 Journal of Business & Economic Statistics, July 1992, Vol. 10, No. 3

Recursive and Sequential Tests of the Unit-Root and Trend-Break Hypotheses: Theory and International Evidence

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This article investigates the possibility, raised by Perron and by Rappoport and Reichlin, that aggregate economic time series can be characterized as being stationary around broken trend lines. Unlike those authors, we treat the break date as unknown a priori. Asymptotic distributions are developed for recursive, rolling, and sequential tests for unit roots and/or changing coefficients in time series regressions. The recursive and rolling tests are based on changing subsamples of the data. The sequential statistics are computed using the full data set and a sequence of regressors indexed by a "break" date. When applied to data on real postwar output from seven Organization for Economic Cooperation and Development countries, these techniques fail to reject the unit-root hypothesis for five countries (including the United States) but suggest stationarity around a shifted trend for Japan.

KEY WORDS: Changepoint; Dickey–Fuller tests: Persistence.

There is a large literature on the persistence exhibited by aggregate output, in particular whether output is well characterized as containing a unit autoregressive root. One alternative to the unit-root (or *integrated*) model, suggested by Perron (1989) and Rappoport and Reichlin (1989), is that log output (y_t) is stationary around a deterministic time trend that has one slope in an initial fraction of the sample and, later, a different slope. Using quarterly data for the postwar United States, Perron (1989) presented evidence against the unit-root null in favor of this trend-shift alternative when the trend shift is associated with the first oil-price shock. Evans (1989) and Perron (1990a) suggested the related model in which there is a shift in the intercept, in Perron's case possibly in conjunction with a shift in the slope of the deterministic trend.

These models and empirical findings are important for four reasons. First, as Perron emphasized, if the stationary/trend-shift model is correct, then studies such as Cochrane's (1988) and Cogley's (1990) have attributed too much persistence to innovations in gross national product, and conventional unit-root test statistics as used by Nelson and Plosser (1982) will incorrectly fail to reject the unit-root null. Second, in the spirit of Harvey (1985) and Watson (1986), this provides a parsimonious model for a slowly changing trend component of output that might be useful as data description. Interpreted literally, the single-shift/deterministic-trend model has little economic appeal, but interpreted more broadly it can be thought of as a metaphor for there being a few large events that determine the growth path of output over a decade or two—in the United States, the Depression or, later, the productivity slowdown. Once these decade-shaping events are taken into account, output exhibits business-cycle properties in the sense of mean-reversion over business-cycle horizons. Third, current empirical research relies heavily on techniques built on the integrated/stationary classification of time series: If series that are stationary with breaking trends are incorrectly classified as integrated, incorrect inferences can follow. Fourth, if the stationary/breaking-trend model fits many time series better than the integrated model, then the empirical relevance of the growing literature in theoretical econometrics on unit roots and cointegration is brought into question.

The empirical focus of this article is on international patterns of persistence and possible permanent shifts in growth trends. We are, however, persuaded by Christiano's (1988) argument that the date of the break ought not be treated as known—Perron's (1989, 1990a) approach—but rather should be treated as unknown a priori. After all, the hypothesis that there might have been a break in the U.S. output process around date of the first oil shock has intuitive appeal precisely because we know, before performing formal tests for breaks, that this major event was followed by a period of slower growth. This article therefore starts with the presumption that, if there is a break, its date is not known a priori but rather is gleaned from the data.

The literature on persistence of output includes several international comparisons (Campbell and Mankiw 1989; Clark 1989; Cogley 1990; Kormendi and Meguire 1990). These are based on standard full-sample techniques rather than procedures that explicitly allow for changing coefficients and thus leave unanswered some intriguing questions. Once the break point is treated as unknown a priori, is there evidence of a break in the drift of output? Is output stationary around a changing deterministic trend? If so, is this pattern consistent across countries, or is it idiosyncratic to specific countries? In particular, if there are identified breaks, are they associated with the productivity slowdown of the mid-1970s, and do they have the same timing across countries?

Our two objectives are first to develop econometric techniques (and the associated distribution theory) appropriate for answering these questions and second to apply these techniques to international data on output (real gross national product or gross domestic product) for seven Organization for Economic Cooperation and Development countries. Once the break point is treated as unknown, the usual distribution theory—which is conditional on a nonrandom known break point—does not apply. The approach taken in this article is to develop a distribution theory for a series of statistics evaluated over a range of possible break dates. This permits analyzing the distribution of continuous functionals of these statistics—for example, the maximum of the sequence of unit-root test statistics, one for each possible break date. Christiano (1988) and Evans (1989) recognized the nonstandard nature of these distributions and used numerical simulations to examine extrema of sequences of test statistics.

The methodological contribution of this article is to provide an asymptotic distribution theory for statistics pertaining to the shifting-root/shifting-trend hypotheses. Three classes of statistics are considered—recursive, rolling, and sequential. Recursive statistics are computed using subsamples $t = 1, \ldots, k$, for $k = k_0, \ldots, T$, where k_0 is a start-up value and T is the size of the full sample. Rolling statistics are computed using subsamples that are a constant fraction δ_0 of the full sample, rolling through the sample. Sequential statistics are computed using the full sample, sequentially incrementing the date of the hypothetical break (or shift).

Recursive and rolling statistics have been historically important tools in the econometric analysis of time series, even though formal distributional results have been limited. The term "recursive" derives from Brown, Durbin, and Evans's (1975) treatment of recursive estimation. Some recursive techniques are currently implemented in Hendry's (1987) statistical package PC-GIVE; also see Dufour (1982). A leading example of the use of recursive statistics is Hendry and Ericsson's (1991) use of plots of recursive coefficient estimates (their figs. 5, 9, 10, and 13) to argue that one of Friedman and Schwartz's (1982) money-demand equations is unstable and to support an improved specification. A second application, more closely related to the empirical problem considered here, is DeLong and Summers's (1988) study of whether U.S. output was more persistent after World War II than before the Depression. In particular, they examined whether a stationary root later changed to a unit root by estimating the largest roots of output for various countries in these subsamples (their table 2). Because Delong and Summers's subsample dates were determined from historical evidence, their break dates, like Perron's, are arguably best thought of as data dependent. In both Hendry and Ericsson's (1991) and DeLong and Summers's (1988) applications, inference was performed without the guidance of a formal distribution theory for the relevant recursive and rolling statistics. One solution in Delong and Summers's (1988) application is to compute a full set of recursive or rolling unit-root statistics and to apply the asymptotic theory developed here, as was done by Banerjee, Dolado, and Galbraith (1990).

The main sequential statistic of interest here is the Perron (1989)/Rappoport–Reichlin (1989) unit-root test with a trend shift at date k, computed sequentially for $k = k_0, \ldots, T - k_0$, where k_0 allows for trimming the initial and final parts of the sample. Another sequential statistic that will prove useful in our empirical analysis is Quandt's (1960) likelihood-ratio statistic, which entails computing the sequence of likelihood-ratio statistics testing for a break in at least one of the coefficients and then taking the maximum. These statistics are shown to have natural representations as stochastic processes defined on the unit interval, and their limiting distributions are characterized by functionals of Wiener processes. These results extend related work by McCabe and Harrison (1980), Sen (1980, 1982), Dufour (1982), James, James, and Siegmund (1987), Krämer, Ploberger, and Alt (1988), and Ploberger, Krämer, and Kontrus (1989). Our primary extension is to the case of a unit root in the regressors and to recursive, rolling, and sequential tests for unit roots. Our theoretical results on sequential Dickey–Fuller tests parallel those obtained independently in closely related work by Zivot and Andrews (1992).

These techniques are applied to data on postwar real output for Canada, France, Germany, Italy, Japan, the United Kingdom, and the United States. In only one case (Japan) is the unit-root hypothesis rejected in favor of the trend-shift hypothesis. We also investigate the possibility that output has a unit root, but that its drift (or mean growth rate) changed at an unknown date over this period. This broken drift could proxy for a permanent shift in technological progress or for some broader productivity slowdown. In three of the seven countries, output appears to be well characterized as having a unit root with a drift that fell in the early 1970s.

The recursive and sequential statistics are described, and their asymptotic properties studied, in Sections 1 and 2. Critical values and a Monte Carlo experiment are presented in Section 3. The empirical results are presented in Section 4, and conclusions are summarized in Section 5.

1. RECURSIVE AND ROLLING TEST STATISTICS

The primary focus of this section is the behavior of sequences of Dickey–Fuller (1979) t tests for a unit root, although the results are more general in that some specialize to stationary time series as well. The observations on y_t are assumed to be generated by

$$\text{Model I: } y_t = \mu_0 + \mu_1 t + \alpha y_{t-1} + \beta(L)\Delta y_{t-1} + \varepsilon_t, \qquad t = 1, \ldots, T, \quad (1)$$

where $\beta(L)$ is a lag polynomial of known order p with the roots of $1 - \beta(L)L$ outside the unit circle. Under the null hypothesis, $\alpha = 1$ and $\mu_1 = 0$. The errors are assumed to satisfy the following assumption:

Assumption A: ε_t is a martingale difference sequence and satisfies $E(\varepsilon_t^2|\varepsilon_{t-1}, \ldots) = \sigma^2$, $E(|\varepsilon_t|^i|\varepsilon_{t-1}, \ldots) = \kappa_i$ $(i = 3, 4)$, and $\sup_t E(|\varepsilon_t|^{4+\gamma}|\varepsilon_{t-1}, \ldots) = \kappa < \infty$ for some $\gamma > 0$.

When (1) is estimated by ordinary least squares (OLS) without restrictions on μ_0, μ_1, or α (i.e., when y_t is regressed on $1, t, y_{t-1}, \Delta y_{t-1}, \ldots, \Delta y_{t-p}$), the t statistic testing $\alpha = 1$ is the standard Dickey–Fuller (1979) $\hat{\tau}_\tau$ test for a unit root against a trend-stationary alternative.

In this section, we extend this Dickey–Fuller t test to the recursive case; that is, we consider the time series of recursively computed estimators and t statistics. Because of the unit root under the null hypothesis, it is convenient to define transformed regressors Z_t and a transformed parameter vector θ so that (1) can be rewritten

$$y_t = \theta' Z_{t-1} + \varepsilon_t, \quad (2)$$

where $Z_t = [Z_t^{1\prime}\ Z_t^2\ Z_t^3\ Z_t^4]'$, where $Z_t^1 = (\Delta y_t - \bar{\mu}_0 \cdots \Delta y_{t-p+1} - \bar{\mu}_0)'$, $Z_t^2 = 1$, $Z_t^3 = (y_t - \bar{\mu}_0 t)$, and $Z_t^4 = t + 1$, where $\bar{\mu}_0 = E\Delta y_t = \mu_0/(1 - \beta(1))$; and where $\theta = (\theta_1'\ \theta_2\ \theta_3\ \theta_4)'$ with $\theta_1 = (\beta_1 \cdots \beta_p)'$, $\theta_2 = \mu_0 + (\beta(1) - \alpha)\bar{\mu}_0$, $\theta_3 = \alpha$, and $\theta_4 = \mu_1 + \alpha\bar{\mu}_0$. Under the null hypothesis that $\alpha = 1$, the transformed regressors Z_t are linear combinations of the original regressors in (1), with the linear combination chosen to isolate the regressors with different stochastic properties; specifically, Z_t^1 are mean zero stationary regressors and $y_t - \bar{\mu}_0 t$ is an integrated process with no deterministic component. (This transformation is adopted from and discussed by Sims, Stock, and Watson [1990].) Because the elements of θ converge at different rates, define the scaling matrix $\Upsilon_T = \text{diag}\,(T^{1/2}I_p, T^{1/2}, T, T^{3/2})$, partitioned conformably with Z_t and θ. Let Ω_p denote the covariance matrix of $\Delta y_t, \ldots, \Delta y_{t-p}$, so $EZ_t^1 Z_t^{1\prime} = \Omega_p$. Moreover, suppose that observations on y_s $(s = -p, \ldots, 0)$ exist so that Z_0 is well defined.

The recursive OLS estimator of the coefficient vector is

$$\hat{\theta}(\delta) = \left(\sum_{t=1}^{[T\delta]} Z_{t-1}Z_{t-1}'\right)^{-1}\left(\sum_{t=1}^{[T\delta]} Z_{t-1}y_t\right), \qquad 0 < \delta_0 \le \delta \le 1. \quad (3)$$

Thus

$$\Upsilon_T(\hat{\theta}(\delta) - \theta) = V_T(\delta)^{-1}\phi_T(\delta), \quad (4)$$

where $V_T(\delta) = \Upsilon_T^{-1}\Sigma_{t=1}^{[T\delta]}Z_{t-1}Z_{t-1}'\Upsilon_T^{-1}$ and $\phi_T(\delta) = \Upsilon_t^{-1}\Sigma_{t=1}^{[T\delta]}Z_{t-1}\varepsilon_t$. (Throughout we adopt the notation that the symbols ^, ¯, and ˉ over characters refer to recursive, rolling, and sequential statistics, respectively.) Thus (3) and (4), respectively, provide representations of the recursive least squares estimator and its scaled deviation from θ as random elements of $D[0,1]$.

There are analogous expressions for a general recursively computed Wald statistic and for the Dickey–Fuller (DF) t statistic testing the hypothesis that $\alpha = 1$. Suppose that the Wald statistic tests the q hypotheses $R\theta = r$, where without loss of generality the hypotheses are ordered so that R is upper block triangular when partitioned conformably with θ; that is, the first restrictions involve coefficients on Z_t^1 (and perhaps Z_t^2, Z_t^3, and Z_t^4), the next restrictions involve coefficients on Z_t^2 (and perhaps Z_t^3 and Z_t^4), and so forth. The test statistics are

$$\hat{F}_T(\delta) = (R\hat{\theta}(\delta) - r)'\left[R\left(\sum_{t=1}^{[T\delta]} Z_{t-1}Z_{t-1}'\right)^{-1} R'\right]^{-1}(R\hat{\theta}(\delta) - r)/q\hat{\sigma}^2(\delta), \qquad \delta_0 \le \delta \le 1, \quad (5)$$

and

$$\hat{t}_{\text{DF}}(\delta) = T(\hat{\theta}_3(\delta) - 1)/[V_T(\delta)^{33}\hat{\sigma}^2(\delta)]^{1/2}, \quad \delta_0 \le \delta \le 1, \quad (6)$$

where $\hat{\sigma}^2(\delta) = ([T\delta] - p - 3)^{-1}\Sigma_{t=1}^{[T\delta]}(y_t - \hat{\theta}(\delta)'Z_{t-1})^2$ and $V_T(\delta)^{ij}$ denotes the (i, j) element of $V_T(\delta)^{-1}$. [Algebraic manipulations have been used to rewrite $\hat{t}_{\text{DF}}(\delta)$ from (1) in terms of the transformed regression (2).] Finally, define R^* (partitioned conformably with θ) so that $R_{ii}^* = R_{ii}$ $(i = 1, \ldots, 4)$, $R_{12}^* = R_{12}$, and $R_{ij}^* = 0$ otherwise.

Let "⇒" denote weak convergence on $D[0,1]$ (for example, see Ethier and Kurtz 1986 or Hall and Heyde 1980). The asymptotic behavior of the recursive estimators and test statistics is summarized in the following theorem.

Theorem 1. Suppose that y_t is generated by Model I in which $\mu_1 = 0$ and $\alpha = 1$ and that Assumption A holds. Then, for $0 < \delta_0 \le \delta \le 1$, (a) $V_T(\cdot) \Rightarrow V(\cdot)$, $\phi_T(\cdot) \Rightarrow \phi(\cdot)$, and $\Upsilon_T(\hat{\theta}(\cdot) - \theta) \Rightarrow \hat{\theta}^*(\cdot)$, where $\hat{\theta}^*(\delta) \equiv V(\delta)^{-1}\phi(\delta)$, where $V(\delta)$ and $\phi(\delta)$ are partitioned conformably with Υ_T and $\phi(\delta) = \sigma[B(\delta), W(\delta),$

$\frac{1}{2}b\sigma(W(\delta)^2 - \delta)$, $\delta w(\delta) - \int_0^\delta W(\lambda)d\lambda]'$, $V_{11} = \delta\Omega_p$, $V_{1j} = 0$, $j = 2, 3, 4$, $V_{22} = \delta$, $V_{23} = \sigma b\int_0^\delta W(\lambda)d\lambda$, $V_{24} = \frac{1}{2}\delta^2$, $V_{33} = \sigma^2 b^2 \int_0^\delta W(\lambda)^2 d\lambda$, $V_{34} = \sigma b\int_0^\delta \lambda W(\lambda)d\lambda$, and $V_{44} = \frac{1}{3}\delta^3$, where $W(\delta)$ is a standard Brownian motion on [0, 1] and $B(\delta)$ is a p-dimensional Brownian motion with covariance matrix Ω_p, W and B are independent, and $b = (1 - \beta(1))^{-1}$. (b) Suppose that $R\theta = r$. Then,

$$\hat{F}_T(\cdot) \Rightarrow [R^*\hat{\theta}^*(\cdot)]'[R^*V(\cdot)^{-1}R^{*\prime}]^{-1}[R^*\hat{\theta}^*(\cdot)]/q\sigma^2 \equiv \hat{F}^*(\cdot).$$

$$\text{(c) } \hat{t}_{\text{DF}}(\cdot) \Rightarrow [\sigma^2 V(\cdot)^{33}]^{-1/2}\hat{\theta}_3^*(\cdot) \equiv \hat{t}_{\text{DF}}^*(\cdot).$$

Appendix A gives proofs of theorems. Some remarks highlight different aspects of this result.

1. Krämer et al. (1988) and Ploberger et al. (1989) considered the case in which the regressors are stationary lagged dependent variables; our results provide explicit proofs in their case that the relevant "denominator" matrices are uniformly consistent.

2. As in the $\delta = 1$ case, $V(\cdot)$ is block diagonal. Thus the recursive estimation of the nuisance parameters ($\beta_1, \ldots, \beta_p$) does not affect the asymptotic distribution of the recursive Dickey–Fuller statistic. The novel feature of these results is that they apply uniformly in δ; the marginal distributions at any fixed δ are those that would be obtained using conventional (fixed δ) asymptotics. For example, $\hat{t}_{\text{DF}}(\delta')$, evaluated at a fixed δ' ($0 < \delta_0 \le \delta' \le 1$), has the "$\hat{\tau}_\tau$" distribution derived by Dickey and Fuller (1979). Thus the limiting stochastic process $\hat{t}_{\text{DF}}(\cdot)$ can be thought of as a Dickey–Fuller t-statistic process. Because $V(\cdot)$ is block diagonal, when the restrictions in R^* involve only coefficients on Z_{t-1}^1, $\hat{F}(\cdot)$ similarly can be thought of as a χ_q^2/q process. Moreover, the distribution of the recursive "demeaned" Dickey–Fuller statistic obtains as a special case by omitting t as a regressor in (1). The asymptotic representations apply for $0 < \delta_0 \le \delta \le 1$, accounting for $[T\delta_0]$ startup observations.

3. The transformation of the original regressors to Z_t is used to obtain a nondegenerate joint limiting representation of the estimators. This is the device used by Fuller (1976) and Dickey and Fuller (1979). As in the $\delta = 1$ case, because t is included as a regressor the distributions of the processes based on $\hat{\beta}(\delta)$ and $\hat{\alpha}(\delta)$ do not depend on the nuisance parameter μ_0. For further discussion in the $\delta = 1$ case, see Sims et al. (1990).

4. Although Theorem 1 is stated for the null model in which $\alpha = 1$ and $\mu_1 = 0$, the results are sufficiently general to handle the case $|\alpha| < 1$, $\mu_1 \ne 0$. This is achieved by redefining the variables in Model I. Specifically, let the left-hand variable be Δy_t rather than y_t and exclude y_{t-1} from the regression. Then the regressors are (Z_{t-1}^1, 1, t), where Z_{t-1}^1 has mean 0 and is stationary. Thus associating the $I(0)$ regressors with $\Delta y_t, \ldots, \Delta y_{t-p+1}$ in the notation of (1) and omitting the terms in y_t in the statement of the theorem provides the limiting process for the recursive estimators for the case of a stationary autoregression when $\mu_1 = 0$. If μ_1 is nonzero, an additional modification so that Z_t^1 remains mean 0 and stationary (by subtracting $\mu_1 t$ from Δy_t) results in Theorem 1 applying to the case of a regression involving an autoregressive process of order p [AR(p)] that is stationary around a time trend. With these modifications, the result concerning $\hat{F}_t(\cdot)$ [Theorem 1(b)] applies directly, although of course the result on $\hat{t}_{\text{DF}}(\cdot)$ is no longer germane.

5. Asymptotic representations for rolling estimators and test statistics obtain as a consequence of Theorem 1. Because a fixed fraction δ_0 of the sample is used, unlike recursive estimators the sampling variability of rolling coefficient estimators is (in expectation) constant through the sample. The rolling estimator $\bar{\theta}$ is

$$\bar{\theta}(\delta; \delta_0) = \left(\sum_{t=[T(\delta-\delta_0)]+1}^{[T\delta]} Z_{t-1}Z_{t-1}'\right)^{-1} \times \left(\sum_{t=[T(\delta-\delta_0)]+1}^{[T\delta]} Z_{t-1}y_t\right)$$

so that $\Upsilon_T(\bar{\theta}(\delta; \delta_0) - \theta) = V_T(\delta; \delta_0)^{-1}\phi_T(\delta; \delta_0)$, where $V_T(\delta; \delta_0) = \Upsilon_T^{-1}(\sum_{t=[T(\delta-\delta_0)]+1}^{[T\delta]} Z_{t-1}Z_{t-1}')\Upsilon_T^{-1} = V_T(\delta) - V_T(\delta - \delta_0)$ and $\phi_T(\delta; \delta_0) = \Upsilon_T^{-1} \times (\sum_{t=[T(\delta-\delta_0)]+1}^{[T\delta]} Z_{t-1}\varepsilon_t) = \phi_T(\delta) - \phi_T(\delta - \delta_0)$. From Theorem 1(a), $V_T(\cdot; \delta_0) \Rightarrow V(\cdot; \delta_0)$, where $V(\delta; \delta_0) = V(\delta) - V(\delta - \delta_0)$ and $\phi_T(\cdot; \delta_0) \Rightarrow \phi(\cdot; \delta_0)$, where $\phi(\delta; \delta_0) = \phi(\delta) - \phi(\delta - \delta_0)$. Thus $\Upsilon_T(\bar{\theta}(\cdot; \delta_0) - \theta) \Rightarrow \theta^*(\cdot; \delta_0)$, where $\theta^*(\delta; \delta_0) = V(\delta; \delta_0)^{-1}\phi(\delta; \delta_0)$. Representations for rolling F and t statistics are obtained using analogous arguments.

6. Another alternative is to compute these statistics in a *reverse recursion*—that is, using data over $t = k + 1, \ldots, T$ for $k = 0, \ldots, T - k_0$. The approach in remark 5 can be modified to handle this situation. Let $\hat{\theta}_{rr}(\delta)$ be the reverse-recursive estimator, so $\Upsilon_T(\hat{\theta}_{rr}(\cdot) - \theta) = V_{T,rr}(\delta)^{-1}\phi_{T,rr}(\delta)$, where $V_{T,rr}(\delta) = \Upsilon_T^{-1} \times \sum_{t=[T\delta]+1}^T Z_{t-1}Z_{t-1}'\Upsilon_T^{-1}$ and $\phi_{T,rr}(\delta) = \Upsilon_T^{-1}\sum_{t=[T\delta]+1}^T Z_{t-1}\varepsilon_t$. Because $V_{T,rr}(\delta) = V_T(1) - V_T(\delta)$ and $\phi_{T,rr}(\delta) = \phi_T(1) - \phi_T(\delta)$, $\Upsilon_T(\hat{\theta}_{rr}(\cdot) - \theta) \Rightarrow \{V(1) - V(\cdot)\}^{-1}\{\phi(1) - \phi(\cdot)\}$. Reverse-recursive Wald statistics are handled similarly.

7. Although this discussion has focused on Dickey–Fuller t statistics, the approach can be used to develop asymptotic representations for other unit-root tests as well. An example is Sargan and Bhargava's (1983) test of the Gaussian random walk against the AR(1) alternative, as extended by Bhargava (1986) to the time-trend case and by Stock (1988) to the case of the general $I(1)$ null. This test is of particular interest because Sargan and Bhargava interpreted it as the uniformly most powerful test in the Gaussian AR(1) case. Asymptotic and selected Monte Carlo properties of recursive and rolling modified Sargan–Bhargava statistics were provided in the working paper version of this article (Banerjee, Lumsdaine, and Stock 1990).

2. SEQUENTIAL TESTS FOR CHANGES IN COEFFICIENTS

The statistics analyzed in this section are computed sequentially using the full sample. These allow for a single shift or break in a deterministic trend at an unknown date. The model considered is

$$\text{Model II: } y_t = \mu_0 + \mu_1\tau_{1t}(k) + \mu_2 t + \alpha y_{t-1} + \beta(L)\Delta y_{t-1} + \omega' x_{t-1}(k) + \varepsilon_t \qquad (7)$$

for $t = 1, \ldots, T$, where $\beta(L)$ is a lag polynomial of known order p. Unlike Model I, Model II allows for an additional m-vector of regressors, $x_{t-1}(k)$, which are assumed to be stationary with a constant zero mean. As in Section 1, transform the regressors to $Z_t = [Z_t^{1\prime}, 1, (y_t - \bar{\mu}_0 t), \tau_{1t+1}(k), t + 1]'$, where $Z_t^1 = (\Delta y_t - \bar{\mu}_0 \cdots \Delta y_{t-p+1} - \bar{\mu}_0 x_t(k)')'$ and $\bar{\mu}_0 = E\Delta y_t$, and let $\theta = [\theta_1' \ \theta_2 \ \theta_3 \ \theta_4 \ \theta_5]'$, where $\theta_1 = [\beta' \ \omega']'$, $\theta_2 = \mu_0 + (\beta(1) - \alpha)\bar{\mu}_0$, $\theta_3 = \alpha$, $\theta_4 = \mu_1$, and $\theta_5 = \mu_2 + \alpha\bar{\mu}_0$.

The deterministic regressor $\tau_{1t}(k)$ captures the possibility of a shift or jump in the trend at period k. Following Perron (1989, 1990a), consider two cases:

$$\text{Case A (shift in trend): } \tau_{1t}(k) = (t - k)1(t > k) \qquad (8)$$

and

$$\text{Case B (shift in mean): } \tau_{1t}(k) = 1(t > k), \qquad (9)$$

where $1(\cdot)$ is the indicator function. For Case A (referred to by Perron [1989, 1990a] as the "changing growth" model), the t statistic testing $\mu_1 = 0$ provides information about whether there has been a shift (change in slope) in the trend. For Case B (Perron's "crash" model), this t statistic provides information about whether there has been a jump or break in the trend.

Let ω_0 denote the value of ω under the null. It is assumed that those x_t terms involving k do not enter under the null. The disturbances and $\{x_t(k)\}$ are assumed to satisfy the following assumption.

Assumption B. (a) Let $M_t(k)$ be the sigma field generated by $\{\varepsilon_t, x_t(k), \varepsilon_{t-1}, x_{t-1}(k), \ldots\}$. Then $E(\varepsilon_t|M_{t-1}) = 0$, $E(\varepsilon_t^2|M_{t-1}) = \sigma^2$, $E(|\varepsilon_t|^i|M_{t-1}) = \kappa_i$ ($i = 3, 4$), and $E(|\varepsilon_t|^{4+\gamma}|M_{t-1}) \le \bar{\kappa} < \infty$ for some $\gamma > 0$ uniformly in k. (b) $\{x_t([T\delta])\}$ is such that $Ex_t([T\delta]) = 0$ for all t, $T^{-1}\Sigma_{t=1}^T Z_{t-1}^1([T\delta])Z_{t-1}^1([T\delta])' \xrightarrow{p} \Sigma(\delta)$, $T^{-1/2}\Sigma_{t=1}^T Z_{t-1}^1([T\delta])\varepsilon_t \Rightarrow \sigma G(\delta)$, $T^{-3/2}\Sigma_{t=1}^T Z_{t-1}^1([T\delta])y_t \Rightarrow 0$, and $(T^{-1/2}\Sigma_{t=1}^{[T\lambda]}\varepsilon_t, T^{-1/2}\Sigma_{t=1}^{[T\lambda]}\omega_0' x_t([T\delta])) \Rightarrow (\sigma W(\lambda), \pi H(\lambda))$, all uniformly in δ, where W and H are standard one-dimensional Brownian motions, W and H are not necessarily independent, $\Sigma(\cdot)$ is a nonrandom positive semidefinite matrix-valued function on [0, 1], $G(\cdot)$ is a square-summable $(p + m)$-dimensional stochastic process in $D[0, 1]$, and π is a constant.

The leading case in which Assumption B is satisfied is when $\alpha = 1$, $1 - \beta(L)L$ has all its roots outside the unit circle, and $\{x_{t-1}(k)\}$ is omitted so that Z_{t-1}^1 consists of $\Delta y_{t-1}, \ldots, \Delta y_{t-p}$. In this case, Model II introduces a break in the deterministic trend in the Dickey–Fuller regression (1). In the notation of Assumption B, in this case $\Sigma(\delta) = \Omega_p$ and $G(\delta) = B(1)$, where $B(\cdot)$ is a $p \times 1$ Brownian motion with covariance matrix Ω_p, independent of (W, H).

The formulation (7) generalizes this leading case to include additional mean zero stationary regressors and certain regressors that depend on k. For example, setting $x_t(k) = ((\Delta y_t - \bar{\mu}_0)1(t > k) \cdots (\Delta y_{t-p+1} - \bar{\mu}_0) \times 1(t > k))'$ (which satisfies Assumption B) permits testing whether the coefficients on $(\Delta y_{t-1}, \ldots, \Delta y_{t-p})$ in Model II are constant against the alternative that at least one changes at an unknown date. In this case, $G(\delta) = (B(1)', (B(1) - B(\delta))')'$, where $B(\cdot)$ is a $p \times 1$ Brownian motion with covariance Ω_p. Moreover, $\Sigma(\delta)$ is $2p \times 2p$, with $p \times p$ blocks $\Sigma_{11}(\delta) = \Omega_p$ and $\Sigma_{12}(\delta) = \Sigma_{21}(\delta) = \Sigma_{22}(\delta) = (1 - \delta)\Omega_p$. Note that, under the null hypothesis, $Ex_t = 0$ is assumed without loss of generality as long as a constant is included in the regression.

The estimators and test statistics are computed using the full T observations for $k = k_0, k_0 + 1, \ldots, T - k_0$, where $k_0 = [T\delta_0]$. The resulting statistics are thus sequential rather than recursive. As usual, let R and r, respectively, be $q \times (m + p + 4)$ and $q \times 1$ matrices of linear restrictions on θ. The stochastic processes constructed from the sequential estimators and Wald test statistic are, for $\delta_0 \le \delta \le 1 - \delta_0$,

$$\tilde{\theta}(\delta) = \left(\sum_{t=1}^{T} Z_{t-1}([T\delta])Z_{t-1}([T\delta])'\right)^{-1}\left(\sum_{t=1}^{T} Z_{t-1}([T\delta])y_t\right), \qquad (10)$$

$$\Upsilon_T(\tilde{\theta}(\delta) - \theta) = \Gamma_T(\delta)^{-1}\Psi_T(\delta), \qquad (11)$$

and

$$\tilde{F}_T(\delta) = [R\tilde{\theta}(\delta) - r]' \times \left[R\left(\sum_{t=1}^{T} Z_{t-1}([T\delta])Z_{t-1}([T\delta])'\right)^{-1} R'\right]^{-1} \times [R\tilde{\theta}(\delta) - r]/q\tilde{\sigma}^2(\delta), \qquad (12)$$

where $\tilde{\sigma}^2(\delta) = (T - p - m - 4)^{-1}\Sigma_{t=1}^T(y_t - \tilde{\theta}(\delta)'Z_{t-1}([T\delta]))^2$, $\Gamma_T(\delta) = \Upsilon_T^{-1}\Sigma_{t=1}^T Z_{t-1}([T\delta]) \times Z_{t-1}([T\delta])'\Upsilon_T^{-1}$, and $\Psi_T(\delta) = \Upsilon_T^{-1}\Sigma_{t=1}^T Z_{t-1}([T\delta])\varepsilon_t$. Here, $\Upsilon_T = \Upsilon_{AT}$ in case A and $\Upsilon_T = \Upsilon_{BT}$ in case B, where $\Upsilon_{AT} = \text{diag}(T^{1/2}I_{p+m}, T^{1/2}, T, T^{3/2}, T^{3/2})$ and $\Upsilon_{BT} = \text{diag}(T^{1/2}I_{p+m}, T^{1/2}, T, T^{1/2}, T^{3/2})$.

The next theorem provides asymptotic representations for the standardized sequential coefficients.

Theorem 2. Suppose that y_t is generated according to Model II with $\mu_1 = \mu_2 = 0$ and $\alpha = 1$ and that Assumption B holds. Then (a) in case A [Eq. (8)], $\Upsilon_{AT}(\tilde{\theta}(\cdot) - \theta) \Rightarrow \Gamma(\cdot)^{-1}\Psi(\cdot)$, where $\Psi(\delta) = \sigma\{G(\delta)', W(1), \int_0^1 J(\lambda)dW(\lambda), (1 - \delta)W(1) - \int_\delta^1 W(\lambda)d\lambda, W(1) -$

$\int_0^1 W(\lambda)d\lambda\}'$, $\Gamma_{11} = \Sigma(\delta)$, $\Gamma_{1j} = 0$ $(j = 2, \ldots, 5)$, $\Gamma_{22} = 1$, $\Gamma_{23} = \int_0^1 J(\lambda)d\lambda$, $\Gamma_{24} = \frac{1}{2}(1-\delta)^2$, $\Gamma_{25} = \frac{1}{2}$, $\Gamma_{33} = \int_0^1 J(\lambda)^2 d\lambda$, $\Gamma_{34} = \int_\delta^1 (\lambda - \delta)J(\lambda)d\lambda$, $\Gamma_{35} = \int_0^1 \lambda J(\lambda)d\lambda$, $\Gamma_{44} = (1-\delta)^3/3$, $\Gamma_{45} = \frac{1}{3} - \frac{1}{2}\delta + \delta^3/6$, and $\Gamma_{55} = \frac{1}{3}$, where $W(\delta)$ is a standard Brownian motion process, $b = (1 - \beta(1))^{-1}$, and $J(\lambda) = b\pi H(\lambda) + \sigma b W(\lambda)$; and (b) in case B [Eq. (9)], $\Upsilon_{BT}(\tilde{\theta}(\cdot) - \theta) \Rightarrow \Gamma(\cdot)^{-1}\Psi(\cdot)$, where Ψ is as in (a) except that $\Psi_4(\delta) = \sigma(W(1) - W(\delta))$, and where Γ is as in (a) except for $\Gamma_{24} = 1 - \delta$, $\Gamma_{34} = \int_\delta^1 J(\lambda)d\lambda$, $\Gamma_{44} = 1 - \delta$, and $\Gamma_{45} = \frac{1}{2}(1 - \delta^2)$, where W, J, and b are as defined in (a).

Several remarks are in order:

1. When $x_{t-1}(k)$ does not appear as a regressor and δ is fixed, this reduces to the model and results presented by Perron (1989). Theorem 2 generalizes this result to the case in which the estimator and test-statistic processes are random elements of $D[0, 1]$, indexed by δ. Note, however, that Perron considered the case of unknown (possibly infinite) AR order p, whereas here p is assumed to be finite and known.

2. This result applies for $0 < \delta_0 \le \delta \le (1 - \delta_0) < 1$. Thus the test for the change in the coefficients is constrained not to be at the ends of the sample. In practice, this requires choosing a "trimming" value $k_0 = [T\delta_0]$, an issue addressed in the next two sections.

3. Formal representations for the $\tilde{F}_T(\delta)$ statistic, or for sequential Dickey–Fuller statistics, obtain using the R^* device used in Theorem 1(b). The limiting processes for $T^{1/2}(\tilde{\theta}_1(\cdot) - \theta)$ and $\tilde{F}_T$ statistics testing q restrictions on Z_{t-1}^1 can be thought of, respectively, as Gaussian and χ_q^2/q processes, with the marginal distribution of each process for fixed evaluation points $\delta = \delta'$ being, respectively, Gaussian or χ_q^2/q.

4. This result provides joint uniform convergence of all the estimators and test statistics. Thus, in particular, it provides the asymptotic representation of continuous functions of one or more of these processes. One example is a rule considered by Christiano (1988): Compute the Dickey–Fuller t statistic in Model II, $\tilde{t}_{\mathrm{DF}}(k/T)$, for $k_0 \le k \le T - k_0$, and let $\tilde{t}_{\mathrm{DF}}^{\min *} = \min_{k_0 \le k \le T-k_0} \tilde{t}_{\mathrm{DF}}(k/T)$.

5. A related sequential statistic is the Quandt (1960) likelihood-ratio (LR) statistic, which tests for a break in any or all of the coefficients. This entails estimating $2(T - 2k_0)$ separate regressions of the form (1) over the subsamples $1, \ldots, [T\delta]$ and $[T\delta] + 1, \ldots, T$. The LR statistic is computed for each possible break point, and the Quandt LR statistic Q_{LR} is the maximum of these. In the notation of Section 1, $Q_{\mathrm{LR}} = \max_{k_0 \le k \le T-k_0} (-2 \ln \tilde{\lambda}(k))$, where $\tilde{\lambda}(k) = \tilde{\sigma}_{1,k}^k \tilde{\sigma}_{k+1,T}^{T-k}/\tilde{\sigma}_{1,T}^T$, where $\tilde{\sigma}_{t_1,t_2}^2$ is the Gaussian maximum likelihood estimator of the regression error variance over observations $t_1, \ldots, t_2$. Although Q_{LR} is based on the full sample and thus is a sequential statistic, the asymptotic distribution is obtained using the results in Section 1. Calculations based on Theorem 1 (provided by Banerjee et al. 1990) show that, for Model I under the null hypothesis with no breaks [Eq. (1) with $\alpha = 1$ and $\mu_0 = 0$], $-2 \ln \tilde{\lambda}([T\cdot]) \Rightarrow -2 \ln \tilde{\lambda}^*(\cdot)$, where

$$\begin{aligned} -2 \ln \tilde{\lambda}^*(\delta) = {} & \sigma^{-2}\phi(\delta)'\{V(\delta)^{-1} \\ & + (V(1) - V(\delta))^{-1}\}\phi(\delta) \\ & - \sigma^{-2}\phi(1)'\{V(1)^{-1} \\ & - (V(1) - V(\delta))^{-1}\}\phi(1) \\ & - 2\sigma^{-2}\phi(\delta)'\{V(1) \\ & - V(\delta)\}^{-1}\phi(1), \end{aligned} \tag{13}$$

where $\phi(\cdot)$ and $V(\cdot)$ are defined in Theorem 1. By the continuous mapping theorem, $Q_{\mathrm{LR}} \Rightarrow \sup_{\delta_0 \le \delta \le 1-\delta_0} (-2 \ln \tilde{\lambda}^*(\delta))$. Because Q_{LR} tests for a break in any of the coefficients, its distribution depends on p.

This specializes to Chu's (1989, sec. 2) result for the Quandt LR statistic when only the p stationary regressors and a constant are included; then $Q_{\mathrm{LR}} \Rightarrow \sup_{\delta_0 \le \delta \le 1-\delta_0} \{W_{p+1}^*(\delta)' W_{p+1}^*(\delta)/(\delta(1 - \delta))\}$, where W_{p+1}^* is a $(p + 1)$-dimensional standard Brownian bridge. Chu (1989) also provided critical values, a Monte Carlo analysis, and alternative related test statistics and discussed related results in the literature. Hansen (1990) recently proposed a related "mean Chow" statistic, the average of these likelihood ratios, the distribution of which is the multivariate generalization of the limiting distribution of the Anderson–Darling (1954) statistic.

6. A statistic that will be used in the empirical analysis is the sequential t statistic, $\tilde{t}_{\tau_1}(\cdot)$, testing the hypothesis that the coefficient on $\tau_{1t}(k)$ is 0 in case B ($\tau_{1t}(k) = 1(t > k)$), under the restriction that $\alpha = 1$ and $\mu_2 = 0$. This corresponds to a shift in the intercept in a pth order autoregression of Δy_t. Because $\Gamma(\cdot)$ is block diagonal, the asymptotic distribution of $\tilde{t}_{\tau_1}(\cdot)$ (obtained from Theorem 2) does not depend on p, and the statistic has the limiting representation $\tilde{t}_{\tau_1}(\cdot) \Rightarrow \tilde{t}_{\tau_1}^*(\cdot)$, where $\tilde{t}_{\tau_1}^*(\delta) = W_1^*(\delta)/(\delta(1 - \delta))^{1/2}$ and $W_1^*(\delta) = W(\delta) - \delta W(1)$ is the one-dimensional standard Brownian bridge.

3. MONTE CARLO RESULTS

This section reports asymptotic critical values and examines the size and power of selected recursive, rolling, and sequential statistics. All regressions include $(1, t)$ to allow for a possible time trend under the alternative, except $\tilde{t}_{\tau_1}(\cdot)$ in the restricted Model II ($\alpha = 1$, $\mu_2 = 0$) model for which t is excluded.

The first four statistics examined are recursive tests for unit roots—the full-sample Dickey–Fuller statistic, $\hat{t}_{\mathrm{DF}}$ [$= \hat{t}_{\mathrm{DF}}(1)$ in the notation of (6)]; the maximal Dickey–Fuller statistic, $\hat{t}_{\mathrm{DF}}^{\max} \equiv \max_{k_0 \le k \le T} \hat{t}_{\mathrm{DF}}(k/T)$; the minimal Dickey–Fuller statistic, $\hat{t}_{\mathrm{DF}}^{\min} \equiv \min_{k_0 \le k \le T} \hat{t}_{\mathrm{DF}}(k/T)$; and $\hat{t}_{\mathrm{DF}}^{\mathrm{diff}} \equiv \hat{t}_{\mathrm{DF}}^{\max} - \hat{t}_{\mathrm{DF}}^{\min}$. For these, $\hat{t}_{\mathrm{DF}}(k/T)$ is computed using (6)—that is, as the t statistic testing $\alpha = 1$ in the regression (1), estimated over $t = 1, \ldots, k$.

The second set of statistics is rolling Dickey–Fuller statistics—namely $\bar{t}_{\mathrm{DF}}^{\max} \equiv \max_{k_0 \le k \le T} \bar{t}_{\mathrm{DF}}(k/T;\ \delta_0)$, $\bar{t}_{\mathrm{DF}}^{\min}$

$\equiv \min_{k_0 \le k \le T} \bar{t}_{DF}(k/T; \delta_0)$, and $\bar{t}_{DF}^{diff} \equiv \bar{t}_{DF}^{max} - \bar{t}_{DF}^{min}$, where $\bar{t}_{DF}(k/T; \delta_0)$ is the t statistic testing $\alpha = 1$ in the regression (1) estimated over $t = k - [T\delta_0] + 1, \ldots, k$, as discussed in remark 5 of Section 1.

The final set of statistics are sequential—the Quandt LR statistic $Q_{LR}(p)$, where p refers to the number of lags of Δy_t in the regression; the maximum of the sequential F statistics, $\tilde{F}_T^{max} \equiv \max_{k_0 \le k \le T-k_0} \tilde{F}_T(k/T)$, testing the hypothesis that $\mu_1 = 0$ in (7); the sequential Dickey–Fuller statistic evaluated at the value of k ($\tilde{k}$, equivalently $\tilde{\delta} = \tilde{k}/T$) that maximizes $\tilde{F}_T(k/T)$, $\tilde{t}_{DF}(\tilde{\delta})$; $\tilde{t}_{DF}^{min*} \equiv \min_{k_0 \le k \le T-k_0} \tilde{t}_{DF}(k/T)$, the minimal Dickey–Fuller statistic over all the sequentially computed Dickey–Fuller statistics; and the absolute extreme t statistic on $\tilde{\tau}_{1t}(k)$, $|\text{ext}_{\delta \in (\delta_0, 1-\delta_0)} \tilde{t}_{\tau 1}(\delta)|$, for the mean break model, under the restriction $\mu_2 = 0$ and $\alpha = 1$. This final statistic corresponds to a sequential t test for a change in the intercept in an AR(p) estimated using Δy_t as the dependent variable as discussed in remark 6 of Section 2. Of these, Christiano (1988) proposed $\tilde{F}_T^{max}$, $\tilde{t}_{DF}(\tilde{\delta})$, and $\tilde{t}_{DF}^{min*}$ to extend Perron's analysis to the case in which k is unknown. Asymptotic critical values for the sequential statistics are presented for both the trend-shift and mean-shift regressions [(7) with τ_{1t} given by (8) and (9), respectively].

The trimming parameters used are as follows: For the recursive and Q_{LR} statistics, $\delta_0 = .25$; for rolling statistics, $\delta_0 = \frac{1}{3}$; and for sequential statistics except Q_{LR}, $\delta_0 = .15$. The choice of δ_0 entails a trade-off between needing enough observations in the shortest regressions to support the Gaussian approximation and wanting to capture possible breaks early and late in the sample. The chosen values are representative of those used in practice and in any event seem appropriate for the empirical application in Section 4. We did not examine the size and power of the statistics as a function of δ_0, nor did we examine statistics based on tapered rather than square cutoffs as discussed by Deshayes and Picard (1986). Rather, investigations into the choice of δ_0 are left for future research.

Approximate asymptotic critical values for these recursive, rolling, and sequential statistics are reported in Tables 1 and 2. Throughout, the convention for the sample size is that T represents the number of observations used in the regression; with (say) four lags of Δy_t, this requires an original data set with $T + 5$ observations. The critical values were computed using artificial discrete realizations of Brownian motions to approximate the various limiting functionals appearing in Theorems 1 and 2. This is equivalent to performing Monte Carlo simulations for the null model $\Delta y_t = \varepsilon_t$, ε_t iid $N(0, 1)$. Of the statistics considered here, only Q_{LR} has a limiting distribution that depends on p, so critical values for all statistics were computed for $p = 0$ except for Q_{LR}, for which critical values are reported for $p = 0$, 4, and 8. Not surprisingly, the critical values for the recursive $\hat{t}_{DF}^{min}$ and rolling $\bar{t}_{DF}^{min}$ statistics are well below the full-sample t_{DF} critical values. The large critical value of the sequential $\tilde{F}_T^{max}$ is comparable to that found by Christiano (1988). Comparison of the percentiles for different T indicates rapid convergence to the asymptotic limits, so the $T = 500$ values can be treated as approximate asymptotic critical values.

Size and nominal power of the recursive, rolling, and trend-break sequential statistics are summarized in Table 3 for $T = 100$. The tests were computed from regressions in which $p = 4$, the base case examined in the empirical work, and were evaluated using the $T = 100$ critical values from Tables 1 and 2. Panel A reports size when the true model is a Gaussian AR(1). With the exceptions of the recursive $\hat{t}_{DF}^{diff}$ and the rolling $\bar{t}_{DF}^{max}$ and especially $\bar{t}_{DF}^{diff}$, all of the statistics have sizes near their levels. As expected, the size distortions diminish as T increases. For example, for $T = 250$ and $\beta = .4$, the rolling $\bar{t}_{DF}^{max}$ has size of 8%. Additional experiments (not tabulated here) were performed to evaluate the size of the other statistics whose distributions are summarized in Tables 1 and 2. For the sequential mean-break unit-root statistic $\tilde{t}_{DF}^{min*}$, the size was close to its level, 8.2% and 8.8% for 10%-level tests with $\beta = .4$ and .6, re-

Table 1. Recursive and Rolling Test Statistics: Critical Values

			Recursive			Rolling		
T	Percentile	t_{DF}	$\hat{t}_{DF}^{max}$	$\hat{t}_{DF}^{min}$	$\hat{t}_{DF}^{diff}$	$\bar{t}_{DF}^{max}$	$\bar{t}_{DF}^{min}$	$\bar{t}_{DF}^{diff}$
100	.025	−3.73	−2.21	−4.62	4.06	−1.66	−5.29	5.13
	.050	−3.45	−1.99	−4.33	3.65	−1.49	−5.01	4.76
	.100	−3.15	−1.73	−4.00	3.23	−1.31	−4.71	4.40
250	.025	−3.69	−2.15	−4.42	3.91	−1.66	−5.07	5.01
	.050	−3.43	−1.94	−4.18	3.61	−1.48	−4.85	4.68
	.100	−3.13	−1.69	−3.91	3.24	−1.27	−4.59	4.36
500	.025	−3.68	−2.17	−4.42	3.91	−1.62	−5.00	4.93
	.050	−3.42	−1.92	−4.18	3.57	−1.47	−4.79	4.65
	.100	−3.13	−1.66	−3.88	3.21	−1.25	−4.55	4.31

NOTE: All critical values were computed using data generated as $\Delta y_t = \varepsilon_t$, ε_t iid $N(0, 1)$ and are based on 10,000 Monte Carlo replications for $T = 100$ and $T = 250$ and 5,000 replications for $T = 500$. The full-sample Dickey–Fuller statistic critical values (t_{DF}) were taken from Fuller (1976, table 8.5.2), for the case of (1, t) being included as regressors. Each recursive statistic $\hat{t}_{DF}(k/T)$ was computed by estimating (1) with $p = 0$ over $t = 1, \ldots, k$, with $\delta_0 = .25$. Each rolling statistic $\bar{t}_{DF}(k/T; \delta_0)$ was computed by estimating (1) with $p = 0$ over $t = k - [T\delta_0] + 1, \ldots, k$, $k = [T\delta_0], \ldots, T$, $\delta_0 = \frac{1}{3}$.

Table 2. Sequential Test Statistics: Critical Values

					Trend-shift statistics (8)			Mean-shift statistics (9)			
T	Percentile	$Q_{LR}(0)$	$Q_{LR}(4)$	$Q_{LR}(8)$	$\bar{F}_T^{max}$	$\tilde{t}_{DF}(\tilde{\delta})$	$\tilde{t}_{DF}^{min*}$	$\bar{F}_T^{max}$	$\tilde{t}_{DF}(\tilde{\delta})$	$\tilde{t}_{DF}^{min*}$	$\lvert ext_\delta \tilde{t}_{\tau 1}(\delta)\rvert$
100	.025	28.96	37.25	46.52	19.15	−4.76	−4.76	20.83	−5.07	−5.07	3.23
	.050	26.45	34.56	43.51	16.30	−4.47	−4.48	18.62	−4.80	−4.80	2.95
	.100	23.86	31.78	40.21	13.64	−4.19	−4.20	16.20	−4.52	−4.54	2.63
250	.025	30.41	36.37	43.27	18.36	−4.66	−4.66	21.31	−5.05	−5.06	3.19
	.050	27.87	34.12	40.36	15.94	−4.39	−4.39	19.01	−4.79	−4.80	2.92
	.100	24.97	30.95	37.06	13.32	−4.12	−4.12	16.72	−4.50	−4.51	2.63
500	.025	30.42	37.00	43.00	18.58	−4.68	−4.69	21.26	−5.05	−5.05	3.23
	.050	27.80	34.00	39.95	16.04	−4.39	−4.39	18.99	−4.77	−4.78	2.93
	.100	25.19	30.72	36.84	13.20	−4.12	−4.13	16.78	−4.49	−4.51	2.66

NOTE: All critical values were computed using data generated as $\Delta y_t = \varepsilon_t$, ε_t iid $N(0, 1)$ and are based on 10,000 Monte Carlo replications for $T = 100$ and $T = 250$ and 5,000 replications for $T = 500$. The $Q_{LR}(p)$ statistic was computed for the regression of y_t onto $(1, t, y_{t-1}, \Delta y_{t-1}, \ldots, \Delta y_{t-p})$ with $\delta_0 = .25$. The trend-shift statistics were computed by estimating (7) sequentially, with $\tau_{1t}(k)$ given by (8) and $p = 0$. The mean-shift statistics were computed by estimating (7) sequentially, with $\tau_{1t}(k)$ given by (9) and $p = 0$. For all sequential statistics except $Q_{LR}(p)$, $\delta_0 = .15$.

spectively. For the restricted statistic $\lvert ext_\delta \tilde{t}_{\tau 1}(\delta)\rvert$ with $p = 1$, the size was 12.6% and 15.1% for $\beta = .4$ and .6, respectively, but for $p = 4$ these rejection rates rose to 17.1% and 18.5%. Because of this size distortion as p increases, in the empirical work (where $p = 4$ is the base case) p values for this statistic are computed using the null distribution, calculated by Monte Carlo, with $p = 4$ and $T = 100$; the 2.5%, 5%, and 10% critical values are, respectively, 3.40, 3.13, and 2.84, somewhat larger than the $p = 0$ values given in the final column of Table 2.

Panel B of Table 3 presents nominal (not size-adjusted) power against the alternative that the largest root is 1 in half the sample and is less than 1 in the other half. The recursive $\hat{t}_{DF}^{diff}$ and $\hat{t}_{DF}^{min}$ statistics and Q_{LR} typically have the best power against ($\alpha_1 < 1$, $\alpha_2 = 1$). Because the recursive statistics always include the initial data, it is perhaps unsurprising that the recursive $\hat{t}_{DF}^{min}$ performs relatively well when $\alpha_1 < 1$. The rolling statistics have less power against the $\alpha_1 < 1$ alternative than the recursive statistics, presumably because at most 33 stationary observations are used for the rolling sta-

Table 3. Size and Power of Recursive, Rolling, and Sequential Tests: Monte Carlo Results

		Recursive			Rolling			Sequential trend-shift statistics				
	t_{DF}	$\hat{t}_{DF}^{max}$	$\hat{t}_{DF}^{min}$	$\hat{t}_{DF}^{diff}$	$\bar{t}_{DF}^{max}$	$\bar{t}_{DF}^{min}$	$\bar{t}_{DF}^{diff}$	$Q_{LR}(4)$	$\tilde{F}_T^{max}$	$\tilde{t}_{DF}(\tilde{\delta})$	$\tilde{t}_{DF}^{min*}$	$\tilde{k} \pm .05T$
A. Size for ARIMA(1, 1, 0): $\Delta y_t = \beta\Delta y_{t-1} + \varepsilon_t$, ε_t NIID(0, 1)												
β												
0.4	9.1	6.9	10.0	21.8	2.5	10.5	28.7	10.8	13.4	11.5	11.8	–
0.6	9.6	7.1	10.5	25.4	3.1	13.2	35.8	13.4	14.3	13.5	13.5	–
B. Power against changing AR coefficients: $y_t = \mu_t + \alpha_t y_{t-1} + \varepsilon_t$, ε_t NIID(0, 1), $\alpha_t = \alpha_1$, $t \le \frac{1}{2}T$, $\alpha_t = \alpha_2$, $t > \frac{1}{2}T$												
(α_1, α_2)												
(0.8, 1.0)	12.6	12.4	14.1	14.5	6.2	9.7	19.1	9.6	11.9	12.8	12.8	–
(0.6, 1.0)	12.5	14.1	21.7	24.7	6.7	10.1	19.8	19.6	14.2	15.5	15.3	–
(0.4, 1.0)	10.2	12.4	28.7	37.6	7.0	13.2	22.1	35.3	17.9	16.5	16.4	–
(1.0, 0.8)	18.5	11.9	9.9	14.7	7.8	11.6	17.6	11.3	12.1	17.7	17.8	–
(1.0, 0.6)	22.1	13.6	11.4	13.4	12.9	14.5	16.6	19.7	16.6	24.5	24.9	–
(1.0, 0.4)	18.9	12.9	9.6	10.3	13.8	17.1	16.6	36.4	18.6	25.5	25.8	–
C. Power against trend shift: $y_t = \mu_1\tau_{1t}([T\delta^*]) + \alpha y_{t-1} + \varepsilon_t$, ε_t NIID(0, 1), $\tau_{1t}([T\delta^*]) = (t - [T\delta^*])1(t > [T\delta^*])$												
$(\alpha, \delta^*, \mu_1)$												
(.9, .4, .2)	0.0	0.0	6.2	94.1	0.0	11.8	71.8	55.5	90.1	74.5	74.3	79.5
(.9, .5, .2)	0.0	0.0	7.8	98.5	0.0	11.5	68.8	60.2	92.2	66.8	67.5	79.8
(.9, .6, .2)	0.0	0.0	9.0	99.4	0.0	11.3	66.5	61.8	92.8	54.1	55.9	84.0
(.9, .4, .4)	0.0	0.0	5.9	96.3	0.1	20.0	90.8	93.3	100.0	99.6	99.6	98.4
(.9, .5, .4)	0.0	0.0	6.7	98.8	0.0	19.2	89.5	95.2	100.0	99.1	99.2	98.8
(.9, .6, .4)	0.0	0.0	8.9	99.8	0.0	17.3	87.8	95.1	100.0	93.8	94.8	98.8
(.8, .4, .2)	0.0	0.0	8.6	80.8	0.0	12.6	47.3	42.0	87.1	73.8	73.3	87.3
(.8, .5, .2)	0.0	0.0	9.9	91.4	0.0	12.1	48.2	48.5	91.2	70.8	70.5	90.0
(.8, .6, .2)	0.0	0.0	12.4	97.3	0.1	11.8	44.8	52.8	92.1	63.5	63.8	91.0

NOTE: Percent rejections at 10% critical values; T = 100. The reported values are the percent rejections by the various statistics based on the $T = 100$ 10% critical values from Tables 1 and 2. The recursive, rolling, and sequential statistics were computed as described in the text, with $p = 4$ and (respectively) $\delta_0 = .25, \frac{1}{3}$, and .15, except for Q_{LR}, for which $\delta_0 = .25$. The final column reports the fraction of $\tilde{F}_T^{max}$'s which attained their maximum at a value $\tilde{k}$ within $.05T$ of the true value of the break date k. In panel B, for designs in which the root changes from unity to stationary, the mean of the stationary process is adjusted to avoid a spurious sharp jump to zero at the break date: If $\alpha_1 = 1$ and $|\alpha_2| < 1$, then $y_t = \mu_t + \alpha_t y_{t-1} + \varepsilon_t$, where $\mu_t = 0$ for $t \le \frac{1}{2}T$ and $\mu_t = y_{(1/2)T}(1 - \alpha_2)$, $t > y\frac{1}{2}T$, while if $|\alpha_1| < 1$ and $\alpha_2 = 1$, $\mu_t = 0$ for all t. All pseudodata were generated including 50 startup values (initial observations not used in computing the statistics). Based on 2,000 replications.

tistics, whereas all 50 stationary observations are used for $\hat{t}_{\mathrm{DF}}^{\min}$. Overall, Q_{LR} performs well against both sets of alternatives. With $T = 50$ and $\alpha = .6$, the conventional Dickey–Fuller detrended t test with $p = 4$ has power of only 33%; relative to this benchmark, the power of the rolling and recursive statistics here seems reasonable.

The nominal power of these statistics against the trend-shift alternative (8) is examined in Panel C of Table 3. The trend shift is large, 20% or 40% of the standard deviation of the innovation. The full-sample Dickey–Fuller test fails to reject the unit-root null against this alternative, as do the recursive and rolling maximal t_{DF}'s. This confirms Perron's (1989, 1990a) results and interpretation: The permanent shift in the deterministic trend is mistaken for a persistent innovation to a stochastic trend. Overall, the sequential F statistic $\tilde{F}_T^{\max}$ (testing the trend coefficient) has high power against this alternative, particularly for breaks toward the end of the sample; Q_{LR} has lower power, not surprisingly because it tests all of the coefficients for breaks. The $\tilde{t}_{\mathrm{DF}}^{\min *}$ and $\tilde{t}_{\mathrm{DF}}(\tilde{\delta})$ statistics perform almost identically. Two interesting features of these results, not pursued here, are that, given δ^* and μ_1, a drop in α is occasionally associated with little or no increase in power for $\tilde{t}_{\mathrm{DF}}^{\min *}$ and that, given α and μ_1, the sequential t_{DF} tests have better power the earlier the break occurs. Finally, the break point is estimated rather accurately: A large fraction of the estimated break dates $\tilde{k}$ fall within $\pm .05T$ of the true break point.

The nominal power of the mean-shift statistics (not tabulated here) against the mean-shift alternative, $y_t = \mu_1 1(t > [T\delta^*]) + \alpha y_{t-1} + \varepsilon_t$, is good in cases of empirical interest. For example, for $\alpha = .9$ and a shift of two standard deviations in the middle of the sample ($\mu_1 = 2$, $\delta^* = .5$), the rejection rates for $\tilde{F}_T^{\max}$, $\tilde{t}_{\mathrm{DF}}(\tilde{\delta})$, and $\tilde{t}_{\mathrm{DF}}^{\min *}$ are 74.8%, 36.6%, and 42.0% at the 10% level; with $\mu_1 = 10$, the rejection rate is 100% for all three statistics. For the restricted $|\mathrm{ext}_\delta \tilde{t}_{\tau 1}(\delta)|$ statistic, power was computed against the alternative model, $\Delta y_t = \mu_1 1(t > [T\delta^*]) + \beta \Delta y_{t-1} + \varepsilon_t$, with $\beta = .4$ and $\delta^* = .5$; the $p = 4$ critical values were used here because of the moderate size distortions (which depend on p) discussed previously. For $\mu_1 = .5$, the rejection rate is 55.9%; for $\mu_1 = 1$, it is 96.7%.

Although these statistics make the break date data dependent, they still require a choice of p, which is rarely known in practice. The size and power calculations therefore were repeated for $p = 8$. The main effects were a deterioration of size for the recursive $\hat{t}_{\mathrm{DF}}^{\max}$ and rolling $\bar{t}_{\mathrm{DF}}^{\max}$ and a reduction in the power of some tests, notably (and unsurprisingly) Q_{LR} against both break alternatives. (These and other results not tabulated here are available from the authors on request.)

These results suggest several conclusions. The recursive $\hat{t}_{\mathrm{DF}}^{\mathrm{diff}}$ and rolling $\bar{t}_{\mathrm{DF}}^{\mathrm{diff}}$ statistics have sizes that are sensitive to the nuisance parameters in moderate samples, making them unattractive for applications. Although the extremal rolling statistics have the advantage of being able to detect multiple breaks, this is associated with reduced power against these single-break alternatives. More generally, once uniform critical values are used, the extremal recursive and rolling statistics have fairly low power. Hence it is particularly important not to interpret nonrejection by these statistics as acceptance of the null. The results also suggest, however, that the Q_{LR} statistic could be a powerful and reliable diagnostic tool. In addition, the sequential trend-break statistics have high power against the alternative they are designed to detect.

4. EMPIRICAL RESULTS

The previous results are used here to examine whether shifts or breaks in trends provide a suitable model for the apparent persistence in seasonally adjusted output in seven OECD countries—Canada, France, West Germany, Italy, Japan, the United Kingdom, and the United States. All statistics were computed using logarithms of these series. Data sources are discussed in Appendix B. Real output, in logarithms, is graphed in part A of Figure 1 for, to save space, only three of the countries, France, Japan, and the United States.

4.1 Main Results

We first computed full-sample statistics for the seven countries, modeling each series as an AR(2) and an AR(4) in first (and second) differences. This produced four Dickey–Fuller t statistics per country, two testing the null hypothesis of one unit root and two testing for two unit roots. A constant and a time trend were included in the regressions with first differences, and a constant was included in the regressions with second differences. For each country, the hypothesis of two unit roots was rejected but the single unit-root hypothesis was not. We therefore adopted the single unit-root model (with nonzero drift) as the relevant null hypothesis for each of the seven countries and proceeded with the computation of other statistics using Models I and II (with no additional regressors $\{x_{t-1}\}$), where the order of the lag polynomial $\beta(L)$ is 4 and the null hypothesis is the existence of a single unit root.

The results are summarized in Tables 4 and 5. As in the previous section, for the recursive statistics and Q_{LR}, $\delta_0 = .25$; for the rolling statistics, $\delta_0 = \frac{1}{3}$; and for the sequential statistics except Q_{LR}, $\delta_0 = .15$. All p values are based on the $T = 100$ distribution summarized in Tables 1 and 2, except for the restricted statistic $|\mathrm{ext}_\delta \tilde{t}_{\tau 1}(\delta)|$ as discussed in Section 3. Recursive and rolling $t_{\mathrm{DF}}^{\mathrm{diff}}$ statistics are not reported because of their unreliable size in the Monte Carlo analysis. In no case is the standard nonrecursive Dickey–Fuller statistic (the first column of statistics in Table 4) significant at the 25% level. Moreover, none of the recursive or rolling tests reject the unit-root null at the 10% level, although the relatively low power of these tests found in

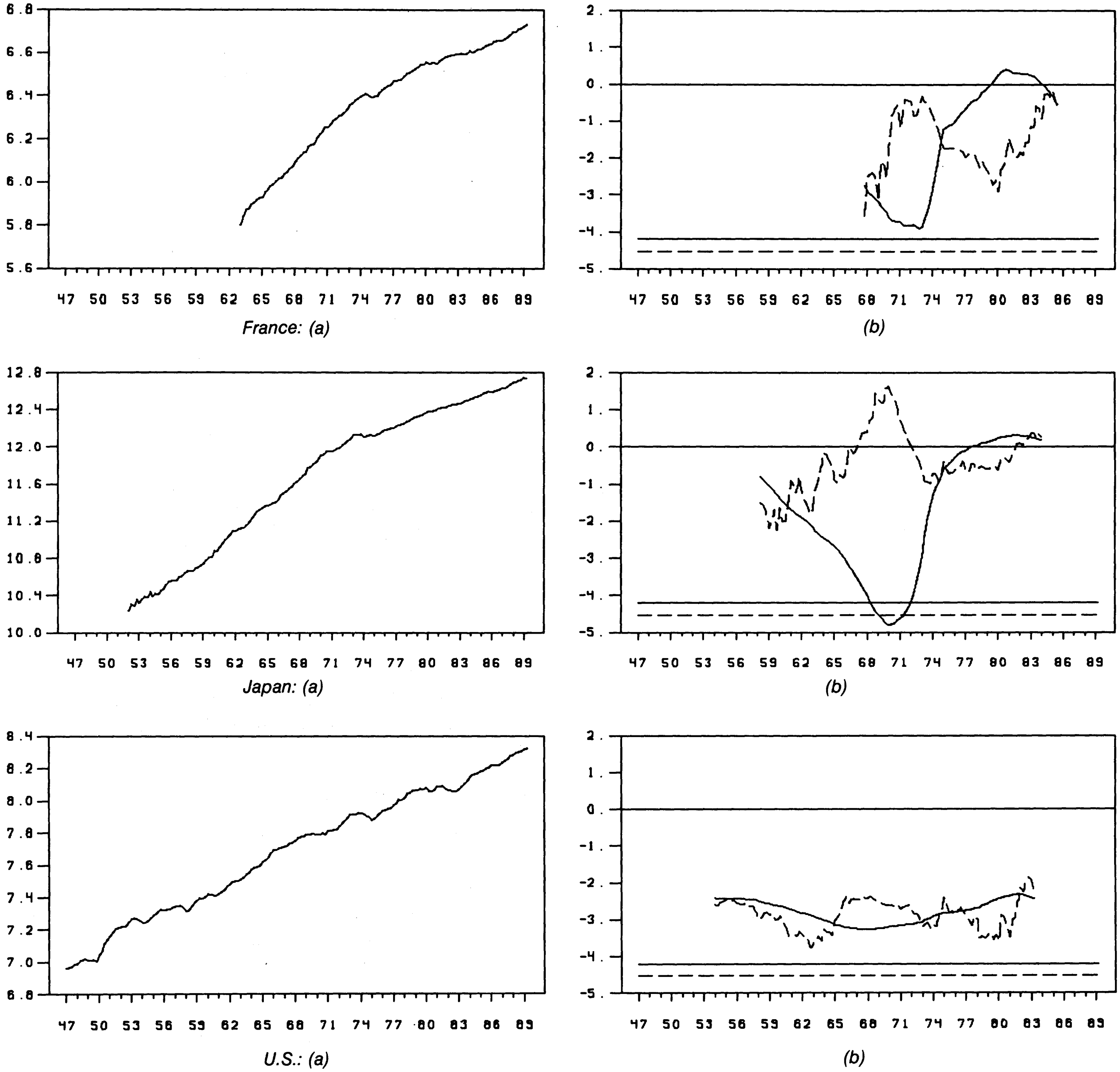

Figure 1. Real Output (A, in logarithms) and Sequential $\tilde{t}_{DF}$ statistics (B) for France, Japan, and the United States: In Panel (B), the Solid Line = Sequential Trend-Shift $\tilde{t}_{DF}$ Statistic; the Dashed Line = Sequential Mean-Shift $\tilde{t}_{DF}$ Statistic; and the Straight Lines = 10% Critical Value for Trend-Shift (solid line) and Mean-Shift (dashed line) statistics.

Section 3 suggests that these nonrejections might be uninformative.

The sequential statistics, reported in Table 5, provide strong evidence for changing coefficients of some form. The Q_{LR} statistic rejects the hypothesis that all of the coefficients are constant at the 5% level for all countries but France, Germany, and the United States. To examine the possibility that this rejection arises from a stationary model with a breaking trend (as opposed to, for example, an integrated model with a change in the drift), the sequential $\tilde{t}_{DF}$'s were computed. Part B of Figure 1 plots the sequential trend-break $\tilde{t}_{DF}$'s (solid lines) and mean-shift $\tilde{t}_{DF}$'s (dashed lines) for France, Japan, and the United States. The unit-root/no-break null can be rejected at the 10% level against the stationary/trend-shift alternative for only one of the seven countries, Japan, for which the p value is 3% [in Fig. 1(B), the solid line for Japan drops below its 10% critical value from 1968:3 to 1971:4]. There is some evidence against the unit-root null in favor of the trend-shift alternative for Canada, with a p value for $\tilde{t}_{DF}^{\min *}$ of 12%. Similar conclusions obtain using $\tilde{t}_{DF}(\hat{\delta})$: In almost all cases $\tilde{t}_{DF}^{\min *} = \tilde{t}_{DF}(\hat{\delta})$, indicating that the same break dates were estimated by the maximal $\tilde{F}_T$ and minimal Dickey–Fuller statistics. The minimal mean-break sequential $\tilde{t}_{DF}$'s are given in the next columns of Table 5. The only 10% rejection is for Canada, with the break in 1981:3.

Table 4. *Empirical Results: International Evidence on Unit Roots Based on Recursive and Rolling Statistics*

			Recursive		Rolling	
Country	Sample	t_{DF}	$\hat{t}_{DF}^{max}$	$\hat{t}_{DF}^{min}$	$\bar{t}_{DF}^{max}$	$\bar{t}_{DF}^{min}$
Canada	48:1–89:2	−1.96	−1.17	−3.63	1.15	−3.45
		(.62)	(.32)	(.21)	(.99)	(.74)
France	63:1–89:2	−1.74	1.39	−2.31	1.73	−3.77
		(.73)	(.99)	(.91)	(.99)	(.53)
Germany	50:1–89:2	−1.96	.20	−1.96	−.73	−3.64
		(.62)	(.93)	(.99)	(.40)	(.61)
Italy	52:1–82:4	.26	.26	−2.78	−.60	−4.36
		(.99)	(.94)	(.67)	(.49)	(.21)
Japan	52:1–89:2	−.09	.42	−2.44	.66	−4.19
		(.99)	(.96)	(.86)	(.97)	(.28)
U.K.	60:1–89:2	−1.88	−.48	−3.09	−.75	−4.54
		(.66)	(.70)	(.48)	(.39)	(.15)
U.S.	47:1–89:2	−2.60	−.52	−2.60	.37	−4.30
		(.29)	(.68)	(.78)	(.93)	(.23)

NOTE: Entries are test statistics with p values in parentheses. For all statistics, $p = 4$. The sample period refers to the full sample of data used, including initial values for lags in the autoregressions. For example, for the United States, the full-sample Dickey–Fuller regression was run over 1948:2–1989:2, with 1947:1–1948:1 providing initial values of the regressors $y_{t-1}, \Delta y_{t-1}, \ldots, \Delta y_{t-4}$. The data and statistics are described in the text. The p values are for tests of the (constant coefficient) unit-root null hypothesis. These p values were calculated from the null distribution for $T = 100$ and $p = 0$, computed by Monte Carlo and summarized in Table 1. A p value of .00 denotes a p value of <.005. The p values for variants of t_{DF} are one-sided against the hypothesis $\alpha < 1$; all other p values are for two-sided tests. The first statistic, t_{DF}, is the full-sample Dickey–Fuller statistic based on (1), including a constant and a time trend. The recursive and rolling statistics were computed as described in the text, with $\delta_0 = .25$ and $\frac{1}{3}$, respectively, using (1) (with a constant and a time trend).

The final column in Table 5 examines the possibility that output for these countries is $I(1)$ but that there has been a single shift in the mean growth rate. This corresponds to Model B of Section 2 [(7) and (9)], with the unit root imposed ($\alpha = 1$) and a zero time trend in first differences ($\mu_2 = 0$). This model is examined by computing $\tilde{t}_{\tau_1}(\cdot)$, the sequential t statistic on $\tau_{1t}(k) = 1(t > k)$ in a regression of Δy_t onto $(1, \tau_{1t}(k), \Delta y_{t-1}, \ldots, \Delta y_{t-p})$. Although the results are presented in Tables 4 and 5 for all countries for completeness, for Canada and Japan our previous rejection of the unit-root null suggests that in these two cases the restricted $\tilde{t}_{\tau_1}(\cdot)$ tests are not meaningful. In three additional cases—France, Germany, and Italy—the restriction of a constant drift is rejected in favor of the hypothesis of a shift in the drift.

4.2 Sensitivity Analysis

This subsection summarizes five additional analyses that were performed to investigate the sensitivity of the results to potential measurement errors or anomalies in the data. First, in 1968:2 France experienced a major strike, over which we have interpolated for the reported results. When the original data are used, the unit-root

Table 5. *Empirical Results: International Evidence on Unit Roots Based on Sequential Statistics*

		Case A: trend shift				Case B: mean shift				Case B: restricted	
Country	$Q_{LR}(4)$	$\tilde{k}$	$\tilde{F}_T^{max}$	$\tilde{t}_{DF}(\tilde{\delta})$	$\tilde{t}_{DF}^{min*}$	$\tilde{k}$	$\tilde{F}_T^{max}$	$\tilde{t}_{DF}(\tilde{\delta})$	$\tilde{t}_{DF}^{min*}$	$\tilde{k}$	$ext_\delta \tilde{t}_{\tau 1}(\delta)$
Canada	39.09	76:3	12.97	−4.14	−4.14	81:3	22.57	−5.14	−5.14	76:3	−2.11
	(.02)		(.12)	(.12)	(.12)		(.02)	(.03)	(.03)		(.38)
France	22.87	73:1	12.44	−3.89	−3.89	68:1	9.73	−3.55	−3.55	74:2	−4.45
	(.48)		(.14)	(.19)	(.20)		(.50)	(.52)	(.55)		(.00)
Germany	25.97	60:2	3.69	−2.64	−2.65	80:2	4.15	−2.84	−2.84	60:4	−3.28
	(.31)		(.69)	(.81)	(.83)		(.96)	(.82)	(.90)		(.04)
Italy	37.46	71:1	15.35	−3.67	−3.67	74:3	6.53	−1.60	−1.60	74:2	−3.47
	(.03)		(.07)	(.28)	(.28)		(.80)	(.98)	(.99)		(.03)
Japan	77.61	70:1	25.03	−4.78	−4.81	73:2	16.72	−.69	−2.23	73:2	−4.85
	(.00)		(.00)	(.03)	(.03)		(.09)	(.99)	(.98)		(.00)
U.K.	51.34	82:1	2.54	−1.42	−2.40	79:3	13.47	−3.98	−3.98	82:4	1.55
	(.00)		(.82)	(.99)	(.92)		(.22)	(.30)	(.31)		(.72)
U.S.	17.75	68:3	3.99	−3.27	−3.27	63:1	7.74	−3.76	−3.76	68:3	−1.39
	(.81)		(.66)	(.48)	(.50)		(.69)	(.41)	(.43)		(.82)

NOTE: Entries are test statistics with p values in parentheses. For all statistics, $p = 4$. The sample period is the same as in Table 4. The p values are for tests of the (constant coefficient) unit-root null hypothesis. These p values were calculated from the null distribution for $T = 100$ and $p = 0$, computed by Monte Carlo and summarized in Table 2, except for $|ext_\delta \tilde{t}_{\tau 1}(\delta)|$, for which p values were obtained from the $T = 100$, $p = 4$ null distribution because of Monte Carlo evidence of moderate size distortions as p increases, as discussed in Section 3. A p value of .00 denotes a p value of <.005. The p values for variants of t_{DF} are one-sided against the hypothesis $\alpha < 1$; all other p values are for two-sided tests. The sequential statistics were computed as described in the text, with $\delta_0 = .15$ ($\delta_0 = .25$ for Q_{LR}).

test results do not change, except for the sequential mean-shift statistics; then the mean-shift $\tilde{t}_{DF}^{\min *}$ rejects the constant-drift unit-root null at the 5% level, with the break sharply identified as 68:3. We view this as an artifact: Reversion to "trend" after the strike is best thought of rather prosaically as people returning to work, not as reflecting trend-stationary behavior in the long-run factors driving French economic growth, such as technical progress and labor productivity.

Second, the results for Germany are for 1950:1–1989:2; for Japan, they are for 1952:1–1989:2. Because these data start near the end of World War II, the earliest observations might have unusually large measurement error. But the conclusions about the significance of unit-root tests do not change upon repeating the analysis over 1955:1–1989:2, although the evidence against the unit-root/constant-drift null in favor of the unit-root/mean-shift alternative (based on $|\text{ext}_\delta \tilde{t}_{\tau 1}(\delta)|$) is less strong for Germany.

Third, to ensure that the results for Canada, Germany, and Japan were not an artifact of combining several data sources, the recursive and sequential statistics were recomputed using the largest subsample of data that came from a single data source (see App. B). For Japan, the trend-shift $\tilde{t}_{DF}^{\min *}$ continues to reject at the 5% level. For Canada, the mean-shift $\tilde{t}_{DF}^{\min *}$ and $\tilde{t}_{DF}(\tilde{\delta})$ tests no longer reject at the 10% level. For Germany, the trend-shift $\tilde{t}_{DF}^{\min *}$ rejects at the 10% (but not 5%) level, with the breakpoint identified at 73:1. These German data, however, exclude the entire 1950s, leading us to put greater weight on the results for the longer spliced series.

Fourth, the sensitivity of the results with respect to p was examined by recomputing the sequential statistics for p chosen (a) by the Akaike information criterion (AIC), (b) by the Schwarz criterion, and (c) by setting $p = 8$. In each case, the results were unchanged, with three exceptions—the Canadian mean-shift $\tilde{t}_{DF}^{\min *}$ is insignificant at the 10% level in two of the three cases; for the AIC and Schwarz choice of $p = 0$ for France, the trend-shift $\tilde{t}_{DF}^{\min *}$ just rejects at the 10% level, although not for $p = 4$ or $p = 8$; and for the United Kingdom for $p = 0$ (the Schwarz choice) and $p = 8$, but not for the AIC choice of $p = 1$, the mean-shift $\tilde{t}_{DF}^{\min *}$ rejects at the 10% but not 5% level. The results for $|\text{ext}_\delta \tilde{t}_{\tau 1}(\delta)|$ for all countries, and all the results for Germany, Italy, Japan, and the United States are robust to these changes in p. (For a different approach to the choice of p in evaluating these statistics, see Perron [1990b].)

Fifth, one could argue that the values of δ_0 used here are too small, for two reasons: Small δ_0 might introduce substantial deviations of the finite sample distributions from their normal-error approximations in Tables 1 and 2, and the alternative of primary empirical interest (a productivity slowdown in the early 1970s) occurred in the middle of the sample so that larger δ_0 could have increased power. The results were therefore recomputed for $\delta_0 = \frac{1}{3}$, .4, and .25 for the recursive and Q_{LR}, rolling, and sequential (except Q_{LR}) statistics, respectively. (This entailed computing new sets of critical values.) The only qualitative change is for Canada, for which the 1981 mean shift is not detected because it falls outside the increased trimming range.

4.3 Summary

These results suggest rather different characterizations of the long-run properties of output across these countries. In two countries, Canada and Japan, these statistics provide evidence against the unit-root/no-break null hypothesis. For Canada, the unit-root null is rejected against the stationary/mean-shift alternative, with the breakpoint in 1981:3. This portrays the recession of the early 1980s as a permanent downward shift in the trend growth path; after the recovery, output again is stationary with its original growth rate. For Japan, the null is rejected against the stationary/trend-shift alternative, with the break in 1970:1. This shift is apparent in Figure 1: From 1952:1 to 1969:4, on average Japanese output grew at 9.2% per year, but since 1970 it has grown at 4.4%.

For the United Kingdom, the evidence against the unit-root null is either weak or nonexistent, with the results somewhat sensitive to the choice of p. The restricted t statistics in Table 5 do not indicate a statistically significant slowdown; indeed, the growth rate increased in the 1980s, although not significantly using these procedures.

The results for the remaining countries provide no evidence against the unit-root hypothesis. Based on the results in Table 5 for the restricted mean-shift model; however, France, Germany, and Italy seem to have suffered a highly persistent reduction in the rate of growth of output. For Italy and France, this slowdown appears around 1974, the time of the first oil shock. For Germany, the sequential t statistic is less precise in identifying a specific break point, although the statistic is significantly negative just before 1974. For these countries then, output is well characterized as being integrated but with a lower average growth rate over the period of the productivity slowdown.

The results for the United States indicate no rejections of the unit-root/no-break null against any of the various alternatives. This parallels Christiano's (1988) failure to reject this null using bootstrapped critical values. The results accord with Banerjee, Dolado and Galbraith's (1990) failure to reject the unit-root/no-break null against the stationary/trend-break alternative for the United States (using the uniform critical values tabulated in Sec. 3) for longer annual data series that include the Depression. They are also consistent with Zivot and Andrews's (1992) failure to reject the unit-root null against a trend-break alternative for U.S. real postwar quarterly GNP when they use uniform critical values. Although Perron (1989) found evidence of stationarity around a trend that shifted in 1973, this conclusion was based on the assumption that the break date is known a priori; when the break date is treated as unknown, our evidence is much weaker.

4.4 Comparison With Previous Literature and Discussion

Several recent works extend Cochrane's (1988) study of persistence in U.S. output to international data. Each study differs in its sample period and, to various degrees, in the statistical measure used. Although point estimates of persistence are not comparable across studies, relative rankings are.

Campbell and Mankiw (1989) examined the same seven countries considered here over 1957–1986 and measured persistence by the size of a (bias-adjusted) variance ratio for long (5–10 year) differences. They concluded that persistence in the United Kingdom was less than in the United States but greater in each of the other countries. Cogley (1990) computed modified variance ratios over 1870–1985 for nine countries, including Canada, France, Italy, the United Kingdom, and the United States. Although his data set is much longer, his conclusions are similar to Campbell and Mankiw's: The United States exhibited the least persistence, followed by Canada; the largest variance ratios were for France and Italy. Kormendi and Meguire (1990) also used variance ratios to analyze long annual data on 12 countries and postwar data for 32 countries, including Canada, France, Germany, Italy, the United Kingdom, and the United States. Of these six, using bias-unadjusted measures they too found U.S. output to exhibit the least persistence (the smallest variance ratios), with French, German, and Italian output exhibiting the most persistence.

Clark (1989) used a different technique—a stochastic trend-cycle decomposition of the form studied by Harvey (1985) and Watson (1986)—to study the relative importance of "cyclical" components for the seven countries we consider, over approximately 1960–1986. A notable feature of his results is that an $I(1)$ trend fit well for five of the countries; for France and especially Japan, however, the fit of the model was substantially improved when an $I(2)$ trend (a stochastic trend with a random-walk drift) was introduced. He interpreted this as providing a flexible way to account for the slower growth in these two countries in the latter half of the sample.

The striking feature of the variance-ratio results is that in each study the variance ratios are highest in the countries for which we identify deterministic breaks [with either $I(1)$ or $I(0)$ stochastic components]—Japan, Germany, France, and Italy—and lowest for the United States, for which we find no evidence against the unit-root/no-break hypothesis. Clark also found evidence of extreme persistence, in the form of an integrated drift, for France and Japan. These results have a consistent explanation. If the deterministic-break specification is valid, then this break is by definition highly persistent. If the break is not taken into account explicitly, then it will be misidentified by variance-ratio statistics as a large but otherwise typical shock to output. In a model in which the drift is forced to be either constant or integrated (Clark's 1989 model), a sufficiently large permanent change will be modeled as an integrated drift. If the deterministic-break view is correct, variance-ratio statistics would give quite misleading views of persistence.

Interpreted more broadly, these results suggest that not all shocks to output are the same: The shocks associated with the oil crisis of 1974–1975 were considerably more persistent than other shocks before and after, so much so that these procedures classify them as deterministic breaks rather than a large negative realization. This interpretation treats the single-break model as a simple device for separating massive, economy-changing shocks—the Depression, World War II, the productivity slowdown in the 1970s—from the other shocks to output that, although persistent, exhibit less permanence. To us, interpreting these broken trends as deterministic is unsatisfying, for this conditions on elements of aggregate activity, such as productivity growth rates or changes in fiscal or monetary management rules, that are unpredictable. We instead prefer to interpret these rejections as metaphors for these countries having long-run trends that are smooth with occasional large shocks (see Perron 1989, 1990a). This is analogous to Blanchard and Watson's (1986) "large shock/small shock" hypothesis, which they developed using U.S. data—except that this better describes Japan and is more accurately termed the "persistent shock/less persistent shock" hypothesis. It is also compatible with Hamilton's (1989) model of random regime switches, although the regimes here last much longer than the business-cycle switches Hamilton identified for U.S. GNP.

5. SUMMARY AND CONCLUSIONS

The results in Sections 1 and 2 provide a framework for obtaining the asymptotic distributions of various recursive, rolling, and sequential statistics. These results resolve some open questions, such as the distribution of the process of the recursive least squares estimators in an autoregression with a unit root. There are, however, several theoretical questions that these results only begin to address. In particular, each of the test statistics discussed here has the flavor of general diagnostic tests against possible changes in coefficients, either autoregressive coefficients or coefficients describing the deterministic components of the process. This suggests the value of obtaining formal results on the power of these tests against various structural break alternatives. In addition, the results here are for reduced-form, single-equation systems; a natural extension is to a system of simultaneous equations, perhaps with different break dates in different equations.

For Germany, Italy, and France, these results provide a new characterization of the productivity slowdown as being a reduction in the drift of the $I(1)$ output process over this period; there is no evidence in favor of the stationary/trend-break hypothesis for 3 of these countries. This slowdown occurred at approximately the same time for each of these countries, 1974. For these countries at least, these results suggest two further avenues of investigation—testing for breaks in the deterministic components without pretesting for unit roots and using multivariate techniques to examine whether the breaks occurred simultaneously, at least for these European countries. Recently, work by Perron (1991) has examined the former issue, and work by Bai, Lumsdaine, and Stock (1991) has pursued the latter. The analysis also provides new insights into growth trends in Japan and Canada. For Japan, the unit-root/no-break null is rejected against the alternative that output is stationary around a trend that slowed significantly around 1970. For Canada, the unit-root null is rejected against the alternative that a deterministic trend growth path shifted downward after the 1979–1982 recession, although this result is sensitive to the number of autoregressive lags used. Finally, the empirical analysis provides little evidence against the unit-root null for the United Kingdom and none for the United States.

ACKNOWLEDGMENTS

We thank T. W. Anderson, Donald Andrews, Bruce Hansen, Lars Peter Hansen, G. S. Maddala, N. Gregory Mankiw, Alain Monfort, Mark Watson, three anonymous referees, and an associate editor for helpful comments. This research was initiated while Banerjee was Visiting Scholar, Kennedy School of Government, Harvard University. Since writing the first draft of this article, we learned of six independent contemporaneous treatments of various aspects of the recursive coefficients and trend-break problems in econometric time series models: Andrews (1989), Bates (1990), Chu (1989), Hansen (1990), Perron (1990b), and Zivot and Andrews (1992). Discussion of these papers has been incorporated into the relevant sections of this revision. Stock thanks the Sloan Foundation and the National Science Foundation (Grant SES-89-10601) for financial support.

APPENDIX A: PROOFS OF THEOREMS

Proof of Theorem 1

To simplify notation, it is assumed that $\bar{\mu}_0 = 0$. This is done without loss of generality, since $Z_t^1 = (\Delta y_t - \bar{\mu}_0, \ldots, \Delta y_{t-p+1} - \bar{\mu}_0)'$ and $Z_t^3 = (y_t - \bar{\mu}_0 t)$. To simplify the algebra, set $Z_s = 0$, $s \le 0$ (this can be relaxed as discussed, for example, by Phillips [1987]). Throughout, the notation $\phi_T(\delta) \Rightarrow \phi(\delta)$ and $\phi_T(\cdot) \Rightarrow \phi(\cdot)$ are used interchangeably.

(a) First consider ϕ_T. Let $C(L) = (1 - \beta(L)L)^{-1}$ so that $\Delta y_t = C(L)\varepsilon_t$, let $b = C(1) = (1 - \beta(1))^{-1}$, and let $s = [T\delta]$. The uniform convergence results $\phi_{2T}(\delta) = T^{-1/2}\Sigma_{t=1}^{s}\varepsilon_t \Rightarrow \sigma W(\delta)$ and $\phi_{4T}(\delta) = T^{-3/2}\Sigma_{t=1}^{s} t\varepsilon_t \Rightarrow \sigma[\delta W(\delta) - \int_0^\delta W(\lambda)d\lambda]$ are immediate consequences of Assumption A and the functional central limit theorem (FCLT) (Herrndorf 1984; also see Hall and Heyde 1980). Consider ϕ_{3T} and write $y_t = C(1)\xi_t + U_t$, where $\xi_t = \Sigma_{r=1}^{t}\varepsilon_r$ and $U_t = C^*(L)\varepsilon_t$, where $C^*(L) = (1 - L)^{-1}[C(L) - C(1)]$. Then

$$\phi_{3T}(\delta) = T^{-1}\sum_{t=1}^{s} y_{t-1}\varepsilon_t$$

$$= C(1)T^{-1}\sum_{t=1}^{s}\xi_{t-1}\varepsilon_t + T^{-1}\sum_{t=1}^{s}U_{t-1}\varepsilon_t.$$

Now $T^{-1}\Sigma_{t=1}^{s}\xi_{t-1}\varepsilon_t = \frac{1}{2}\{T^{-1}\xi_s^2 - T^{-1}\Sigma_{t=1}^{s}\varepsilon_t^2\}$. Because $\nu_t \equiv \varepsilon_t^2 - \sigma^2$ is a martingale difference sequence (MDS) with $\sup_t E|\nu_t|^{2+\gamma} \le \bar{\kappa} < \infty$ by Assumption A, $T^{-1}\Sigma_{t=1}^{s}\varepsilon_t^2 = (s/T)\sigma^2 + T^{-1}\Sigma_{t=1}^{s}\nu_t \Rightarrow \delta\sigma^2$. Because $T^{-1}\xi_s^2 \Rightarrow \sigma^2 W(\delta)^2$, $T^{-1}\Sigma_{t=1}^{s}\xi_{t-1}\varepsilon_t \Rightarrow \frac{1}{2}\sigma^2\{W(\delta)^2 - \delta\}$. Because $U_{t-1}\varepsilon_t$ is an MDS with $\sup_t E|U_{t-1}\varepsilon_t|^4 < \infty$ (this from the moment assumptions on ε_t and from the 1-summability of $C(L)$ [e.g., Stock 1987] and thus the absolute summability of $C^*(L)$), $T^{-1/2}\Sigma_{t=1}^{s}U_{t-1}\varepsilon_t$ obeys an FCLT. Thus $T^{-1}\Sigma_{t=1}^{s}U_{t-1}\varepsilon_t \Rightarrow 0$, so $\phi_{3T}(\delta) \Rightarrow \frac{1}{2}b\sigma^2\{W(\delta)^2 - \delta\}$. Finally, because $Z_{t-1}^1\varepsilon_t$ is an MDS with $\sup_t E(Z_{it-1}^1\varepsilon_t)^4 \le \kappa_4^2(\Sigma_{j=0}^{\infty}|c_j|)^4 < \infty$, $\phi_{1T}(\delta) = T^{-1/2}\Sigma_{t=1}^{s}Z_{t-1}^1\varepsilon_t \Rightarrow \sigma B(\delta)$, where $B(\delta)$ has covariance matrix $EZ_t^1Z_t^{1\prime} = \Omega_p$. It follows from Chan and Wei (1988, theorem 2.2) that B and W are independent.

Next consider $V_T(\delta)$. The uniform convergence of each element of V_T, with the exception of V_{11T} and V_{13T}, either obtains by direct calculation or is a consequence of $T^{-1/2}y_s \Rightarrow b\sigma W(\delta)$ and the continuous mapping theorem. For example, $V_{33T}(\delta) = T^{-2}\Sigma_{t=1}^{[T\delta]}y_t^2 \Rightarrow b^2\sigma^2\int_0^\delta W(r)^2dr$. Note, however, that this formally holds only for δ fixed; to show convergence of the process $V_{33T}(\cdot) \Rightarrow V_{33}(\cdot)$, it further must be shown that $g(\delta; f) = \int_0^\delta f(r)^2dr$ is a continuous mapping from $D[0, 1]$ to $D[0, 1]$. This argument is made by Zivot and Andrews (1992) and is not repeated here.

To demonstrate the convergence of $V_{11T}(\delta)$ it is sufficient to consider its (1, 1) element; the argument for the other elements is similar. Now $(V_{11T})_{11} = T^{-1}\Sigma_{t=1}^{s}(\Delta y_t)^2$ (recall $\bar{\mu}_0 = 0$). Define $\gamma_0 \equiv E(\Delta y_t)^2$, $X_t \equiv (\Delta y_t)^2 - \gamma_0$, and $S_i \equiv \Sigma_{t=1}^{i}X_t$. With these definitions, $T^{-1}\Sigma_{t=1}^{i}(\Delta y_t)^2 = (i/T)\gamma_0 + T^{-1}S_i$ $(i = 1, \ldots, T)$. Thus the desired result follows if $\Pr[\max_{i\le T}|T^{-1}S_i| > \delta] \to 0$ for all $\delta > 0$. This will be shown by, first, showing that X_t is a mixingale and, second, applying the mixingale extension of Doob's inequality.

From Hall and Heyde (1980, p. 19), X_t is a mixingale if there exist sequences of nonnegative constants d_t and L_m such that $L_m \to 0$ as $m \to \infty$ and (*i*) $\|E(X_t|F_{t-m})\|_2 \le L_m d_t$ and (*ii*) $\|X_t - E(X_t|F_{t+m})\|_2 \le L_{m+1}d_t$ for all $t \ge 1$ and $m \ge 0$, where $\|X\|_2 \equiv (EX^2)^{1/2}$. Condition (*ii*) is automatically satisfied by $\{X_t\}$ because $\{F_t\}$ is an increasing sequence of σ-fields and X_t is an adapted stochastic process so that $E(X_t|F_{t+m}) = X_t$; thus $\|X_t - E(X_t|F_{t+m})\|_2 = 0$.

Next turn to condition (i). Now

$$\{\|E(X_t|F_{t-m})\|_2\}^2$$
$$= E\{(E(X_t|F_{t-m}))^2\}$$
$$= E\{(E[(C(L)\varepsilon_t)^2 - \gamma_0|\varepsilon_{t-m},\varepsilon_{t-m-1}, \ldots])^2\}$$
$$= E\left\{\left[\sum_{j=0}^{m-1} C_j^2\sigma^2 + \left(\sum_{j=m}^{\infty} C_j\varepsilon_{t-j}\right)^2 - \sum_{j=0}^{\infty} C_j^2\sigma^2\right]^2\right\}$$
$$= E\left\{\left[\left(\sum_{j=m}^{\infty} C_j\varepsilon_{t-j}\right)^2 - \sum_{j=m}^{\infty} C_j^2\sigma^2\right]^2\right\}$$
$$= E\left[\left(\sum_{j=m}^{\infty} C_j\varepsilon_{t-j}\right)^4\right] - \left(\sum_{j=m}^{\infty} C_j^2\sigma^2\right)^2$$
$$= \sum_{j=m}^{\infty}\sum_{k=m}^{\infty}\sum_{l=m}^{\infty}\sum_{r=m}^{\infty} C_jC_kC_lC_rE(\varepsilon_{t-j}\varepsilon_{t-k}\varepsilon_{t-l}\varepsilon_{t-r}) - \left(\sum_{j=m}^{\infty} C_j^2\sigma^2\right)^2$$
$$\leq \kappa_4\left(\sum_{j=m}^{\infty} |C_j|\right)^4,$$

where the final inequality obtains by Assumption A. Now $\Sigma_{j=m}^{\infty}|C_j| \leq K_1\Sigma_{j=m}^{\infty}\lambda^j = K_1\lambda^m/(1 - \lambda)$, where K_1 is a constant and λ is the absolute value of the largest root of $1 - L\beta(L)$, which satisfies $|\lambda| < 1$ by assumption. Thus $\|E(X_t|F_{t-m})\|_2 \leq d_tL_m$, where $d_t = \{\kappa_4K_1^4/(1 - \lambda)^4\}^{1/2}$ and $L_m = \lambda^{2m}$. Thus condition (i) is satisfied with $L_m \to 0$ as $m \to \infty$ and $(\Delta y_t)^2$ is a mixingale.

Next, apply Chebyschev's inequality and the mixingale extension of Doob's inequality (Hall and Heyde 1980, lemma 2.1) to show that $T^{-1}S_{[T\lambda]} \Rightarrow 0$. The condition of this lemma is that L_m be $0(m^{-1/2}(\log(m))^{-2})$, which is satisfied here. Thus

$$\Pr[\max_{i\leq T}|T^{-1}S_i| > \delta] \leq T^{-2}\delta^{-2}E[\max_{i\leq T}S_i^2]$$
$$\leq \delta^{-2}K_2T^{-2}\sum_{i=1}^{T} d_i^2$$
$$= \delta^{-2}K_2(\kappa_4K_1^4/(1 - \lambda)^4)/T,$$

(where K_2 is a constant), which tends to 0 for all $\delta > 0$, where the second inequality obtains by lemma 2.1 of Hall and Heyde (1980). Thus $T^{-1}S_{[T\lambda]} \Rightarrow 0$, so $T^{-1}\Sigma_{t=1}^{[T\delta]}(\Delta y_t)^2 \Rightarrow \delta\gamma_0$.

The final term is the $p \times 1$ vector V_{13T}, of which consider the ith element, $(V_{13T}(\delta))_i$. Recall that by assumption $Z_t = 0$, $t \leq 0$. For $s = [T\delta]$,

$$(V_{13T}(\delta))_i = T^{-3/2}\sum_{t=1}^{s} y_{t-1}\Delta y_{t-i}$$
$$= T^{-3/2}\sum_{t=1}^{s} \left(y_{t-i-1} + \sum_{r=1}^{i}\Delta y_{t-r}\right)\Delta y_{t-i}$$
$$= T^{-3/2}\sum_{t=1}^{s-i} y_{t-1}\Delta y_t + \sum_{r=1}^{i} T^{-3/2}\sum_{t=1}^{s-r}\Delta y_t\Delta y_{t-i+r}.$$

Thus $T^{1/2}(V_{13T}(\delta))_i = \nu_{iT}^1(\delta) + \nu_{iT}^2(\delta)$ $(i = 1, \ldots, p)$, where

$$\nu_{iT}^1(\delta) = \tfrac{1}{2}\{(T^{-1/2}y_{[T\delta]-i})^2 - T^{-1}\sum_{t=1}^{[T\delta]-i}(\Delta y_t)^2\}$$
$$= \tfrac{1}{2}\{(T^{-1/2}y_{[T\delta]-i})^2 - (V_{11T}(([T\delta] - i)/T))_{11}\}$$
$$\nu_{iT}^2(\delta) = \sum_{r=1}^{i} T^{-1}\sum_{t=1}^{[T\delta]-r}\Delta y_t\Delta y_{t-i+r}$$
$$= \sum_{j=1}^{i}(V_{11T}(([T\delta] - i + j)/T))_{1j}.$$

It follows from $V_{11T}(\delta) \Rightarrow V_{11}(\delta)$ (just shown) and (for fixed i) $T^{-1/2}y_{[T\delta]-i} \Rightarrow b\sigma W(\delta)$ that $\nu_{iT}^1(\delta) \Rightarrow \frac{1}{2}\{b^2\sigma^2W(\delta)^2 - (V_{11}(\delta))_{11}\}$ and $\nu_{iT}^2(\delta) \Rightarrow \Sigma_{j=1}^{i}(V_{11}(\delta))_{1j}$. Thus $(V_{13}(\cdot))_i \Rightarrow 0$ $(i = 1, \ldots, p)$, so $V_T(\cdot) \Rightarrow V(\cdot)$.

(b) Given the convergence results in (a) and the moment conditions in Assumption A, it follows that $\hat{\sigma}^2(\delta) \Rightarrow \sigma^2$. The asymptotic representation in Theorem 1 follows directly from the results in (a) and from theorem 2 of Sims, Stock, and Watson (1990).

(c) Part (c) follows directly from (a) and (b).

Proof of Theorem 2

(a) The proof is similar to the proof of Theorem 1. First consider Γ. The convergence of Γ_{11T} and Γ_{13T} is by Assumption B(b). The results involving exclusively deterministic terms (Γ_{ijT}, $i, j = 2, 4, 5$) obtain by direct calculation. The remaining limits, which involve y_t, obtain by noting that under H_0 ($\alpha = 1$, $\mu_1 = \mu_2 = 0$), $\Delta y_t - \bar{\mu}_0 = C(L)\omega_0'x_{t-1} + C(L)\varepsilon_t$, where $C(L) = (1 - L\beta(L))^{-1}$. Thus by Assumption B and the fact that $C(L)$ is 1-summable, $T^{-1/2}\Sigma_{t=1}^{[T\lambda]}(\Delta y_t - \bar{\mu}_0) \Rightarrow b\pi H(\lambda) + b\sigma W(\lambda) \equiv J(\lambda)$. The results for the remaining terms follow from this limit and Assumption B.

Next consider Ψ. The convergence of Ψ_{1T} is assumed in Assumption B(b). The terms Ψ_{2T} and Ψ_{4T} obtain by direct calculation. For example, in case A, $\Psi_{4T}(\delta) = T^{-3/2}\Sigma_{t=1}^{T}(t - [T\delta])1(t > [T\delta])\varepsilon_t = T^{-3/2}\Sigma_{t=[T\delta]}^{T}(t - [T\delta])\varepsilon_t$. The result $\Psi_{4T}(\cdot) \Rightarrow \Psi_4(\cdot)$ obtains from this final expression by applying Assumption B and the FCLT. The convergence $\Psi_{3T} = T^{-1}\Sigma_{t=1}^{T}(y_{t-1} - \bar{\mu}_0(t - 1))\varepsilon_t \Rightarrow \int_0^1 J(\lambda)dW(\lambda)$ follows from Chan and Wei (1988), theorem 2.4).

To obtain the expressions for $G(\cdot)$and $\Sigma(\cdot)$ stated in the text following Assumption B, let $z_t = (\Delta y_t \cdots \Delta y_{t-p+1})'$ and consider $x_{t-1}(k) = z_{t-1}1(t > k)$. Then $\Psi_{1T} = T^{-1/2}\Sigma_{t=1}^{T}Z_{t-1}^1\varepsilon_t = ((T^{-1/2}\Sigma_{t=1}^{T}z_{t-1}\varepsilon_t)', (T^{-1/2}\Sigma_{t=1}^{T}z_{t-1}\varepsilon_t - T^{-1/2}\Sigma_{t=1}^{[T\delta]}z_{t-1}\varepsilon_t)')' \Rightarrow \sigma(B(1)', (B(1) - B(\delta))')'$, where $B(\cdot)$ is a $p \times 1$ Brownian motion with covariance matrix Ω_p, where the convergence follows from $z_{t-1}\varepsilon_t$ being an MDS with $2 + \gamma$ moments. The independence of $B(1)$ and (H, W) follows from Chan and Wei (1988, theorem 2.2). The results for $\Sigma(\delta)$ stated in the text follow from the uniform consistency of partial sums of Δy_t^2 shown in the proof of Theorem 1.

(b) The argument for Case B is analogous to that for Case A.

These calculations show convergence of the processes $\Gamma_T(\cdot)$ and $\Psi_T(\cdot)$. Results for $\Upsilon_T(\tilde{\theta}(\cdot) - \theta) = \Gamma_T(\cdot)^{-1}\Psi_T(\cdot)$ and statistics such as $\inf_{\delta_0 \le \delta \le 1-\delta_0} \tilde{t}_{DF}(\delta)$ follow from these results and the continuous mapping theorem.

APPENDIX B: DATA SOURCES

Data for the United States are GNP from Citibase, for 1947:I to 1989:II. The data for the other six countries come from two sources, the OECD Main Economic Indicators data base maintained by Data Resources, Inc. (DRI) and Moore and Moore (1985). In most cases, two series have been spliced together to construct a longer time series of data. Where this has involved an adjustment because the real series are indexed to different base years, they have been adjusted using the earliest available ratio of the two series.

The data for Canada are GNP, with 1948:I to 1960:IV from Moore and Moore (1985) and 1961:I to 1989:II from DRI. The data for France are GDP, 1963:I to 1989:II, and are from DRI. The French data contain a large negative spike (a strike) in 1968:II; this spike was eliminated by replacing the GDP datum for 1968:II with the geometric average of the data for 1968:I and 1968:III. The data for Germany are GNP, with 1950:I to 1959:IV from Moore and Moore (1985) and 1960:I to 1989:II from DRI. The data for Italy from DRI were nominal values, so we have used GDP from Moore and Moore (1985) for 1952:I to 1982:IV. The GNP data for Japan is from Moore and Moore (1985) for 1952:I to 1964:IV and from DRI for 1965:I to 1989:II The data for the United Kingdom are GDP at factor cost and are from DRI for 1960:I to 1989:II. All data were seasonally adjusted at the source.

[*Received June 1990. Revised January 1992.*]

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