The Intertemporal Keynesian Cross

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Abstract

We derive a microfounded, dynamic version of the traditional Keynesian cross, which we call the *intertemporal Keynesian cross*. It characterizes the mapping from all partial equilibrium demand shocks to their general equilibrium outcomes. The aggregate demand feedbacks between periods can be interpreted as a network, and the linkages in the network can be generalized to reflect both the feedback from consumption and other dynamic forces, such as fiscal and monetary policy responses. We explore the general equilibrium amplification and propagation of impulses, and show how they vary with features of the economy. General equilibrium amplification is especially strong when agents are constrained, face uncertainty, or are unequally exposed to aggregate fluctuations, and it plays a crucial role in the transmission of monetary policy.

1 Introduction

The Keynesian cross is a staple of introductory macroeconomics, and one of the central ideas in the analytical tradition that began with Keynes (1936). It spells out a simple feedback mechanism: when some shock leads to a rise in demand for aggregate output, part of the income earned from producing that output goes back into consumption demand, leading to a further rise in demand for aggregate output, and so on.

Locally, the traditional Keynesian cross equation can be written as

$$dY = \partial Y + MPC \cdot dY \tag{1}$$

where ∂Y is the impulse to aggregate demand, dY is the equilibrium change in output, and MPC is the aggregate marginal propensity to consume out of income. Equation (1) can be solved to obtain $dY = (1 - MPC)^{-1}\partial Y$, where $(1 - MPC)^{-1} = 1 + MPC + MPC^2 + \ldots$ is called the *multiplier* and reflects the accumulated consumption feedback amplifying the original impulse.

Although the traditional Keynesian cross embodies a useful intuition, it has a number of weaknesses. It is static, not dynamic. It does not address budget constraints: an impulse to demand from government spending comes out of thin air, rather than being offset by taxation or a cut in spending at some other date. It does not directly correspond to any standard, microfounded model.

This paper presents a modernized alternative, the *intertemporal Keynesian cross*, that addresses these weaknesses and captures the general equilibrium feedback mechanisms in a variety of dynamic macroeconomic models. It has the form

$$d\mathbf{Y} = \partial\mathbf{Y} + \mathbf{M}d\mathbf{Y} \tag{2}$$

which is an intertemporal generalization of (1), with $\mathbf{Y}$ being the *vector* of output at different dates and $\mathbf{M}$ being a *matrix* of aggregate marginal propensities to consume, where each entry shows the fraction of aggregate income in one date that will be spent, at the margin, in another. $\mathbf{M}$ can be generalized to include intertemporal feedbacks other than household consumption, such as investment, fiscal policy, or monetary policy.

The solution to (2) gives a *general equilibrium multiplier* that maps impulses $\partial\mathbf{Y}$ to equilibrium output changes $d\mathbf{Y}$, reminiscent of the traditional $(1 - MPC)^{-1}$ multiplier. The multiplier, however, is now a matrix, and it reflects rich higher-order interactions: for instance, some income earned in period 1 will be spent in period 2, from which some income will be spent again in period 1. This multiplier matrix characterizes simultaneously the general equilibrium consequences of *all* partial equilibrium shocks to the intertemporal pattern of demand—including those coming from preference shocks, changes to fiscal or monetary policy, or changes in the distribution of income among heterogenous agents.

In deriving the general equilibrium multiplier from the intertemporal Keynesian cross, we uncover subtleties not present in the traditional, static analysis. One crucial observation is that in net present value terms, all income is eventually consumed. In analytical terms, this implies that the matrix $\mathbf{M}$ eigenvalue of one, and that it is impossible to invert $\mathbf{I} - \mathbf{M}$ to solve (2) for arbitrary impulses $\partial\mathbf{Y}$. Intuitively, the problem is that if an impulse has positive net present value, then applying $\mathbf{M}$ will preserve that net present value, and the series $\mathbf{I} + \mathbf{M} + \mathbf{M}^2 + \ldots$ of iterated consumption feedbacks will diverge to $+\infty$.

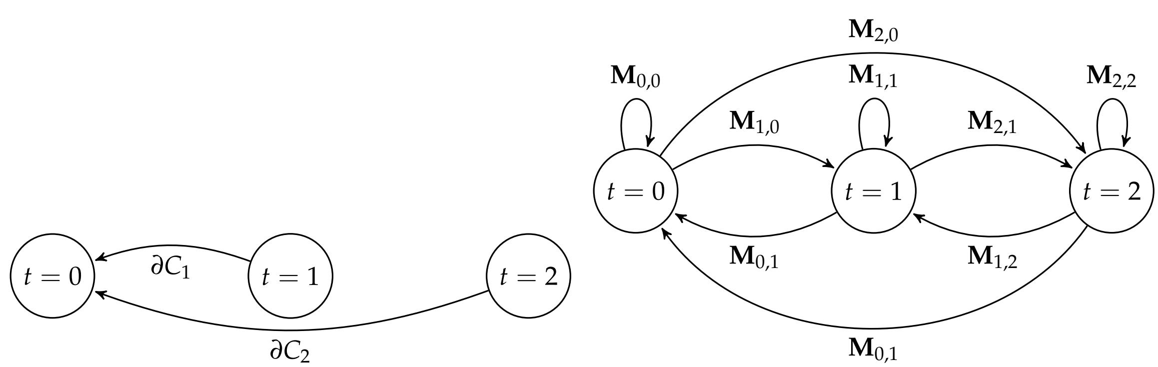
This difficulty, however, is mitigated by another observation: since individual agents must respect budget constraints, a well-defined partial equilibrium impulse to demand must have zero net present value. For instance, the substitution effect from a shock to interest rates changes the pattern of household consumption over time without changing its net present value; assuming asset market clearing and Ricardian fiscal policy, the net present value of the combined income effects must also be zero. Due to this zero net present value property of demand shocks $\partial\mathbf{Y}$, it is possible to solve the intertemporal Keynesian cross (2) for general equilibrium $d\mathbf{Y}$.

Since $\mathbf{M}$ has an eigenvalue of one, however, there is at least one nonzero solution to the equation $d\mathbf{Y} = \mathbf{M}d\mathbf{Y}$, meaning that the general equilibrium solution is *indeterminate* using (2) alone. This indeterminacy is inherent to models with nominal rigidities, and can be resolved by monetary policy. Different policy rules will lead to different outcomes—monetary policy can determine, for instance, whether a given impulse from government spending will result in an increase in the net present value of output or leave it unchanged.

These analytical tools provide us with a variety of results. In a very simple case, we show that they reduce to a version of the traditional Keynesian cross. In more general cases, however, there are rich consequences for the general equilibrium amplification or dampening of various shocks. We find, for instance, that there is greater amplification when households face tighter financial constraints and more uncertain incomes. This directly increases the impact of fiscal policy, while for monetary policy it offsets a decline in the partial equilibrium effect, since constraints and uncertainty make agents less directly responsive to interest rates. As anticipated monetary policy shocks are pushed further into the future, the initial partial equilibrium effect does less and less, while the general equilibrium amplification does more and more.

Network interpretation. As we will see, one useful way to understand the amplification mechanisms inherent to our model is to think of time periods as nodes of a network. Each new unit of income generated at a given node is spent, partly on itself and partly on every other node, according to relationships given by the matrix $\mathbf{M}$. This round of spending generates additional income at each node, which is again spent according to the same pattern, and so on. The final outcome for the distribution of income across nodes is our main object of interest. It is the general equilibrium effect on consumption after the intertemporal Keynesian cross has run its course.

If the matrix $\mathbf{M}$ is written in net present value units—as we generally do—then the fact that all income is spent in net present value terms means that $\mathbf{M}$ is a left-stochastic matrix, with columns summing to one. It can then be interpreted as the transition matrix for a Markov chain, which provides a even more evocative view of the network. A number of analytical results can be usefully viewed in light of this interpretation: for instance, the indeterminacy of GE solutions arises because $\mathbf{M}$ has a (generally unique) Perron-Frobenius eigenvector with eigenvalue one. This eigenvector corresponds to the stationary distribution of the Markov chain.



(a) Partial equilibrium demand shock

(b) General equilibrium amplification

Figure 1: The Intertemporal Keynesian Cross

2 A deterministic framework

In this section we write down a dynamic general equilibrium environment. Agents are heterogeneous in terms of their preferences, their income, the taxes they pay, and their ability to access financial markets. Production results from an aggregate of different types of labor. Hence, our environment incorporates many of the sources of heterogeneity that the recent literature has considered.

In this environment we derive, in Proposition 1, the *Intertemporal Keynesian Cross*: there exists a matrix $\mathbf{M}$ such that the general equilibrium effect on output $d\mathbf{Y}$ of a shock to aggregate demand (from preferences, government spending or monetary policy) satisfies

$$d\mathbf{Y} = \partial\mathbf{Y} + \mathbf{M}d\mathbf{Y} \tag{3}$$

where $\partial\mathbf{Y}$ is the partial equilibrium effect of the shock. Furthermore, we have $\mathbf{1}'\mathbf{M} = \mathbf{1}'$ (Lemma 1) and, for any shock, $\mathbf{1}'\partial\mathbf{Y} = \mathbf{0}$ (Lemma 2). In the next sections, we will characterize the solutions to (3) and illustrate the usefulness of this proposition in a number of important contexts.

2.1 Environment and flexible price equilibrium

We consider a $T+1$ period economy populated by a finite set of I types of agents who face no uncertainty. There are μ_i agents of type $i = 1 \dots I$, with masses normalized such that $\sum_{i=1}^{I} \mu_i = 1$. All agents within a type have the same preferences and face the same environment, so they behave identically.

Agents. Each agent type $i \in I$ has utility over consumption

$$U^i\left(c_0^i, c_1^i, \dots, c_T^i; \theta\right) \tag{4}$$

where θ is an aggregate preference parameters, whose effect on agent i depends on the utility function U^i. Agent i consumes c_t^i goods and works n_t^i units of time in period t. Work provides disutility $V^i\left(n_0^i, n_1^i, \dots, n_T^i\right)$. Utility in consumption and leisure is separable, so overall utility is $U^i - V^i$. The agent can trade in nominal assets, and faces borrowing and saving limits which may be i- and t-specific. Specifically, his budget constraint in period t is

$$\begin{aligned} P_t c_t^i + A_t^i &= (1+i_{t-1}) A_{t-1}^i + W_t^i n_t^i - P_t T_t^i \quad \forall t, i \\ \frac{A_t^i}{P_t} &\in \left[\underline{a_t^i}, \overline{a_t^i}\right] \end{aligned} \tag{5}$$

In (5), i_t is the nominal interest rate, and P_t the nominal price of goods, equal for all agents. W_t^i is the wage of type i labor, and T_t^i are taxes, which are lump sum but may be i and t specific. As we will see shortly, in equilibrium there are no firm profits to be distributed. Agents are born and die with no wealth: we impose $A_{-1}^i = A_T^i = 0$ for all i.

Recursively substituting in (5) and imposing our initial and terminal conditions yields the intertemporal budget constraint

$$\sum_{t=0}^{T} Q_t c_t^i = \sum_{t=0}^{T} Q_t \left(\frac{W_t^i}{P_t} n_t^i - T_t^i \right) \quad \forall i \tag{6}$$

where the real discount rate Q_t is the time-0 price of a unit of consumption at time t,

$$Q_t \equiv \prod_{s=0}^{t-1} \frac{1}{R_s} \tag{7}$$

and the gross real interest rate between periods s and $s+1$ is defined as

$$R_s \equiv (1 + i_s) \frac{P_s}{P_{s+1}} \tag{8}$$

Maximization of (4) subject to (5) implies the Euler equations

$$\left(\frac{U^i_{c_t}\left(c_0^i, c_1^i, \dots, c_T^i; \theta\right)}{U^i_{c_{t+1}}\left(c_0^i, c_1^i, \dots, c_T^i; \theta\right)} - R_t \right) \left(\frac{A_t^i}{P_t} - \underline{a}_t^i \right) \left(\overline{a}_t^i - \frac{A_t^i}{P_t} \right) = 0 \quad \forall i, t \tag{9}$$

Our preference setup is very general. It accomodates arbitrary preferences, asset market participation, and population structure, and can generate a wide array of equilibrium time paths for consumption and labor supply, as well as marginal propensities to consume.

Production. Each period, a perfectly competitive firm produces the unique final good in this economy using a technology that aggregates labor from each type of worker

$$Y_t = F_t \left(l_t^1, \dots l_t^I \right) \quad \forall t$$

where, in each period t, $F_t(\cdot)$ has constant returns to scale and diminishing returns to each labor type. Firm prices are perfectly flexible. Profit maximization implies

$$P_t = \frac{W_t^i}{F_{l^i,t}\left(l_t^1, \dots l_t^I\right)} \quad \forall t, i \tag{10}$$

where W_t^i is the wage of worker type i. Constant returns then imply that firms make zero profits at all times.

Government. The government spends G_t in period t. It raises taxes T_t^i to pay for this spending, and adjusts the stock of nominal public debt B_t, so as to satisfy its budget constraint

$$P_t \left(\sum_{i=I}^{n} \mu_i T_t^i \right) + B_t = (1 + i_{t-1}) B_{t-1} + P_t G_t \quad \forall t \tag{11}$$

We impose $B_{-1} = B_T = 0$, to enforce market asset market clearing at the initial and terminal date.

Flexible wage equilibrium. In a flexible wage equilibrium, firms optimize, implying (10). Households optimize, implying (5) and (9) for each i together with

$$\frac{V^i_{n_t}\left(n^i_0, n^i_1, \ldots, n^i_T\right)}{U^i_{c_t}\left(c^i_0, c^i_1, \ldots, c^i_T; \theta\right)} = \frac{W^i_t}{P_t} \quad \forall t, i \tag{12}$$

Further, labor markets clear, implying

$$l^i_t = \mu_i n^i_t \quad \forall t, i \tag{13}$$

Finally, goods markets clear:

$$\sum_i \mu_i c^i_t + G_t = Y_t \quad \forall t \tag{14}$$

Equivalently, asset markets clear, $\sum_i \mu_i A^i_t = B_t, \forall t$. Fix an allocation $\left\{Y^n_t, G^n_t, c^{i,n}_t, n^{i,n}_t, T^{i,n}_t, A^{i,n}_t, R^n_t, W^{i,n}_t, P^n_t\right\}$ satisfying these equations and call it the "natural allocation" (in some cases, there may be multiple such allocations, but we will consider perturbations away from a particular one).

Note that, given our initial and terminal conditions on debt, the flow government constraints (11) imply that the following intertemporal constraint must hold in equilibrium:

$$\sum_{t=0}^{T} Q_t \left(\sum_{i=I}^{n} \mu_i T^i_t\right) = \sum_{t=0}^{T} Q_t G_t \tag{15}$$

2.2 Sticky wage equilibrium

Consider a natural allocation $\left\{Y^n_t, G^n_t, c^{i,n}_t, n^{i,n}_t, T^{i,n}_t, A^{i,n}_t, R^n_t, W^{i,n}_t, P^n_t\right\}$. We define a sticky wage equilibrium *relative* to that allocation. Following the lead of the large New Keynesian literature, we are then also interested in characterizing equilibrium outcomes relative to that allocation.

Consider next an arbitrary path $\{\theta, i_t, G_t\}$ for preferences, nominal interest rates and fiscal policy, as well as a transfer rule

$$\mathbf{T^i} = \mathcal{T}^i\left(\mathbf{G}, \mathbf{R}\right) \tag{16}$$

specifying transfers T^i_t at time t for individual i as a function of the path for spending and real interest rates. The rule (16) is constrained to satisfy $T^{i,n}_t = \mathcal{T}^i\left(\mathbf{G}^n, \mathbf{R}^n\right)$ as well as (15) for any path $\mathbf{G}$ and $\mathbf{R}$.

Given these paths and our initial natural allocation, a *sticky wage equilibrium* is defined as a set of equilibrium prices and quantities $\left\{Y_t, c^i_t, n^i_t, A^i_t, R_t, W^i_t, P_t\right\}$ such that wages have to remain at their at their natural level:

$$W^i_t = W^{i,n}_t \quad \forall i, t \tag{17}$$

firms optimize, implying (10), households optimize their consumption plan, implying (5), (6) and

(9) for each i, and labor, goods and asset markets clear, implying (13) and (14).

Hence the only difference with a flexible wage equilibrium is that equation (17) replaces the requirement that househols be on their labor supply curves, i.e. equation (12). By construction, the natural allocation under the baseline level of parameters $\left\{\theta^n, i_t^n, G_t^n, T_t^{i,n}\right\}$ is a sticky wage equilibrium.

The following remark simplifies the analysis of equilibrium.

Remark 1. In a sticky wage equilibrium, prices remain at their natural-allocation level at all times:

$$P_t = P_t^n \quad \forall t \tag{18}$$

Proof. Since the production function F_t has constant returns to scale, its derivatives $F_{l^i,t}$ are homogeneous of degree 0. Equation (10) implies that

$$\frac{F_{l^i,t}\left(l_t^1, \dots l_t^I\right)}{F_{l^j,t}\left(l_t^1, \dots l_t^I\right)} = \frac{W_t^i}{W_t^j} = \frac{W_t^{i,n}}{W_t^{j,n}} = \frac{F_{l^i,t}\left(l_t^{1,n}, \dots l_t^{I,n}\right)}{F_{l^j,t}\left(l_t^{1,n}, \dots l_t^{I,n}\right)}$$

and hence there exists a sequence $\{\lambda_t\}$ such that $l_t^i = \lambda_t l_t^{i,n}$ for all i, t. Applying (10) again for any i, we find that $P_t = P_t^n$ for all t. □

Given this remark, the analysis of monetary policy in this model is particularly simple: the exogenous path for the nominal interest rate i_t translates into a path for the real interest rate equal to $R_t = (1 + i_t)\frac{P_t^n}{P_{t+1}^n}$, and we can alternatively think of a sticky wage equilibrium as being defined given such a path.

Our framework is therefore appropriate to study the three main types of "demand" shocks considered in the business cycle literature: preference shocks θ, government spending shocks G_t, and monetary policy shocks i_t.[1]

Towards our derivation of the intertemporal Keynesian cross (3), we now define a number of objects of interest. These correspond to certain combinations of derivatives of policy functions evaluated at the natural allocation.

2.3 Aggregate demand and MPCs

We start by defining marginal propensities to consume.

2.3.1 MPCs out of individual income

Since agents in a sticky wage equilibrium are not able to choose their labor supply, we define MPCs taking this constraint into account. Specifically, we consider a modified problem for each agent, in which we treat income as an exogenous stream $\left\{y_t^i\right\}$. Agent i maximizes (4) subject to

[1] A few simple modifications to this setup would allow us to study productivity shocks as well.

the constraints

$$\begin{aligned} c_t^i + a_t^i &= R_{t-1} a_{t-1}^i + y_t^i \\ a_t^i &\in \left[\underline{a_t^i}, \overline{a_t^i}\right] \end{aligned}$$

and $a_{-1}^i = a_T^i = 0$. The solution determines Marshallian demand functions $c_t^i\left(\left\{y_t^i\right\}, \{R_t\}; \theta\right)$. In particular, the following intertemporal budget constraint holds for each i

$$\sum_{t=0}^{T} Q_t c_t^i \left(\left\{y_t^i\right\}, \{R_t\}; \theta\right) = \sum_{t=0}^{T} Q_t y_t^i \tag{19}$$

For every pair s, t, define $Q_{t,s} \equiv \frac{Q_t}{Q_s}$ as the time-s price of a unit of consumption at date t.

Definition 1. The *marginal propensity to consume* of individual i at time t, for income received at date s is

$$MPC_{t,s}^i \equiv Q_{t,s} \left.\frac{\partial c_t}{\partial y_s}\right|_{y_t = \frac{W_t^{i,n}}{P_t^n} n_t^{i,n} - T_t^i, R_t = R_t^n, \theta = \theta^n}$$

It is the derivative of the Marshallian demand function, discounted back to the date of the income receipt s, and evaluated at the natural allocation, where income is defined as $y_t^i = \frac{W_t^{i,n}}{P_t^n} n_t^{i,n} - T_t^i$ and the real interest rate is $R_t = R_t^n$.

2.3.2 Individual income response to aggregate macroeconomic changes

In order to define aggregate demand, we need to determine how individual income $y_t^i = \frac{W_t^i}{P_t^i} n_t^i - T_t^i$ is affected by macroeconomic aggregates Y_t, G_t and R_t. We have already established that $\frac{W_t^i}{P_t^i} = \frac{W_t^{i,n}}{P_t^{i,n}}$, The fiscal rule (16) implies that $T_t^i = \mathcal{T}_t^i(\mathbf{G}, \mathbf{R})$. We now turn to the determinants of n_t^i .

Consider the firm problem in a sticky wage equilibrium. Since F_t has constant returns and is therefore homothetic, with constant input prices (17), labor demand for each type scales linearly in the level of production

$$l_t^i = l_t^{i,n} \frac{Y_t}{Y_t^n} \tag{20}$$

Since the labor market for each type of labor clears (13), this implies that individual hours worked scale linearly in the amount of aggregate output Y_t

$$n_t^i = n_t^{i,n} \frac{Y_t}{Y_t^n}$$

Combining this with (10), (17), and (18), the gross labor earnings of individual i at time t are therefore

$$\frac{W_t^i}{P_t} n_t^i = \frac{W_t^{i,n}}{P_t^n} n_t^{i,n} \frac{Y_t}{Y_t^n} = \frac{\gamma_t^{i,n}}{\mu_i} Y_t \tag{21}$$

where $\gamma_t^{i,n} = \frac{F_{l^i,t} l_t^{i,n}}{Y_t^n}$ is the share of labor type i in production at time t (which is identical in the natural and the sticky wage allocation).

2.3.3 Aggregate demand

Having established the way in which individual income responds to aggregates $\mathbf{Y} \equiv \{Y_t\}$, $\mathbf{G} \equiv \{G_t\}$ and $\mathbf{R} \equiv \{R_t\}$, we naturally define the consumption demand $c_t^{i,d}$ of individual i at time t as follows

$$c_t^{i,d}(\mathbf{Y}, \mathbf{G}, \mathbf{R}, \theta) = c_t^i\left(\left\{\frac{\gamma_t^{i,n}}{\mu_i} Y_t - \mathcal{T}_t^i(\mathbf{G}, \mathbf{R})\right\}, \mathbf{R}; \theta\right) \tag{22}$$

It is the level of consumption that individual i chooses when macroeconomic aggregates are $\mathbf{Y}$, $\mathbf{G}$ and $\mathbf{R}$, taking into account the effects these aggregates have on his income. We use these functions to define *aggregate demand* as

$$Y_t^d(\mathbf{Y}, \mathbf{G}, \mathbf{R}, \theta) = \sum_i \mu_i c_t^{i,d}(\mathbf{Y}, \mathbf{G}, \mathbf{R}, \theta) + G_t \tag{23}$$

Goods market clearing (14) implies that in sticky wage equilibrium $Y_t^d = Y_t$. Hence solving for equilibrium involves solving for the fixed point in which aggregate output is equal to aggregate demand:

$$Y_t = Y_t^d(\mathbf{Y}, \mathbf{G}, \mathbf{R}, \theta) \quad \forall t$$

Our main proposition below characterizes this fixed point. We need two further definitions.

2.3.4 MPC matrix

We first define the matrix $\mathbf{M}$ with elements

$$\mathbf{M}_{t,s} \equiv \sum_{i=1}^{I} \gamma_s^i MPC_{t,s}^i \tag{24}$$

Notice from differentiating (22) that

$$MPC_{t,s} \gamma_s^{i,n} = \mu_i Q_{t,s} \frac{\partial c_t^{i,d}(\{Y_t\}, \{R_t\}, \theta)}{\partial Y_s} \tag{25}$$

Hence, $\mathbf{M}_{t,s}$ is also the discounted response of aggregate consumption to an increase in aggregate income Y_s at date s. Our assumptions on production and transfers guarantee that the endogenous response of consumption at time t to such an increase happens only via the response of individual income in that period. Moreover, the aggregate income sensitivity of individual i's income in period s is γ_s^i, which is this type's share in production.

The incidence-weighted MPC matrix $\mathbf{M}$ has the following important property.

Lemma 1. *The vector* $\mathbf{1}$ *is a left eigenvector of* $\mathbf{M}$ *with eigenvalue 1, that is*

$$\mathbf{1}'\mathbf{M} = \mathbf{1}' \tag{26}$$

Proof. Differentiating the budget constraint (6) for each s, we see that

$$\sum_{t=0}^{T} MPC_{t,s}^{i} = 1 \quad \forall s, i \tag{27}$$

Applying (25) and (27), we therefore find

$$\sum_{t=0}^{T} \mathbf{M}_{t,s} = \sum_{t=0}^{T} \left(\sum_{i=1}^{I} \gamma_s^{i,n} MPC_{t,s}^{i} \right) = \sum_{i=1}^{I} \gamma_s^{i,n} \sum_{t=0}^{T} \left(MPC_{t,s}^{i} \right) = \sum_{i=1}^{I} \gamma_s^{i,n} = 1$$

□

In words, Lemma 1 comes from the fact that our economy is "closed", in the sense that any additional money earned by the agents in some period s will be spent.

2.3.5 Partial equilibrium effects

We next define the partial equilibrium response $\partial \mathbf{Y}^X$ to any shock $X \in \{\theta, G_0 \cdots G_T, R_0 \cdots R_T\}$ as discounted value of the partial derivative of the aggregate demand function (23).

$$\partial \mathbf{Y}_t^X \equiv Q_t \frac{\partial Y_t^d}{\partial X} dX \tag{28}$$

The following lemma is a fundamental property of partial equilibrium shocks.

Lemma 2. *Vectors of partial equilibrium responses have zero present value, ie*

$$\mathbf{1}' \partial \mathbf{Y}^X = 0 \tag{29}$$

for any shock $X \in \{\theta, G_0 \cdots G_T, R_0 \cdots R_T\}$.

The proof, written separately for each shock in appendix A.1, is a consequence of the agent's and the government's budget constraints holding with equality in partial equilibrium. Since these are "pure" demand shocks that leave total earnings unchanged, all the partial equilibrium responses have to net out to have a zero net present value.

For example, a preference shock alters the pattern of intertemporal spending of each agent but, with labor supply constrained to be constant, does not generate more income. Hence, the present value of consumption is constant, implying (29). Similarly, the government's intertemporal budget constraint (15) implies that an extra unit of spending must be paid for by an equivalent increase in the present value of taxes. Since each agent responds to a unit-size increase in the present value of taxes by reducing the present value of their spending by a unit, the overall reduction in the present value of aggregate consumption must exactly offset the increase in that of government spending, so that (29) holds. Finally, a monetary policy shock can generate present-value redistribution between various individuals or the government depending on their patterns of intertemporal trade in the natural allocation, but when summed up, these individual responses net out to a zero aggregate present value effect.

2.4 The Intertemporal Keynesian Cross

We are now ready for our main proposition of this section.

Proposition 1. *Consider any variable $X \in \{\theta, G_0 \cdots G_T, R_0 \cdots R_T\}$. The change in output $d\mathbf{Y}^X$ that results from a change dX is characterized to first order by*

$$d\mathbf{Y}^X = \partial \mathbf{Y}^X + \mathbf{M} d\mathbf{Y}^X \tag{30}$$

where $\mathbf{M}$ is defined in (24), $\partial \mathbf{Y}^X$ is defined in (28), and $d\mathbf{Y}_t^X \equiv Q_t dY_t$ is the discounted value of the change in output at time t relative to the natural allocation.

Proof. Start by totally differentiating the individual consumption demand function (22)

$$dc_t^{i,d} = \frac{\partial c_t^i}{\partial X} dX + \sum_{s=0}^{T} \frac{\partial c_t^i}{\partial Y_s} dY_s$$

aggregating, we therefore have

$$dC_t = \sum_i \mu_i dc_t^{i,d} = \sum_i \mu_i \frac{\partial c_t^i}{\partial X} dX + \sum_{s=0}^{T} \mu_i Q_t \frac{\partial c_t^i}{\partial Y_s} dY_s$$

But from (24) and (25),

$$Q_t dY_t = Q_t d\left(C_t + G_t\right) = Q_t \frac{\partial Y_t^d}{\partial X} dX + \sum_{s=0}^{T} \mathbf{M}_{t,s} Q_s dY_s$$

which, by (28) and our definition $d\mathbf{Y}_t^X \equiv Q_t dY_t$, results in

$$d\mathbf{Y}^X = \partial \mathbf{Y}^X + \mathbf{M} d\mathbf{Y}$$

as we set out to prove. □

Proposition 1 derives our intertemporal analogue of the traditional Keynesian Cross, describing any general equilibrium response $d\mathbf{Y}^X$ as sum of the partial equilibrium response and the feedback of $d\mathbf{Y}^X$ through the MPC matrix. As it turns out, equation (30) describes the determination of aggregated demand in even more more general settings than the one we have introduced in this section. One could therefore think of it as a "fundamental law" that underlies many modern macroeconomic models.

The next sections explore the implications of Proposition 1.

3 Solving the Intertemporal Keynesian Cross

We now investigate the intertemporal Keynesian cross (30) in detail. Our main result is a characterization of the solution(s) of (30). As it will turn out, since our MPC network is *closed*—that is, there is no demand "flowing" in or out of the system—an equation like (30) admits *infinitely* many solutions. This requires an equilibrium selection rule. In analogy to policy experiments in standard New-Keynesian models, where monetary policy implements a zero output gap after either finite time or asymptotically, we shall select the equilibrium that sets the terminal output gap to zero, $d\mathbf{Y}_T^X = 0$.[2]

Throughout this section, we will drop the superscript X and instead regard (30) independently of its derivation. This requires us to be precise about the objects involved in it. In particular, we assume that $\mathbf{M}$ is a column-stochastic matrix in $\mathbb{R}^{(T+1)\times(T+1)}$, $\sum_{t=0}^{T} \mathbf{M}_{t,s} = 1$, and that $\partial\mathbf{Y} \in \mathbb{R}^{T+1}$ has zero net present value (NPV), that is, $\sum_{t=0}^{T} \partial\mathbf{Y}_t = 0$, which we also write as $\partial\mathbf{Y} \in \mathbf{1}^{\perp}$, using "$\perp$" to denote the orthogonal complement.

An equation of the form (30) does not necessarily admit any solution, even with the assumption in place so far. For example, suppose the MPC matrix $\mathbf{M}$ is the identity matrix. In that case, no solution $\mathbf{dY}$ exists whenever $\partial\mathbf{Y} \neq 0$. Clearly, such an MPC matrix would not be a very convincing description of an aggregate economy for it would imply that any additional income $d\mathbf{Y}_t$ earned in some period t is entirely spent in that period. In other words, there is no money being spent *across* periods. We now introduce two restrictions on the MPC matrix $\mathbf{M}$ that rule out such strict "within period" spending, the first slightly more general than the second.

Assumption 1. $\mathbf{M}$ *is non-negative and irreducible: that is, for each* $s, t \in \{0, \ldots, T\}$, $\mathbf{M}_{t,s} \geq 0$ *and there exists an* $m \in \mathbb{N}$ *such that* $(\mathbf{M}^m)_{t,s} > 0$.

According to this assumption, $\mathbf{M}$ needs to be such that, for each two periods s and t, an increase in aggregate income in period s will be partially spent in period t after m iterations. To give an example of $m > 1$, take a world in which additional aggregate income in period $t+1$ is spent in period t but not period $t-1$. Yet, since spending in period t is equal to income in period t, it is true—after two iterations—that some additional income in period $t+1$ *will* be spent in period $t-1$ despite the lack of a direct link.

If there is such a direct link, we refer to a matrix as $\mathbf{M}$ as *primitive*, as specified in the follwing, more demanding assumption.

Assumption 2. $\mathbf{M}$ *is primitive: that is, it is non-negative, and there exists an* $m \in \mathbb{N}$ *such that* $(\mathbf{M}^m)_{t,s} > 0$ *for any* $s, t \in \{0, \ldots, T\}$.

Utilizing the link of our methodology to network theory and Markov chains, we note that Assumptions 1 and 2 are also commonly used in the theory of Markov chains: Viewing $\mathbf{M}$ as the transition matrix of a Markov chain, Assumption 1 implies the existence of a unique stationary

[2] In future versions of this paper, we intend to provide a formal argument to establish this connection.

distribution of that Markov chain; and Assumption 2 implies convergence to the stationary distribution starting from any other initial distribution. Both of these properties will come in handy in our characterization of the solution to the intertemporal Keynesian cross (30).

Even though these two assumptions may appear restrictive at first glance, both are in fact satisfied under minimal restrictions on the environment presented in Section 2, as we demonstrate in the following lemma.

Lemma 3 (Properties of the MPC matrix)**.** *Suppose that for each pair of periods $s, t \in \{0, \ldots, T\}$ there exists a an agent i with positive marginal utility of consuming in period s, $U^i_{c_s} > 0$, with positive income sensitivity in period t, $\gamma^i_t > 0$, and, with no binding borrowing or savings constraint between periods s and t. Then, the MPC matrix $\mathbf{M}$ is primitive, and Assumptions 1 and 2 are satisfied.*

Proof. The proof is immediate: Under the stated assumptions, $\mathbf{M}_{t,s} > 0$ for each pair of periods $s, t \in \{0, \ldots, T\}$. Thus, $\mathbf{M}$ is primitive (with $m = 1$ in the Assumption 2). □

Having introduced the restrictions $\mathbf{M}$ needs to satisfy, we next describe the family of solutions to (30), before the imposition of a monetary policy rule.

Theorem 1 (Solving the Intertemporal Keynesian Cross)**.** *Let $\mathbf{M}$ satisfy Assumption 1. Then, there exists a matrix $\mathbf{A} \in \mathbb{R}^{(T+1)\times(T+1)}$ mapping the set of zero-NPV vectors $\mathbf{1}^\perp$ into itself, with the following property: For any solution $\mathbf{dY}$ to the intertemporal Keynesian cross (30) there exists a scalar $\lambda \in \mathbb{R}$ such that*

$$d\mathbf{Y} = \mathbf{A}\partial\mathbf{Y} + \lambda\mathbf{v}, \tag{31}$$

and for any $\lambda \in \mathbb{R}$ this is a solution. Here, $\mathbf{v} \in \mathbb{R}^{T+1}$ is the unique and positive right-eigenvector of $\mathbf{M}$ with respect to eigenvalue 1 that is normalized to $\mathbf{1}'\mathbf{v} = 1$. Under Assumption 2, $\mathbf{A}\partial\mathbf{Y}$ can be expressed by the infinite sum

$$\mathbf{A}\partial\mathbf{Y} = \partial\mathbf{Y} + \mathbf{M}\partial\mathbf{Y} + \mathbf{M}^2\partial\mathbf{Y} + \mathbf{M}^3\partial\mathbf{Y} + \ldots. \tag{32}$$

Proof. Since $\mathbf{M}$ is column-stochastic, it has an eigenvalue of 1 and admits a right-eigenvector $\mathbf{v} \in \mathbb{R}^{T+1}$. By the Perron-Frobenius Theorem for non-negative, irreducible, (column-)stochastic matrices, this eigenvalue is unique and weakly exceeds the absolute value of every other eigenvalue. Moreover, $\mathbf{v}$ can be chosen to be positive in all entries. We normalize it henceforth so that $\mathbf{1}'\mathbf{v} = 1$.

Rewrite (30) as $(1 - \mathbf{M})d\mathbf{Y} = \partial\mathbf{Y}$. Using the fact that the Kernel of $1 - \mathbf{M}$ is single-dimensional, the rank-nullity theorem implies that the image of $1 - \mathbf{M}$ must be T dimensional. But, as every vector in the image of $1 - \mathbf{M}$ is orthogonal to a vector of ones 1, it follows that the image of $1 - \mathbf{M}$ is exactly $\mathbf{1}^\perp$—the space of zero NPV vectors. Moreover, note that $\mathbb{R}^{T+1}$ can be decomposed into $\mathbf{1}^\perp \oplus Ke(1 - \mathbf{M})$ since $\mathbf{1}'\mathbf{v} \neq 0$.

Together, this means that $1 - \mathbf{M}$ defines a linear bijection (automorphism) from $\mathbf{1}^\perp$ into itself. Let $\mathbf{A}$ be any matrix that is equal to the inverse of this bijection when restricted to $\mathbf{1}^\perp$. There exists a one-dimensional family of such matrices; an obvious choice is $A = 1 - \mathbf{M} + \mathbf{v}\mathbf{1}'$. Then, any solution to (30) is of the general form

$$\mathbf{dY} = \mathbf{A}\partial\mathbf{Y} + \lambda\mathbf{v}$$

with $\lambda \in \mathbb{R}$.

If $\mathbf{M}$ is a primitive matrix (Assumption 2), we know that any non-unit eigenvalue of $\mathbf{M}$ has an absolute value strictly below 1. Since $\mathbf{M}$ maps $\mathbf{1}^{\perp}$ into itself, any vector $\mathbf{w} \in \mathbf{1}^{\perp}$ can be decomposed into a linear combination of eigenvectors of $\mathbf{M}$, $\mathbf{w} = \sum_{i=1}^{T-1} \lambda_i \mathbf{w}_i$, where $\mathbf{M}\mathbf{w}_i = \mu_i \mathbf{w}_i$, $\mathbf{w}_i \neq 0$, and μ_i is a (possibly complex) eigenvalue with $|\mu_i| < 1$. Therefore, the infinite sum

$$\mathbf{w} + \mathbf{M}\mathbf{w} + \mathbf{M}^2\mathbf{w} + ...$$

has the finite (and real) limit $\mathbf{w}^{lim} = \sum_{i=1}^{T-1} \frac{1}{1-\mu_i} \lambda_i \mathbf{w}_i$, which clearly satisfies $(1 - \mathbf{M})\mathbf{w}^{lim} = \mathbf{w}$. This concludes our proof of Theorem 1. □

According to Theorem 1, the general solution (31) to the intertemporal Keynesian cross consists of two terms: A zero-NPV term $\mathbf{A}\partial\mathbf{Y}$ and an eigenvector term $\lambda\mathbf{v}$. We describe each in turn. As equation (32) illustrates in the special case where $\mathbf{M}$ is primitive, the first term is analogous to the MPC sum when solving the traditional Keynesian cross. The key difference, however, is that the elements in our MPC sum are vectors: $\partial\mathbf{Y}$ is the direct partial equilibrium impact; $\mathbf{M}\partial\mathbf{Y}$ is the second order impact, including the fact that one agent's spending is another agent's income; $\mathbf{M}^2\partial\mathbf{Y}$ is the third order impact, and so on. All these effects are zero-NPV as $\mathbf{M}$ preserves zero-NPV vectors.

As a side note, this is precisely why the infinite sum (32) converges when $\mathbf{M}$ is primitive: In that case, $\mathbf{M}^n\mathbf{x}$ approaches $\kappa\mathbf{v}$ for any vector $\mathbf{x} \in \mathbb{R}^{T+1}$ where the scale κ of $\mathbf{v}$ is just the sum of the elements of $\mathbf{x}$, $\kappa = \mathbf{1}'\mathbf{x}$. In the case at hand, $\partial\mathbf{Y}$ has a zero NPV, that is, $\kappa = 0$, so $\mathbf{M}^n\partial\mathbf{Y} \to 0$. This is a requirement for the infinite sum (32) to be well-defined, which also turns out to be sufficient.

The fact that the first term $\mathbf{A}\partial\mathbf{Y}$ is zero NPV raises an important question: Does our intertemporal Keynesian cross imply that all partial equilibrium shocks $\partial\mathbf{Y}$, which *have* to respect the budget constraints and are therefore zero NPV, can only have zero NPV general equilibrium effects? Effectively, this would mean that any positive demand shock today is necessarily followed by a negative demand shock in the future.

That this is not the case is due to the second term $\lambda\mathbf{v}$ in our expression for $d\mathbf{Y}$, (31). This second term captures how the equilibrium selection rule (or in other words, the monetary policy rule) can crucially influence the dynamics of consumption. Our approach clarifies that any such equilibrium selection rule acts by shifting the level of $d\mathbf{Y}$ up or down proportional to $\mathbf{v}$. This vector can be regarded as a measure of (eigenvector) centrality in the Markov chain described by transition matrix $\mathbf{M}$: For each period t it gives a value $\mathbf{v}_t$ of income that corresponds to the additional income earned in period t if spending in all other periods increases according to $\mathbf{v}$. In that sense, $\mathbf{v}$ is "self-sustaining". As all of its elements are positive, it is naturally positive NPV and we normalize its NPV to 1, implying that λ corresponds to the net present value of the total general equilibrium demand response that was caused by the partial equilibrium shock $\partial\mathbf{Y}$. We summarize this insight in the following corollary.

Corollary 1. *The NPV of the general equilibrium aggregate demand response to partial equilibrium shock* $\partial\mathbf{Y}$ *is given by*

$$\sum_t d\mathbf{Y}_t = \lambda,$$

where λ *is the unique scalar in the decomposition* (31).

Of course, when evaluating the general equilibrium transmission of a partial equilibrium shock, the equilibrium selection rule needs to be taken into account. As we mentioned above, we assume monetary policy to implement a zero output gap in the terminal period. This means λ adjusts endogenously to $\partial\mathbf{Y}$ in order to ensure $d\mathbf{Y}_T = 0$, giving the following result.

Corollary 2. *When including the equilibrium selection rule* $d\mathbf{Y}_T = 0$*, the solution* (31) *becomes*

$$d\mathbf{Y} = \left(1 - \frac{\mathbf{v}e_T'}{\mathbf{v}_T}\right)\mathbf{A}\partial\mathbf{Y} \tag{33}$$

where $e_T = (0, ...0, 1)'$*. In particular, the total demand generated by the partial equilibrium shock* $\partial\mathbf{Y}$ *is given by*

$$\lambda = -\frac{(\mathbf{A}\partial\mathbf{Y})_T}{\mathbf{v}_T}.$$

The expression in (33) combines the zero NPV feedback inside $\mathbf{A}\partial\mathbf{Y}$ with the nonzero NPV shift coming in through the equilibrium selection rule. One way to think about the combination is that $\mathbf{A}\partial\mathbf{Y}$ implies a certain path for consumption, which is then lifted up or down along the vector $\mathbf{v}$, proportional to the value of $(\mathbf{A}\partial\mathbf{Y})_T$.

4 Examples

To illustrate the power of the decomposition provided in the previous section, we now provide specific examples of economies and partial equilibrium shocks $\partial\mathbf{Y}$. We start with a simple 2 period model that provides an intertemporal formalization of the traditional Keynesian cross. Then, we consider a more general economy in which all agents are unconstrained, and lastly move to a study what happens when hand-to-mouth agents enter the picture. Throughout, we will assume that $\mathbf{M}$ satisfies Assumption (1).

4.1 Intertemporal and traditional Keynesian cross

In this subsection, we consider a special case of the analysis in Section 3, where there are only two periods, that is, $T = 1$. One may view this as the most straightforward microfoundation of a traditional Old-Keynesian cross. We will prove that in this case, any nonzero partial equilibrium equilibrium response will be amplified into a general equilibrium one of the same sign, with a factor that precisely corresponds to the one in the Old-Keynesian cross, $1/(1-MPC)$, with MPC being the MPC to spend income earned in the initial period on the initial period.

With two periods and due to its stochasticity property, the MPC matrix $\mathbf{M}$ only has two degrees of freedom and can be written as

$$\mathbf{M} = \begin{bmatrix} MPC_{0,0} & 1-MPC_{1,1} \\ 1-MPC_{0,0} & MPC_{1,1} \end{bmatrix} \tag{34}$$

where we assume $MPC_{0,0}$ and $MPC_{1,1}$ to lie strictly inside $(0,1)$, ensuring that $\mathbf{M}$ is indeed a primitive matrix (see Assumption 2). As one can easily verify, the eigenvector $\mathbf{v}$ in this case is

$$\mathbf{v} = \frac{1}{2-MPC_{1,1}-MPC_{0,0}} \begin{bmatrix} 1-MPC_{1,1} \\ 1-MPC_{0,0} \end{bmatrix}.$$

Moreover, there is a second eigenvector $\mathbf{w}$, which will help in deriving the matrix $\mathbf{A}$. It is given by

$$\mathbf{w} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

and has eigenvalue $\eta = MPC_{00} + MPC_{11} - 1 < 1$. Notice that $\mathbf{w}$ is zero NPV and therefore any partial equilibrium response $\partial\mathbf{Y}$ will be proportional to $\mathbf{w}$. Finally, we find $\mathbf{A}$ by computing the infinite sum (32) using $\mathbf{w}$ as zero NPV vector,

$$\mathbf{A}\mathbf{w} = \left(1+\eta+\eta^2+\ldots\right)\mathbf{w} = \frac{1}{1-\eta}\mathbf{w} = \frac{1}{2-MPC_{1,1}-MPC_{0,0}}\mathbf{w}.$$

We summarize these steps in the following proposition.

Proposition 2 (Intertemporal and traditional Keynesian Cross.)**.** *Let $T = 1$ and assume $\mathbf{M}$ is of the form in (34) with $MPC_{0,0}, MPC_{1,1} \in (0,1)$. Then, our general decomposition in (31) takes the form*

$$d\mathbf{Y} = \frac{1}{1-\eta}\partial\mathbf{Y} + \frac{\lambda}{1-\eta}\begin{bmatrix} 1-MPC_{1,1} \\ 1-MPC_{0,0} \end{bmatrix}$$

where $\eta \equiv MPC_{00} + MPC_{11} - 1 < 1$. When it is imposed that the terminal output gap is zero, $d\mathbf{Y}_1 = 0$, this gives

$$d\mathbf{Y}_0 = \frac{1}{1-MPC_{0,0}}\partial\mathbf{Y}_0. \tag{35}$$

Propostion 2 is the analogue of Theorem 1 and Corollary 2 for the two period case. A very simple result emerges: The first period output response behaves precisely as predicted by a traditional Old-Keynesian cross. One may wonder how this is possible given that the Old-Keynesian cross is neither microfounded nor closed? The logic is slightly different in the intertemporal version: Any partial equilibrium response $\partial\mathbf{Y}$ by itself is zero NPV. Yet if for example the second period response $\partial\mathbf{Y}_1$ is negative, then the equilibrium selection rule requires an intervention (in practice this might happen through a Taylor rule, as mentioned above), lifting both periods' aggregate demand upwards. Through this mechanism, the zero NPV partial equilibrium response ∂Y is turned into a nonzero NPV general equilibrium response, with an amplified first period response $d\mathbf{Y}_0$. Remarkably, this amplification factor is the exact same as in the traditional Keynesian cross, $1/(1 - MPC_{0,0})$.

We wish to point out that this discussion treated the partial equilibrium response $\partial\mathbf{Y}$ as exogenous. Yet, as we shall see in the following subsections, to evaluate the general equilibrium responses from specific shocks or interventions, such as a rise in government spending, the specific assumptions on the underyling economic environment (preferences, constraints, etc) will matter. Not just does the partial equilibrium response $\partial\mathbf{Y}$ to a given shock depend on the environment, but the MPCs themselves might also influence $\partial\mathbf{Y}$, possibly undoing the amplification we found in (35).

4.2 Unconstrained agents with constant income sensitivities

We start with an economy of N unconstrained agents whose income sensitivities γ_t^i are constant over time, that is, for each agent i, $\gamma_t^i = \gamma^i$ for some γ^i. In that case, all agents only care about the present value of their income, and therefore choose to spend income in the same way, no matter in what period it was earned. Together with the fact that income sensitivities are equal across periods, the definition of the MPC matrix $\mathbf{M}$ in equation (24) shows that in this case the columns of $\mathbf{M}$ are the same. This turns out to simplify our decomposition a lot.

Proposition 3 (Equal columns in M.)**.** *Suppose that all columns in $\mathbf{M}$ are alike and positive, that is, for all s, t it holds that $\mathbf{M}_{t,s} = M_t$ for some $M_t > 0$. Then, $A = 1$ and $\mathbf{v}_t = M_t$ in the decomposition in Theorem (1). In particular, the general equilibrium response to a shock $\partial\mathbf{Y}$ is given by*

$$d\mathbf{Y} = \partial\mathbf{Y} - \frac{\partial\mathbf{Y}_T}{M_T}\mathbf{v}. \tag{36}$$

Proof. Since $\mathbf{M}$ has equal columns, it maps zero NPV vectors to zero. Therefore, $\mathbf{A}\partial\mathbf{Y} = \partial\mathbf{Y}$ for all zero NPV $\partial\mathbf{Y}$, using (32). This means $\mathbf{A} = 1$ is a feasible choice for the decomposition in (31). Moreover, it is straightforward to prove that the column vector (M_t) itself is a right-eigenvector of $\mathbf{M}$ if $\mathbf{M}$ has equal columns. $\square$

The proposition establishes a simple general equilibrium correction in the case of unconstrained agents and constant income sensitivities. There are no higher order effects in $\mathbf{A}$, because the first order effect $\partial\mathbf{Y}$ has no effect the present value of income, which here is the only thing that matters for any higher order effects. The total NPV of the general equilibrium demand response is given by $-\partial\mathbf{Y}_T/M_T$, so, for example, backloaded financing of a stimulative policy tends to increase the NPV of demand more than frontloaded financing. The NPV of $d\mathbf{Y}$ is also larger if agents are more impatient and tend to spend their marginal income in earlier periods, lowering M_T. Finally, any monetary policy rule changes $d\mathbf{Y}$ according to the spending propensities in $\mathbf{v} = (M_t)$.

To bring this decomposition to more life, we now consider two specific policy examples: A government spending shock and a monetary policy shock.

Tax-financed government spending shock. Suppose the government levies lump-sum taxes to pay for an increase in government spending by $G > 0$ in the first period. For simplicity, assume that the tax paid by agent i is proportional to its income sensitivity γ^i. Since agents reduce their consumption accordingly, the partial equilibrium shock is given by $\partial \mathbf{Y}_0 = G(1 - M_0) > 0$ in the first period, and by $\partial \mathbf{Y}_t = -GM_t$ thereafter. By design, $\sum \partial \mathbf{Y}_t = 0$. According to (36), the general equilibrium impact is then

$$d\mathbf{Y}_t = \partial \mathbf{Y}_t + GM_t = \begin{cases} G & \text{for } t = 0 \\ 0 & \text{for } t > 0 \end{cases}. \tag{37}$$

Moreover, the total demand created is precisely equal to G. This result seems to stand in remarkable contrast to our previous two-period result in (35). After all, our government spending result (37) also holds in two periods, so why are they different? The answer to this question lies in the way the partial equilibrium response $\partial \mathbf{Y}$ is determined. For our Old-Keynesian result (35), we assumed a fixed partial equilibrium response $\partial \mathbf{Y}$. Yet, as we illustrate in this simple example, $\partial \mathbf{Y}$ after a government spending shock will depend on MPCs itself. In particular, $\partial \mathbf{Y}_0 = G(1 - M_0)$ is *smaller* if the agents' average MPC to spend in period 0, M_0, is larger. Intuitively, this is because (unconstrained) high MPC agents tend to reduce their consumption more in response to future tax cuts. This effect precisely undoes the amplification effect highlighted in (35).[3]

The effect of government spending here is also reminiscent of the analogous result in Woodford (2010), where he studies government spending in a New-Keynesian model while keeping real rates constant. Our approach emphasizes that this response comes about because the partial equilibrium response $\partial \mathbf{Y}$ is canceled to zero in every period after the first, by an endogenous response of monetary policy in the future.

As we will show below, a key driving force behind the result in (37)—one that is shared by the textbook New-Keynesian model—is the assumption of unconstrained agents. This assumption is also what caused the multiplier in the traditional Keyensian cross (35) to be exactly canceled by the $1 - M_0$ term in the partial equilibrium response.

Monetary policy shock. To tractably illustrate the PE and GE mechanics of a monetary policy shock, we shall make further restrictions on the agents' utility functions U^i. In particular, we assume that all preferences are equal $U^i = U$, and have a constant elasticity of intertemporal substitution σ^{-1}. Letting β_t be the share of income spent on period t, we then have $M_t = \beta_t = \mathbf{v}_t$. Let $d \log R_1$ be the exogenous interest rate change between periods $t = 0$ and $t = 1$. Finally,

[3] We shall note that the government could alternatively also finance its expenditure by cutting future government spending, in which case the partial equilibrium response is not mitigated by any MPC terms and the Old Keynesian cross formula (35) fully applies.

note that aggregate consumption in period t is just given by $\mathbf{Y}_t$. The shock then induces a partial equilibrium response equal to

$$\partial \mathbf{Y}_t = \begin{cases} -(1-\beta_0)\mathbf{Y}_0 \frac{1}{\sigma} d\log R_1 & \text{for } t=0 \\ \beta_t \mathbf{Y}_0 \frac{1}{\sigma} d\log R_1 & \text{for } t>0 \end{cases}.$$

Again, note that $\sum_t \partial C_t = 0$. The total amount of demand generated is now given by $\lambda = -Y_1 \frac{1}{\sigma} d\log R_1$, and the general equilibrium demand response is given by

$$d\mathbf{Y}_t = \begin{cases} -\mathbf{Y}_0 \frac{1}{\sigma} d\log R_1 & \text{for } t=0 \\ 0 & \text{for } t>0 \end{cases},$$

where we note that, again, the monetary policy rule at $t = T$ implies that the simple partial equilibrium response is offset by $(\lambda \mathbf{v})_t = -\beta_t \mathbf{Y}_0 \frac{1}{\sigma} d\log R_1$ in any period $t > 0$. Even if this result, according to which only the initial period is affected, looks analogous to the previous one, it is less robust. It heavily depends on the fact all agents' preferences are the same and homothetic, which is basically equivalent to the assumption of a representative agent.

An interesting insight that can be gleaned from this decomposition is that the general equilibrium impact $d\mathbf{Y}_0/\mathbf{Y}_0$ of the monetary policy shock is independent of the agents' impatience β_0, even though higher impatience reduces the partial equilibrium response $\partial \mathbf{Y}_0/\mathbf{Y}_0$. The reason is an offsetting general equilibrium multiplier: Whenever β_0 is larger, agents are more likely to spend any additional demand created by the monetary policy rule in period $t = 0$.

Forward guidance. We can also turn to our framework to decompose the effect of forward guidance as well. To do this, consider the environment we used for monetary policy. In this environment, an interest rate shock between periods τ and $\tau + 1$ (our simplified forward guidance policy) $d\log R_{\tau+1}$ causes the following partial equilibrium response,

$$\frac{\partial \mathbf{Y}_t}{\mathbf{Y}_t} = \begin{cases} -\left(\sum_{s>\tau} \beta_s\right) \frac{1}{\sigma} d\log R_{\tau+1} & \text{for } t \le \tau \\ \left(1-\sum_{s>\tau} \beta_s\right) \frac{1}{\sigma} d\log R_{\tau+1} & \text{for } t > \tau \end{cases}.$$

It is straightforward to check that this response is indeed zero NPV. Clearly, its percentage impact on the initial period falls in τ, and is therefore smaller, the later the intervention is announced. To calculate the general equilibrium impact of forward guidance, recall that the monetary policy rule dictates that $d\mathbf{Y}_T = 0$ and therefore, the total amount of demand generate is

$$\lambda = -\mathbf{Y}_0 \frac{1-\sum_{s>\tau}\beta_s}{\beta_0} \frac{1}{\sigma} d\log R_{\tau+1}.$$

The general equilibrium demand response is then

$$\frac{d\mathbf{Y}_t}{\mathbf{Y}_t} = \begin{cases} -\frac{1}{\sigma} d \log R_{\tau+1} & \text{for } t \leq \tau \\ 0 & \text{for } t > \tau \end{cases}.$$

This is a remarkable result. However far in the future we announce a monetary policy intervention of a given size, this baseline model predicts the same initial response. This is an incarnation of the forward guidance puzzle.[4] Our approach illustrates that the share of the response being due to partial equilibrium effects is given by $\sum_{s>\tau} \beta_s$ and falls to zero as τ becomes larger. In that sense, the forward guidance puzzle is entirely due to general equilibrium effects.

4.3 Hand to mouth agents

Consider any economy like the one introduced in Section (**??**), with MPC matrix M and a partial equilibrium shock $\partial\mathbf{Y}$. We now explore what happens to $d\mathbf{Y}$ if hand to mouth agents are added to this economy, so that they constitute a fraction $\mu > 0$ of the total population and have a constant income sensitivity γ^{h2m}. We have the following formal result.

Proposition 4 (Adding hand to mouth agents.)**.** *If a measure μ of hand to mouth agents with income sensitivity γ^{h2m} is added to an economy with population size $1 - \mu$ and MPC matrix $\mathbf{M}$, it holds that:*

$$\mathbf{M}^{new} = (1 - \mu\gamma^{h2m})\mathbf{M} + \mu\gamma^{h2m}\mathbf{I},$$

where $\mathbf{I}$ is the $T+1$ dimensional identity matrix. Moreover, in the decomposition in Theorem (1), we now have $\mathbf{A}^{new} = \frac{1}{1-\mu\gamma^{h2m}}\mathbf{A}$ and $\mathbf{v}^{new} = \mathbf{v}$. The decomposition is therefore given by

$$d\mathbf{Y}^{new} = \frac{1}{1 - \mu\gamma^{h2m}}\mathbf{A}\partial\mathbf{Y} + \lambda^{new}\mathbf{v} = \frac{1}{1 - \mu\gamma^{h2m}} d\mathbf{Y}.$$

Proposition 4 shows that adding hand to mouth agents scales up the general equilibrium response by a factor $1/(1 - \mu\gamma^{h2m})$, where $\mu\gamma^{h2m} \in (0, 1)$ is the share of income going to hand to mouth agents in each period. It is important to recognize that his result holds *conditional* on the same partial equilibrium response $\partial\mathbf{Y}$. Of course, in general one would expect $\partial\mathbf{Y}$ to change as well. To illustrate the role of hand to mouth agents more concretely, we assume for the remainder of this subsection that the economy is populated by $1 - \mu$ unconstrained agents with constant income sensitivities as in Section 4.2 and $\mu > 0$ hand to mouth agents. We repeat the two experiments from before.

Government spending shock. Again, suppose the government increases its spending by $G > 0$ in the initial period but now levies lump-sum taxes in future periods on both unconstrained and hand to mouth agents. In particular, suppose unconstrained agents pay a share of taxes $\chi \in [0, 1]$,

[4] See also Del Negro Giannoni Patterson (2015), McKay Nakamura Steinsson (2016), Angeletos Lian (2016), cite more?

again in proportion to their respective income sensitivities, while hand to mouth agents pay share $(1-\chi)\varphi_t$ in period t, where $\sum_{t=1}^{T}\varphi_t = 1$. For simplicity, assume that $\varphi_T = 0$. Here, the distribution across periods is important as hand to mouth agents cannot consolidate their budget constraints and therefore the specific income stream matters, rather than just its present value. Then, the partial equilibrium spending shock is given by $\partial \mathbf{Y}_1 = G - \chi G M_1 - (1-\chi)\varphi_1 G$ in the first period, and by $\partial \mathbf{Y}_t = -\chi G M_t - (1-\chi)\varphi_t G$ thereafter. Applying Proposition 4, the general equilibrium response is

$$d\mathbf{Y}_t = \frac{1}{1-\mu\gamma^{h2m}}\left(\partial \mathbf{Y}_t + \frac{-\partial \mathbf{Y}_T}{M_T}\mathbf{v}_t\right) = \begin{cases} \frac{1}{1-\mu\gamma^{h2m}}\left(G-(1-\chi)G\varphi_t\right) & \text{for } t=0 \\ -\frac{1}{1-\mu\gamma^{h2m}}(1-\chi)G\varphi_t & \text{for } t>0 \end{cases},$$

and the total NPV of demand created is

$$\lambda = \frac{1}{1-\mu\gamma^{h2m}}\chi G.$$

These expressions carry a few important insights. First, assume the tax share on hand to mouth agents is zero, $\chi = 1$. In that case, $d\mathbf{Y}$ is only nonzero in the initial period, but now the initial impact is scaled up by $1/(1-\mu\gamma^{h2m})$. Government spending therefore has a multiplier in excess of 1, different from the textbook New-Keynesian model. This shows that the presence of hand to mouth agents can bring back an "Old-Keynesian" feature that is often used as a criticism of the New-Keynesian approach.

Second, assume the tax share on hand to mouth is close to 1, i.e. $\chi \to 0$. Somehwat surprisingly, the total NPV of demand created then tends zero as well. To explain why notice that the partial equilibrium response has $\partial \mathbf{Y}_T = 0$ in that limit. Therefore the monetary policy rule enforces $d\mathbf{Y}_T = 0$ with $\lambda = 0$. In other words, such a policy implies a temporary boom in the initial period, followed by a bust.

Monetary policy shock. To study monetary policy, we use the same setup that we used to study monetary policy in Section 4.2. In particular, we assume that all unconstrained households share the same preferences $U^i = U$ with a constant elasticity of intertemporal substitution σ^{-1} and share β_t of income spent on consumption in period t. Again we let $d\log R_1$ be an exogenous interest change between periods $t=1$ and $t=2$. Moreover, let ψ^{h2m} be the income share earned by a hand to mouth agent on average, net of possible debt payments. Notice that $\mu\psi^{h2m}$ is then the *average* share of income going to hand to mouth agents, while $\mu\gamma^{h2m}$ is the *marginal* share of any additional income going to hand to mouth agents.

Since hand to mouth agents do not respond to monetary policy, we are in fact in the exact situation of Proposition 4. Therefore, the total initial response after a monetary policy shock is then given by

$$d\mathbf{Y}_1 = -\frac{1-\mu\psi^{h2m}}{1-\mu\gamma^{h2m}}\mathbf{Y}_1\frac{1}{\sigma}d\log R_1,$$

and zero in all other periods. Here, we already expressed the income earned (and spent) by unconstrained agents $\mathbf{Y}_1^{unc}$ as share of the total income, $(1-\mu\psi^{h2m})\mathbf{Y}_1$, after interest payments. This leads us to the following insights.

First, contrary to their role as amplifier for government spending, hand to mouth agents have to opposing effects on the effectiveness of monetary policy: A higher share of hand to mouth agents reduces its effectiveness since those agents do not respond to monetary policy, but it also amplifies any demand increases or decreases in the initial period. A case where they exactly balance is when $\psi^{h2m} = \gamma^{h2m}$, a case that is satisfied if hand to mouth agents do not have any initial assets and where their average income is equal to their marginal income.[5] The wealthier hand to mouth agents are, the larger is their income share (after interest payments) ψ^{h2m}, and therefore the weaker is the aggregate monetary policy response.

Second, our approach sheds light on the role of GE amplification for $d\mathbf{Y}_1$. Even in the case where $\psi^{h2m} = \gamma^{h2m}$, a higher share μ of hand to mouth agents will lead to a smaller partial equilibrium response, $\partial\mathbf{Y}_t$ scales with $1-\mu\psi^{h2m}$, but a larger general equilibrium response, scaling with $1/(1-\mu\gamma^{h2m})$. This decomposition can be used to interpret the recent findings in Kaplan Moll Violante (2016).

5 Infinite-horizon stochastic model

In this section, we generalize the model of the previous sections to an infinite-horizon environment, allowing for uninsured idiosyncratic risk and introducing investment and other elements to make the model more quantitatively realistic.

We show that the key steps of our analysis carry over to this more general environment, as the intertemporal Keynesian cross can be solved to obtain a general equilibrium multiplier operator that carries partial equilibrium impulses $\{\partial Y_t\}$ to general equilibrium outcomes $\{dY_t\}$. There are, however, a few key differences that come with the richer model. First, there is no longer necessarily the same multiplicity of solutions, at least if we require these solutions to be bounded: although the matrix $\mathbf{M}$ still has an eigenvalue of one, this often corresponds to an eigenvector that grows explosively as $t\to\infty$. We show that this is, in particular, the case when the monetary policy feedbacks embedded in $\mathbf{M}$ take a form conventionally associated with determinacy, such as a Taylor rule.

Second, when a partial equilibrium shock directly affects the supply side of the economy—as with, for instance, the capital accumulation response to a monetary shock—it is necessary to frame the analysis in terms of *net* demand impulses $\{\partial Y_t\}$. Although many of the steps remain the same, the interpretation becomes more subtle.

Although our primary analysis is of unanticipated shocks at date 0, this also characterizes to first order the impulse responses with respect to aggregate shocks in a stochastic equilibrium. We observe that the partial equilibrium output volatility in response to a shock is directly tied

[5] The logic here is related to the benchmark case in Werning (2016), where borrowing constraints relax with output.

to the 2-norm $\|\partial Y\|$ of its partial equilibrium impulse response, and the general equilibrium output volatility correspondingly depends on the 2-norm $\|dY\|$ of the resulting general equilibrium sequence. The relationship between $\|\partial Y\|$ and $\|dY\|$—the extent to which the propagation mechanisms in the economy amplify the volatility from partial equilibrium shocks—depends on the structure of the general equilibrium multiplier, which we explore quantitatively.

We show how this and other features of the general equilibrium multiplier can be decomposed between the various components of the model, including heterogenous agents and different components of the matrix $\mathbf{M}$.

We particularly emphasize the transmission of fiscal and monetary policy, and how the general equilibrium amplification of both becomes larger when idiosyncratic risk and constraints are severe. The partial equilibrium impulse from monetary policy, however, generally is attenuated under the same circumstances, implying that the general equilibrium effect from fiscal policy grows in relative terms.

(To be completed.)

A Proofs

A.1 Proof of lemma 2

In this section, we define vectors of partial equilibrium responses $\partial \mathbf{Y}^X$ to different shocks X, and prove that they each satisfy a fundamental NPV-0 property (29), thereby establishing lemma 2.

A.1.1 Preference shocks

Differentiating the intertemporal budget constraint (19) with respect to θ, we see that for all i,

$$\sum_{t=0}^{T} Q_t \frac{\partial c_t^i \left(\left\{y_t^i\right\}, \left\{R_t\right\}; \theta\right)}{\partial \theta} = 0 \tag{38}$$

In the case of preference shocks, we have $\partial \mathbf{Y}_t^\theta = Q_t \sum_i \mu_i \frac{\partial c_t^{i,d}}{\partial \theta}$. Hence, by (38) and (27), we obtain

$$\sum_{t=0}^{T} \partial \mathbf{Y}_t^\theta = \sum_{t=0}^{T} Q_t \left(\sum_i \mu_i \frac{\partial c_t^i}{\partial \theta}\right) = \sum_i \mu_i \left(\sum_{t=0}^{T} Q_t \frac{\partial c_t^i}{\partial \theta}\right) = 0$$

A.1.2 Government spending shocks

Consider a change G_s in government spending at date s. The partial equilibrium effect is defined as

$$\partial \mathbf{Y}_t^{G_s} = Q_t \left(\sum_i \mu_i \frac{\partial c_t^i}{\partial G_s} + 1_{\{t=s\}}\right)$$

Now, for each individual i, we have, using the definition (22) together with (27),

$$\sum_{t=0}^{T} Q_t \frac{\partial c_t^{i,d}}{\partial G_s} = -\sum_{t=0}^{T} Q_t \sum_{u=0}^{T} \frac{\partial c_t^i}{\partial y_u} \frac{\partial \mathcal{T}_u^i}{\partial G_s} = -\sum_{u=0}^{T} Q_u \frac{\partial \mathcal{T}_u^i}{\partial G_s} \sum_{t=0}^{T} \frac{Q_t}{Q_u} \frac{\partial c_t^i}{\partial y_u} = -\sum_{u=0}^{T} Q_u \frac{\partial \mathcal{T}_u^i}{\partial G_s} \sum_{t=0}^{T} MPC_{t,u} = -\sum_{u=0}^{T} Q_u \frac{\partial \mathcal{T}_u^i}{\partial G_s}$$

In other words, the present value reduction in consumption of individual i resulting from an increase in government spending at date s is equal to the present value of the increase in taxes for that individual.

But, differentiating the government budget constraint (15)

$$\sum_{u=0}^{T} Q_u \left(\sum_{i=I}^{n} \mu_i \frac{\partial \mathcal{T}_u^i}{\partial G_s} \right) = Q_s$$

Together, these relationships imply that the date-s value of the aggregate consumption response to an increase in government spending at date s is one:

$$\sum_{t=0}^{T} Q_t \left(\sum_i \mu_i \frac{\partial c_t^{i,d}}{\partial G_s} \right) = -Q_s$$

and hence

$$\sum_{t=0}^{T} \partial \mathbf{Y}_t^{G_s} = -Q_s + Q_s = -Q_s + Q_s = 0$$

A.1.3 Monetary policy shocks

Consider a change in the nominal interest rate at date s, inducing a change in the real interest rate R_s by remark 1. The partial equilibrium effect is defined as

$$\partial \mathbf{Y}_t^{G_s} = Q_t \left(\sum_i \mu_i \frac{\partial c_t^i}{\partial R_s} \right)$$

From (22), the response of individual i at time t has two components: i's direct response to R_s and his indirect response through the effect of the change in interest rates on tranfers.

$$\frac{\partial c_t^{i,d}}{\partial R_s} = \frac{\partial c_t^i}{\partial R_s} - \sum_{u=0}^{T} \frac{\partial c_t^i}{\partial y_u} \frac{\partial \mathcal{T}_u^i}{\partial R_s}$$

Hence, using the same steps as above,

$$\sum_t Q_t \frac{\partial c_t^{i,d}}{\partial R_s} = \sum_t Q_t \frac{\partial c_t^i}{\partial R_s} - \sum_{u=0}^{T} Q_u \frac{\partial \mathcal{T}_u^i}{\partial R_s}$$

Now, for the first part, differentiating the intertemporal budget constraint (19) with respect to R_s, we find

$$\sum_{t=0}^{T} Q_t \frac{\partial c_t^i}{\partial R_s} = \sum_{t=0}^{T} \frac{\partial Q_t}{\partial R_s} \left(y_t^i - c_t^i \right)$$

This implies that, aggregating across agents,

$$\sum_{t=0}^{T} \partial \mathbf{Y}_t^{G_s} = \sum_{t=0}^{T} \frac{\partial Q_t}{\partial R_s} \left(\sum_i \mu_i \left(y_t^i - c_t^i \right) \right) - \sum_{u=0}^{T} Q_u \left(\sum_i \mu_i \frac{\partial \mathcal{T}_u^i}{\partial R_s} \right)$$

We follow the same steps as in section A.1.2, differentiating (15) to find

$$\sum_{u=0}^{T} Q_u \left(\sum_i \mu_i \frac{\partial \mathcal{T}_u^i}{\partial R_s} \right) = \sum_{u=0}^{T} \frac{\partial Q_u}{\partial R_s} \left(G_u - T_u \right)$$

Notice also that

$$\sum_i \mu_i y_t^i = \sum_i \mu_i \left(\frac{W_t^{i,n}}{P_t^n} n_t^{i,n} - T_t^i \right) = \sum_i F_i l_t^i - T_t = Y_t - T_t$$

Combining, we therefore obtain

$$\begin{aligned} \sum_{t=0}^{T} \partial \mathbf{Y}_t^{G_s} &= \sum_{t=0}^{T} \frac{\partial Q_t}{\partial R_s} \left(Y_t - T_t - C_t - G_t + T_t \right) \\ &= \sum_{t=0}^{T} \frac{\partial Q_t}{\partial R_s} \left(Y_t - C_t - G_t \right) \\ &= 0 \end{aligned}$$

where we used goods market clearing at each date (14) to obtain the final equation.