances $\varepsilon_t^\dagger$ in the state transition equation involve multiple integrals of different continuous time Brownian noise terms. Implementation of the Kalman filter requires the computation of the covariance matrix of these disturbances. This task is simplified by transforming the expressions for $\varepsilon_t^\dagger$ so that they contain only single integrals of the Brownian noise terms, which in turn can be obtained by changing the order of integration in the expressions for the disturbances.[3] For example,

$$\begin{aligned}\int_{\tau=t-1}^{t} \int_{s=t-1}^{\tau} e^{A(\tau-s)} R \, d\zeta(s) \, d\tau &= \int_{s=t-1}^{t} \left[\int_{\tau=s}^{t} e^{A(\tau-s)} d\tau \right] R \, d\zeta(s) \\ &= A^{-1} \int_{s=t-1}^{t} e^{A(t-s)} R \, d\zeta(s) \\ &\quad - A^{-1} \int_{s=t-1}^{t} R \, d\zeta(s). \qquad (14)\end{aligned}$$

Performing similar calculations for the other stochastic integrals comprising $\varepsilon_t^\dagger$ permits rewriting (13) as

$$\alpha_t^\dagger = \gamma^\dagger + T^\dagger \alpha_{t-1}^\dagger + S^\dagger v_t^\dagger, \qquad (15)$$

[3] Since Fubini's theorem does not apply directly to stochastic integrals, these calculations are a heuristic device to obtain simplified expressions for evaluating the covariance matrix of the discrete time state space disturbances. These results can alternatively be obtained (with more effort) by evaluating directly the moments expressed in terms of multiple integrals such as appear on the left-hand side of (14).

where

$$S^{\dagger}=\begin{bmatrix} I & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & I & 0 & 0 & \theta_2 & 0 \\ 0 & 0 & 0 & 0 & I & 0 & 0 \\ Z^f Z^{\alpha} A^{-1} & -Z^f Z^{\alpha} A^{-1} R & 0 & Z^f\theta_1 & 0 & 0 & Z^f\theta_1\theta_2 \end{bmatrix},$$

and $v_t^{\dagger}=[v_t^{[0]\prime}\quad v_t^{[1]\prime}\quad v_t^{[2]\prime}\quad v_t^{[3]\prime}\quad v_t^{[4]\prime}\quad v_t^{[5]\prime}\quad v_t^{[6]\prime}]'$, where

$$v_t^{[0]}=\int_{s=t-1}^{t} \mathrm{e}^{A(t-s)} R \,\mathrm{d}\zeta(s),$$

$$v_t^{[1]}=\int_{s=t-1}^{t} \mathrm{d}\zeta(s),$$

$$v_t^{[2]}=\int_{s=t-1}^{t} \mathrm{d}\eta(s),$$

$$v_t^{[3]}=\int_{s=t-1}^{t} (t-s)\,\mathrm{d}\eta(s),$$

$$v_t^{[4]}=\int_{s=t-1}^{t} \mathrm{d}\nu(s),$$

$$v_t^{[5]}=\int_{s=t-1}^{t} (t-s)\,\mathrm{d}\nu(s),$$

$$v_t^{[6]}=\tfrac{1}{2}\int_{s=t-1}^{t} (t-s)^2\,\mathrm{d}\nu(s).$$

It is now straightforward to evaluate the integrals involved in the covariance matrix of $v_t^{\dagger}$. Employing the assumption that the noise processes $\zeta(t)$, $\eta(t)$, and $\nu(t)$ have mutually uncorrelated increments, one obtains

$$\mathrm{E}v_t^{\dagger}v_t^{\dagger\prime}=\begin{bmatrix} \Omega & W^{[1]}R\Sigma_{\zeta} & 0 & 0 & 0 & 0 & 0 \\ \Sigma_{\zeta}R'W^{[1]\prime} & \Sigma_{\zeta} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \Sigma_{\eta} & \tfrac{1}{2}\Sigma_{\eta} & 0 & 0 & 0 \\ 0 & 0 & \tfrac{1}{2}\Sigma_{\eta} & \Sigma_{\eta}/3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \Sigma_{\nu} & \tfrac{1}{2}\Sigma_{\nu} & \Sigma_{\nu}/6 \\ 0 & 0 & 0 & 0 & \tfrac{1}{2}\Sigma_{\nu} & \Sigma_{\nu}/3 & \Sigma_{\nu}/8 \\ 0 & 0 & 0 & 0 & \Sigma_{\nu}/6 & \Sigma_{\nu}/8 & \Sigma_{\nu}/20 \end{bmatrix}, \tag{16}$$

where

$$\Omega = \int_{s=t-1}^{t} e^{A(t-s)} R \Sigma_{\zeta} R' e^{A(t-s)'} \mathrm{d}s.^{4}$$

3.2. Measurement equation and likelihood function

The measurement equation relating the augmented state vector $\alpha_t^{\dagger}$ to the observable variable Y_t is obtained from (1), (6), and (7), and can be summarized by

$$Y_t = Z^{\dagger}\alpha_t^{\dagger}, \tag{17}$$

where

$$Z^{\dagger} = \left[\begin{array}{c|c|c|c} Z^s Z^{\alpha} & Z^s\theta_1 & 0 & 0 \\ \hline 0 & 0 & 0 & I \end{array}\right].$$

The Gaussian likelihood function can be calculated by applying the Kalman filter to the discrete time state space form (15) using the innovation covariance matrix (16) and measurement equations (17). The autoregressive component is initialized as in Harvey and Stock (1985, eq. 18) by the mean and covariance matrix of its unconditional distribution. The initial value Y_0^f equals zero by definition.

3.3. Cointegration in mixed stock–flow systems

The remarks made in section 2 concerning the ability of this model to handle cointegrated processes extends to the case that some variables are stocks and some are flows. For simplicity, consider the case that $k_2 = 0$, so that the stochastic trend $\mu(t)$ has a nonrandom (but possibly nonzero) drift δ_1. In addition, suppose that $0 < k_1 < n$ and that each row of θ_1 has at least one nonzero element. We now show that Y_t is CI(1,1) with $n - k_1$ cointegrating vectors even if Y_t contains both stocks and flows.

Without loss of generality, order the variables so that $Z^s = [I_s \quad 0]$ and $Z^f = [0 \quad I_f]$, where I_s and I_f are the $n_s \times n_s$ and $n_f \times n_f$ identity matrices. From the transition equation (13) and the measurement equation (17), it follows that Y_t has the representation,

[4] For a discussion of the computation of the matrix exponential $\exp(A)$ and its integrals, see Van Loan (1978) or Melino (1985).

follows that Y_t has the representation,

$$Y_t = \begin{bmatrix} Z^s\theta_1\delta_1 + Z^sZ^\alpha W^{[1]}R\gamma \\ \frac{1}{2}Z^f\theta_1\delta_1 \end{bmatrix} + \theta_1\mu_{t-1} + u_t, \tag{18}$$

where u_t is the n-dimensional stationary ARMA process implied by the autoregressive component α_t, plus additional unit integrals of $\mathrm{d}\eta(s)$ arising from the transition equation for $\mu(t)$ and its integral. Since each row of θ_1 has nonzero elements by assumption, each element of Y_t is integrated of order 1. However, since θ_1 is $n \times k_1$, there is a $(n-k_1)$-dimensional subspace of $\mathscr{R}^n$ containing vectors α such that $\alpha'\theta_1 = 0$. Thus $\alpha'Y_t$ is stationary (albeit with a possibly nonzero mean), so by definition Y_t is CI(1,1).

4. Empirical results

The preceding algorithms were used to estimate univariate and bivariate UC–AR models of the logarithm of quarterly U.S. real per capita GNP (y_t) and the logarithm of real per capita total consumption (c_t) from 1952:I to 1985:IV.[5] The unit root and cointegration properties of these series were examined by King, Plosser, Stock and Watson (1987); they concluded that each of the series could be characterized as containing a unit root and that they appear to be cointegrated with a cointegrating vector of approximately $(1\ \ -1)$. Building on these results, we consider two models. The first is a univariate model of y_t in which the stochastic trend is modeled as Brownian motion with possibly nonzero but constant drift, so that $k_1 = 0$ and $k_2 = 0$ in (2). The second is a bivariate model in which GNP and consumption are modeled as being cointegrated in continuous time, so that they share a common Brownian motion trend with constant (but possibly nonzero) drift. A cointegrating vector of $(1\ \ -1)$ is imposed in this formulation: in the notation of section 3, we set $\theta_1 = (1\ \ 1)'$ so that the stochastic trend enters both series with the same weight (or 'factor loading'). In both the univariate and multivariate cases the unobserved stationary component $\xi(t)$ was modeled as being generated by a second-order stochastic differential equation, so that $p = 2$ in (2).[6]

[5] The data were obtained from the Citibase data base.

[6] The computations were performed using the programming language MATLAB. Van Loan's (1978) recommended Padé expansion was used to calculate $\exp(A)$, and Ω [given following (16)] was computed by solving the continuous time Lyapunov equation, $A\Omega + \Omega A' = -R\Sigma_\xi R'$. The covariance matrix $P_{0|0}$ for the unconditional estimate of α_0 was computed as the solution to $\Omega = P_{0|0} - \exp(A)P_{0|0}\exp(A')$. Attempts to estimate similar models with $p = 3$ led to negatively exploding roots of the transition matrix A of the $\xi(t)$, corresponding to zero roots of the associated discrete time transition matrix. Moreover, the estimated models with $p = 3$ exhibited no statistically significant (at the 20% level) improvement in the value of the likelihood over the $p = 2$ models.

Table 1

Summary statistics for discrete and continuous time models of GNP and consumption, 1952:I–1985:IV.[a]

	Univariate			Bivariate			
	AR(2)	AR(6)	Cts. UC–AR(2)	VAR(6)		Cts. UC–VAR(2)	
	y	y	y	y	c	y	c
$Q(8)$	6.51	5.93	9.88	3.73	6.78	3.50	6.54
$Q(12)$	8.04	6.38	11.11	6.45	7.98	7.55	7.36
SEE ($\times 10^{-2}$)	0.996	0.993	1.021	0.960	0.732	0.965	0.733

[a] y denotes log real per capita GNP and c denotes log real per capita consumption. The SEE is $[(T-k)^{-1}\Sigma_{t=1}^{T}\hat{e}_t^2]^{1/2}$, where $\hat{e}_t^2$ is the one-step-ahead in-sample forecast error and k is the number of estimated parameters. Q is the Box–Ljung Q-statistic based on the indicated number of estimated autocorrelation coefficients. The discrete time autoregressions, estimated using data prior to 1952 as initial conditions, were estimated using log levels of the variables. The UC–AR and UC–VAR models were estimated using only data from 1952:I to 1985:IV.

Table 2

Estimated continuous time UC–AR models of GNP and consumption, 1952:I–1985:IV.[a]

A. Univariate UC–AR(2): $Y_t = y_t$

$$y(t) = \mu(t) + \xi(t),$$

$$\mathrm{d}\mu(t) = 0.0034\,\mathrm{d}t + \mathrm{d}\eta(t), \qquad \Sigma_\eta = 1.63 \times 10^{-4}$$

$$\mathrm{dD}\xi(t) = [0.0020 - 0.16\mathrm{D}\xi(t) - 0.525\xi(t)]\,\mathrm{d}t + \mathrm{d}\zeta(t), \quad \Sigma_\zeta = 0.178 \times 10^{-4}$$

Roots of A: -0.08 ± 0.721

Roots of e^A: $0.69 \pm 0.61i$

$\mathscr{L} = 448.24$

B. Bivariate UC–VAR(2): $Y_t = (y_t\ c_t)'$

$$y(t) = \theta_1\mu(t) + \xi(t), \qquad \theta_1 = (1.00\ 1.00)'$$

$$\mathrm{d}\mu(t) = 0.0045\,\mathrm{d}t + \mathrm{d}\nu(t), \qquad \Sigma_\nu = 0.612 \times 10^{-4}$$

$$\mathrm{dD}\xi(t) = \left\{\begin{bmatrix} 0.017 \\ 0.036 \end{bmatrix} + \begin{bmatrix} -0.076 & -8.761 \\ 0.739 & -3.695 \end{bmatrix}\mathrm{D}\xi(t) + \begin{bmatrix} 1.608 & -7.55 \\ 2.650 & -10.38 \end{bmatrix}\xi(t)\right\}\mathrm{d}t + \mathrm{d}\zeta(t), \quad \Sigma_\zeta = \begin{bmatrix} 0.206 & 0.122 \\ 0.122 & 0.073 \end{bmatrix} \times 10^{-4}$$

Roots of A: $-0.16,\ -0.90 \pm 3.30i,\ -1.83$

Roots of e^A: $0.86,\ -0.40 \pm 0.065i,\ 0.16$

$\mathscr{L} = 1070.23$

[a] The models were estimated by maximum likelihood using the algorithm described in the text with $\mu(0) = 0$. The data were shifted arbitrarily so that $y_{1951:\mathrm{IV}} = c_{1951:\mathrm{IV}} = 0$.

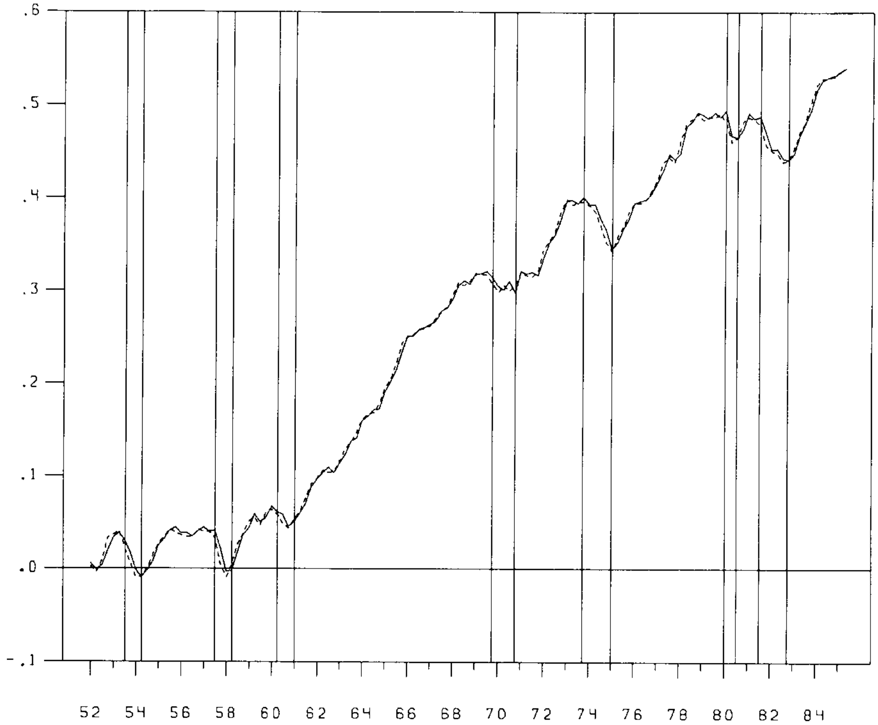

Fig. 1a. GNP (solid line) and its trend component (dashed line) estimated using the univariate UC–AR(2) in table 2, panel A. The vertical lines denote business cycle turning points identified by the NBER.

Measures of the fit of the two estimated models are given in table 1. In addition, similar measures are presented for conventional discrete time autoregressions estimated using the levels of the data including a constant but no time trend. The fit of the continuous time UC–AR(2) models is broadly comparable to the fit of the unconstrained autoregressions, which do not impose the restriction that each system contains a single discrete time unit root. The estimated parameters of the UC–AR(2) models are presented in table 2. Both the univariate and bivariate models have complex roots in the transition matrix of the unobserved stationary component, indicating pseudo-cyclical behavior.

Since the estimated parameters are difficult to interpret directly, the estimated trend and stationary components of GNP from the univariate UC–AR(2) model are presented in figs. 1a and 1b, respectively. The corresponding components of GNP from the bivariate model are shown in figs. 2a and 2b. The estimated trend and stationary components in the two models are

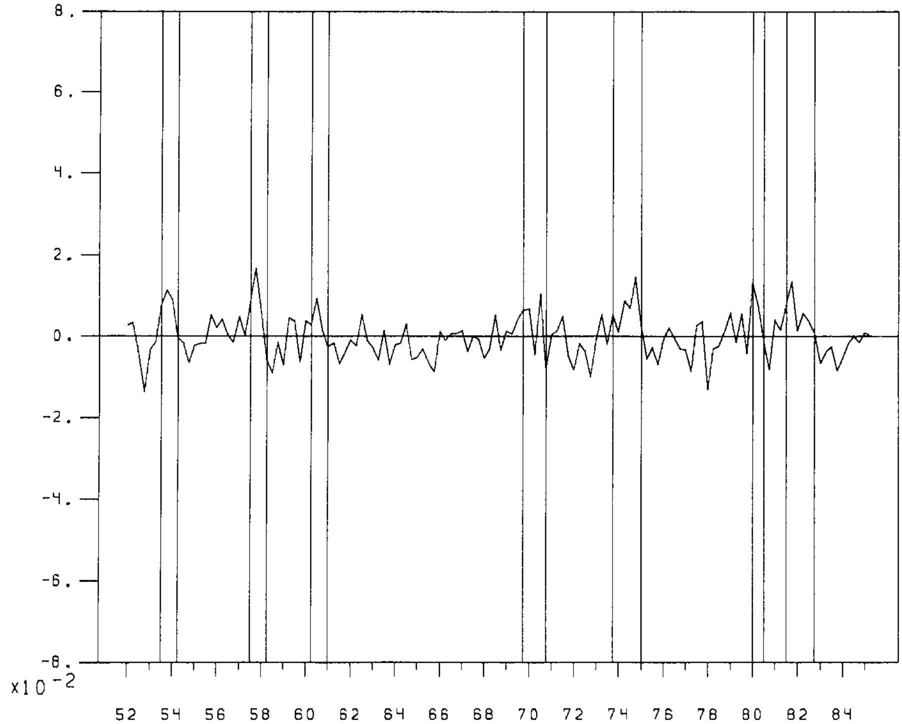

Fig. 1b. The stationary component of GNP estimated using the univariate UC–AR(2) in table 2, panel A. The vertical lines denote business cycle turning points identified by the NBER.

strikingly different. In the univariate model, the trend essentially equals log GNP itself; the fluctuations in the stationary component are small and exhibit no clear relation to the patterns of expansion and recession indicated by the NBER business cycle turning points. In contrast, GNP and its estimated trend component differ substantially when the bivariate model is used to extract the trend. For example, according to these estimates during much of the expansion of the late 1960's GNP was in excess of its trend value, while the recession of the 1980's and the subsequent expansion indicate a substantial drop below trend. Moreover, the peaks in the estimated stationary component (fig. 2b) generally coincide closely with the NBER-dated turning points.

There is a simple explanation for the apparent incompatibility of the estimated trends from the univariate and bivariate models. As other researchers [including Nelson and Plosser (1982) and Watson (1986)] have noted, the first-order sample autocorrelation in GNP growth is positive; for the period we examine it is 0.34. However, following from Working (1960), the first difference of the time average of Brownian motion has a discrete time

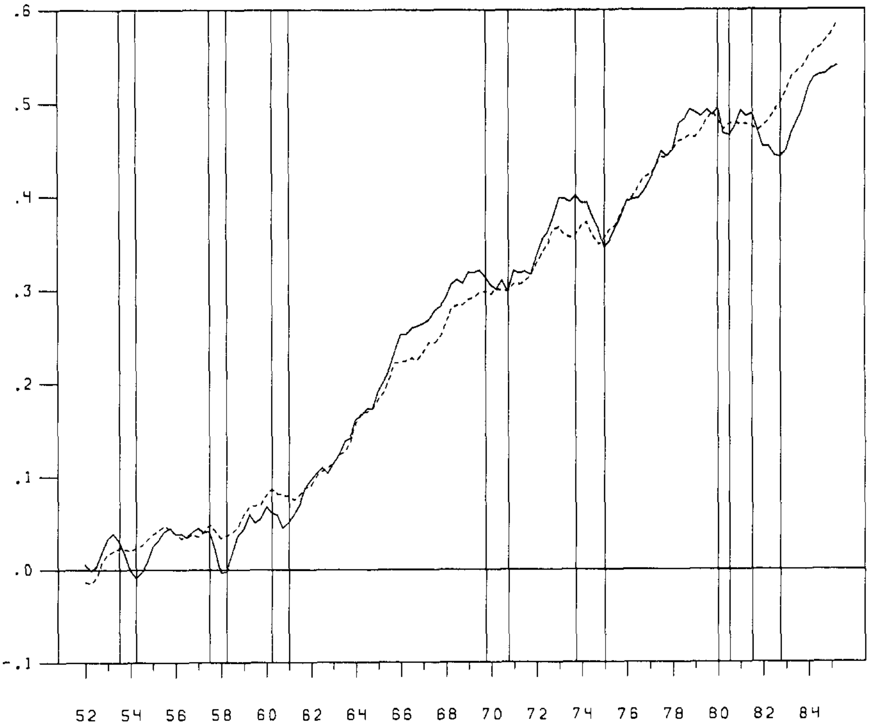

Fig. 2a. GNP (solid line) and its trend component (dashed line) estimated using the bivariate UC–AR(2) in table 2, panel B. The vertical lines denote business cycle turning points identified by the NBER.

MA(1) representation with a first-order autocorrelation coefficient of 0.25. This result does not apply directly here, since it is the level of GNP rather than its logarithm which is time-averaged. Nevertheless, modeling log GNP simply as the discrete time realization of time-averaged Brownian motion captures most of the dependence in the process, leaving little role for a remaining stationary component.[7] In contrast, upon imposing the restriction that the trend in GNP is the same as that in consumption, the imperfect correlation between changes in GNP and changes in consumption means that changes in the common trend can only account for part of the movements in each of the series. Indeed, the estimated trend in the bivariate model is substantially

[7]A discrete time random walk plus noise is not a satisfactory model for log GNP because of the positive first-order autocorrelation in GNP growth; nor does the addition of a stationary AR(2) process improve matters much. A comparable fit to the continuous time model can, however, be obtained in discrete time by allowing the disturbance in the level of the trend component to follow an AR(2) or a stochastic cycle process as in Harvey (1985).

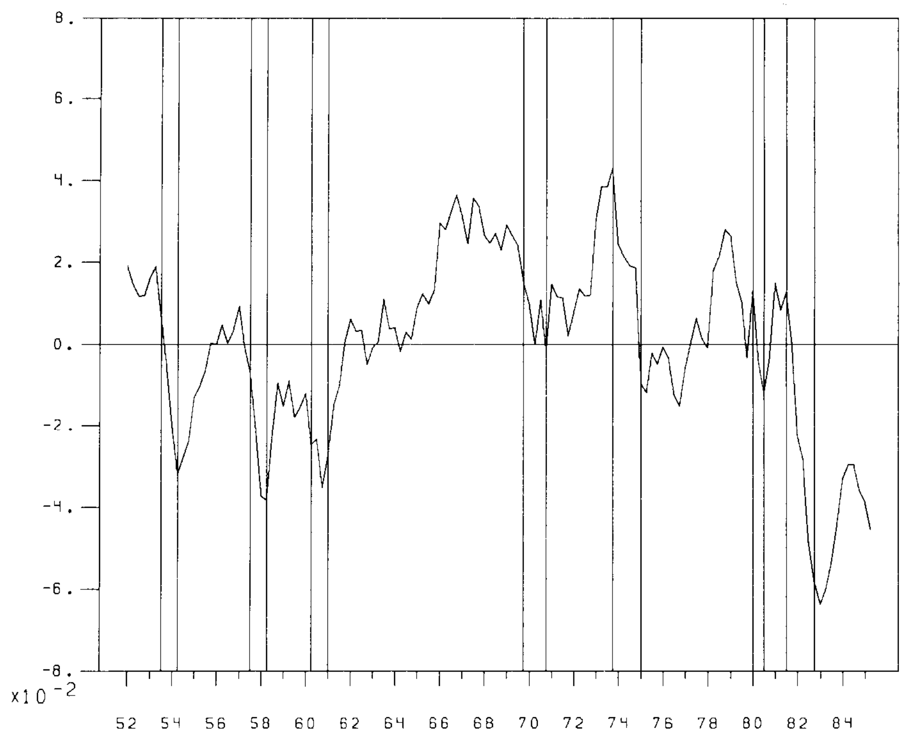

Fig. 2b. The stationary component of GNP estimated using the bivariate UC–AR(2) in table 2, panel B. The vertical lines denote business cycle turning points identified by the NBER.

'closer' to the actual consumption series than to GNP: the standard deviation of the estimated stationary component in consumption is 0.0074, while for GNP this is 0.0225. From a theoretical perspective, to the extent that consumption decisions incorporate forecasts of GNP over the longer run – as they would under the Permanent Income Hypothesis – consumers in large part have solved the trend extraction problem that is difficult when one examines GNP alone. This reasoning suggests that consumption should provide a good measure of the trend in GNP, and the empirical results are consistent with this interpretation.

5. Conclusions

These findings suggest two broad conclusions. First, imposing the overidentifying restriction that a stochastic trend is common to several time series can alter significantly the estimated permanent and stationary components. For consumption and output, we would argue that imposing this restriction makes for a more convincing framework in which to interpret these components.

Second, the results of the maximum likelihood estimation of the univariate model suggest that GNP is, in a statistical sense, well approximated as the time average of a continuous time random walk with drift. This raises questions about the ability of other exactly identified models to provide useful information about the trend in aggregate time series variables. While such models might be useful for making univariate forecasts, their value as devices for interpreting the trend and stationary components is less clear. However, when an additional series arguably containing the same trend is used – here, when consumption is included – much richer dynamics emerge for the joint process than would be suggested simply by looking at its univariate properties. One interpretation of this is that movements in consumption in large part reflect movements in forecasts of future income.

Appendix

Extension of continuous time UC–AR(p) Kalman filter to mixed frequencies and missing observations

It might be of interest to estimate a UC–AR(p) model using data observed at different sampling intervals, say, using monthly unemployment and quarterly GNP. In addition, there may be missing observations on some variables. In this appendix, the estimation algorithm of section 3 is extended to handle these two types of data irregularities. For expositional convenience, we develop the algorithm for the case of no random drift in the trend, i.e., $k_2 = 0$ in (2).

The modification for missing observations is straightforward and simply entails noting that the selection matrices differ from period to period; for example, the selection matrix Z^s is replaced by Z_t^s. This approach permits estimation when some stock variables are observed at a coarser frequency than the other variables. We modify Zadrozny's (1984) approach to handling the temporal aggregation arising when flow variables are observed at the coarser frequency. Specifically, of a total of n_f flow variables, we suppose that n_{f1t} are observed at $t = 1, 2, 3, \dots$, while n_{f2t} are observed at $t = m, 2m, 3m, \dots$. These coarser observation times are denoted by $t_\tau \equiv \tau m$, $\tau = 1, 2, 3, \dots$. The observable variable vector Y_t can thus be partitioned into stocks and the two types of flows:

$$Y_t = \begin{bmatrix} Y_t^s \\ Y_t^{f1} \\ Y_t^{f2} \end{bmatrix}.$$

In addition, define the n_c-dimensional cumulator variable y_t^c by

$$y_t^c = Z^c \sum_{s=t_{\tau-1}+1}^{t} y^f(s), \qquad t_{\tau-1} < t \le t_\tau, \tag{A.1}$$

where Z^c is a $n_c \times n_f$ matrix of ones and zeros, selecting all the flow variables that are observed as integrals over the coarser interval for at least part of the sample. The various elements of the observed process Y_t are thus related to the state variables α_t, μ_t, y_t^f, and y_t^c by

$$Y_t^s = Z_t^s Z^\alpha \alpha_t + Z_t^s \theta_1 \mu_t, \tag{A.2a}$$

$$Y_t^{f1} = Z_t^{f1} y_t^f, \tag{A.2b}$$

$$Y_t^{f2} = Z_t^{f2} y_t^c, \tag{A.2c}$$

where y_t^f is given by (1b), where Z^f is now the $n_f \times n$ matrix of ones and zeros selecting all variables measured as flows at either frequency, and where Z_t^s, Z_t^{f1}, and Z_t^{f2} are respectively $n_{st} \times n$, $n_{f1t} \times n_f$, and $n_{f2t} \times n_c$ selection matrices of ones and zeros.

The Kalman filter is derived for these data irregularities by augmenting the state vector in section 3 by the cumulator variable, y_t^c. First, note that (A.1) can be written

$$y_t^c = Z^c y_t^f + \pi_t y_{t-1}^c, \tag{A.3}$$

where $\pi_t = 1$, $t \neq t_{\tau-1} + 1$, and $\pi_t = 0$, $t = t_{\tau-1} + 1$. Thus the transition equation for y_t^c can be derived from (12) and (A.3):

$$\begin{aligned} y_t^c = {} & \tfrac{1}{2} Z^c Z^f \theta_1 \delta_1 + Z^c Z^f Z^\alpha W^{[2]} R\gamma + Z^c Z^f Z^\alpha W^{[1]} \alpha_{t-1} + Z^c Z^f \theta_1 \mu_{t-1} \\ & + Z^c Z^f Z^\alpha A^{-1} v_t^{[0]} - Z^c Z^f Z^\alpha A^{-1} R v_t^{[1]} + Z^c Z^f \theta_1 v_t^{[3]} + \pi_t y_{t-1}^c. \end{aligned} \tag{A.4}$$

The augmented state transition equation obtains by combining (13) and (A.4) yielding

$$\begin{aligned} \begin{bmatrix} \alpha_t \\ \mu_t \\ y_t^f \\ y_t^c \end{bmatrix} = {} & \begin{bmatrix} e^A & 0 & 0 & 0 \\ 0 & I & 0 & 0 \\ Z^f Z^\alpha W^{[1]} & Z^f \theta_1 & 0 & 0 \\ Z^c Z^f Z^\alpha W^{[1]} & Z^c Z^f \theta_1 & 0 & \pi_t \end{bmatrix} \begin{bmatrix} \alpha_{t-1} \\ \mu_{t-1} \\ y_{t-1}^f \\ y_{t-1}^c \end{bmatrix} \\ & + \begin{bmatrix} W^{[1]} R\gamma \\ \delta_1 \\ \tfrac{1}{2} Z^f \theta_1 \delta_1 + Z^f Z^\alpha W^{[2]} R\gamma \\ \tfrac{1}{2} Z^c Z^f \theta_1 \delta_1 + Z^c Z^f Z^\alpha W^{[2]} R\gamma \end{bmatrix} \\ & + \begin{bmatrix} I & 0 & 0 & 0 \\ 0 & 0 & I & 0 \\ Z^f Z^\alpha A^{-1} & -Z^f Z^\alpha A^{-1} R & 0 & Z^f \theta_1 \\ Z^c Z^f Z^\alpha A^{-1} & -Z^c Z^f Z^\alpha A^{-1} R & 0 & Z^c Z^f \theta_1 \end{bmatrix} \begin{bmatrix} v_t^{[0]} \\ v_t^{[1]} \\ v_t^{[2]} \\ v_t^{[3]} \end{bmatrix}, \end{aligned} \tag{A.5}$$

or, more compactly,

$$\alpha_t^* = T^*\alpha_{t-1}^* + \gamma^* + S^*v_t^*, \tag{A.6}$$

using obvious notation. The measurement equation is

$$Y_t = Z_t^*\alpha_t^*, \tag{A.7}$$

where

$$Z_t^* = \begin{bmatrix} Z_t^s Z^\alpha & Z_t^s\theta_1 & 0 & 0 \\ 0 & 0 & Z_t^{f1} & 0 \\ 0 & 0 & 0 & Z_t^{f2} \end{bmatrix}.$$

Estimation proceeds as in section 3. Initial conditions for $(y_t^{f\prime}\ y_t^{c\prime})$ are the same as those for $Y_t^{f\prime}$ in section 3.

References

Bergstrom, A.R., 1983, Gaussian estimation of structural parameters in higher-order continuous time dynamic models, Econometrica 51, 117–152.

Bergstrom, A.R., 1985, The estimation of parameters in nonstationary higher-order continuous time dynamic models, Econometric Theory 1, 369–385.

Blanchard, O.J. and D. Quah, 1987, The dynamic effects of aggregate demand and supply disturbances, Manuscript (MIT, Cambridge, MA).

Campbell, J.Y. and N.G. Mankiw, 1987a, Are output fluctuations transitory?, Quarterly Journal of Economics 102, 857–880.

Campbell, J.Y. and N.G. Mankiw, 1987b, Permanent and transitory components in macroeconomic fluctuations, American Economic Review 77, 111–117.

Christiano, L.J. and M. Eichenbaum, 1987, Temporal aggregation and structural inference in macroeconomics, Carnegie-Rochester Conference Series on Public Policy 26, 63–130.

Clark, P.K., 1987a, The cyclical component of U.S. economic activity, Quarterly Journal of Economics 102, 797–814.

Clark, P.K., 1987b, Trend reversion in real output and unemployment, Discussion paper (Graduate School of Business, Stanford University, Stanford, CA).

Cochrane, J.H., 1986, How big is the random walk component in GNP, Journal of Political Economy, forthcoming.

Engle, R.F. and C.W.J. Granger, 1987, Co-integration and error correction: Representation, estimation and testing, Econometrica 55, 251–276.

Fernandez-Macho, F.J., A.C. Harvey and J.H. Stock, 1987, Forecasting and interpolation using vector autoregressions with common trends, Annales d'Economie et de Statistique 6/7, 279–287.

Harvey, A.C., 1985, Trends and cycles in macroeconomic time series, Journal of Business and Economic Statistics 3, 216–227.

Harvey, A.C. and J.H. Stock, 1985, The estimation of higher order continuous time autoregressive models, Econometric Theory 1, 97–112.

Harvey, A.C. and J.H. Stock, 1987, Estimating integrated higher order continuous time autoregressions with an application to money-income causality, Discussion paper no. 87-28 (Hoover Institution, Stanford University, Stanford, CA).

Kalman, R.E. and R.S. Bucy, 1961, New results in linear filtering and prediction theory, Transactions of the AMSE Journal of Basic Engineering 83D, 95–108.
King, R., C.I. Plosser, J.H. Stock and M.W. Watson, 1987, Stochastic trends and economic fluctuations, NBER discussion paper no. 2229.
Jones, R.H., 1981, Fitting a continuous time autoregression to discrete data, in: D.F. Findley, ed., Applied time series analysis II (Academic Press, New York).
Melino, A., 1985, Estimation of unit averaged diffusion processes, Department of Economics working paper no. 8502 (University of Toronto, Toronto).
Nelson, C.R. and C.I. Plosser, 1982, Trends and random walks in macroeconomic time series, Journal of Monetary Economics 10, 129–162.
Van Loan, C.F., 1978, Computing integrals involving the matrix exponential, IEEE Transactions on Automatic Control AC-23, 395–404.
Watson, M.W., 1986, Univariate detrending methods with stochastic trends, Journal of Monetary Economics 18, 49–75.
Working, H., 1960, A note on the correlation of first differences of averages in a random chain, Econometrica 28, 916–918.
Zadrozny, P., 1984, Gaussian likelihood of continuous-time ARMAX models when data are stocks and flows at different frequencies, Econometric Theory, forthcoming.