No holdup in dynamic markets*

Matthew Elliott[1] and Eduard Talamàs[2]

[1]*Faculty of Economics, University of Cambridge.*
[2]*Department of Economics, University of Pennsylvania.*

Abstract

In many settings, heterogenous agents make non-contractible investments before bargaining over both who matches with whom and the terms of trade. In thin markets, the *holdup problem*—that is, underinvestment caused by agents receiving only a fraction of the returns from their investments—is ubiquitous. Using a non-cooperative investment and bargaining game, we show that holdup need not be a problem in markets with dynamic entry—even if they are thin at every point in time. This provides non-cooperative foundations for the standard price-taking assumption in matching markets, and shows that intertemporal competition can perfectly substitute for intratemporal competition.

1 Introduction

In many markets, crucial investments are sunk by the time agents bargain over prices and allocations. For example, workers and employers invest in human and physical capital well before bargaining over who will match with whom and for what wages. This can lead to holdup problems—that is, agents underinvesting because they do not expect to fully appropriate the returns from their investments (e.g., Williamson 1975; Grout 1984; Grossman and Hart 1986; Tirole 1986; Hart and Moore 1990) and severely limit the efficiency of these markets (e.g., Hosios 1990; Acemoglu 1996, 1997; Cole, Mailath, and Postlewaite 2001a; de Meza and Lockwood 2010; Elliott 2015; Felli and Roberts 2016).

**Date Printed:* April 17, 2019. We thank Benjamin Golub, George Mailath, Andrew Postlewaite and Larry Samuelson for useful conversations, and various audiences for useful feedback. Financial support from the European Research Council under the grant "EMBED" #757229 (Elliott), and the Warren Center for Network & Data Sciences and the Rockefeller Foundation #2017PRE301 (Talamàs) is acknowledged. All errors are ours.

What market forces might prevent such holdup problems? Informally, the standard answers involve the market being sufficiently thick—in the sense that each market participant has many close substitutes, and hence can plausibly take the prices that she faces as being independent from her investments.[1] For example, the literature investigating the efficiency of investments under competitive matching (e.g., Cole, Mailath, and Postlewaite 2001b; Peters and Siow 2002; Mailath, Postlewaite, and Samuelson 2013 and 2017; Nöldeke and Samuelson 2015; Chiappori, Salanié, and Weiss 2017; Chiappori, Dias, and Meghir 2018; Dizdar 2018) assumes holdup problems away by focusing on markets featuring a continuum of price-taking agents on each side, and Acemoglu and Shimer (1999) show that holdup need not be a problem in directed search environments also with a continuum of agents on each side.[2]

While, in practice, many markets are not often particularly thick—so the arguments above do not readily apply—many markets feature substantial inflows and outflows of agents. For example, at any given time, most employers cannot find more than a handful of appropriate available candidates and, similarly, most candidates cannot find more than a handful of appropriate job openings. Over time, however, most employers can interview a large number of appropriate candidates and, similarly, most candidates can interview for a large number of appropriate job openings. This motivates the questions that we investigate in this paper: To what extent can future entry generate present competition? Can intertemporal competition substitute for intratemporal competition? In particular, can markets that appear thin at every point in time nevertheless be sufficiently competitive to prevent holdup?

The contribution of this paper is twofold. First, we show that intertemporal competition can perfectly substitute for intratemporal competition. In particular, we show that dynamic entry in a two-sided matching market can allow everyone to obtain the full returns from their marginal investments—and hence eliminate the holdup problem. Intuitively, even if there are only a few close substitutes for a given agent today, the expected entry of close substitutes in the future creates present competition for this agent. Second, we provide non-cooperative foundations for the widely used price-taking assumption in matching markets (e.g., Chiappori, Iyigun, and Weiss 2009; Nöldeke and Samuelson 2015; Eeckhout and

[1] Gretsky, Ostroy, and Zame (1999) formalize this idea in the context of the transferable utility assignment game by showing that (i) in finite economies, agents generically do not get their marginal products, and (ii) in continuum economies, (small groups of) agents generically get their marginal products. They also formalize the idea that sufficiently large finite assignment economies are generally approximately perfectly competitive.

[2] The competitive matching literature assumes that agents are price takers—precluding holdup—in order to investigate other possible sources of investment inefficiencies like coordination failures, participation constraints, and imperfect information.

Kircher 2018; Chade and Eeckhout 2019). This clarifies the conditions under which the price-taking assumption is appropriate, and suggests that dynamic entry can be an important force driving price-taking behavior. To the best of our knowledge, this paper is the first to deliver either of these two objectives.

We model two-sided matching markets as a non-cooperative investment and bargaining game featuring stochastic inflows and outflows of agents. There are finitely many types of agents—with potentially few agents of each type in the market at any point in time (that is, limited intratemporal competition within types). Prior to entering the market, agents on both sides have access to a rich investment technology. Agents' investments shape their matching surpluses. These investments can include general and type-specific investments. Different types can have access to different investments. Investments are not contractible, and hence sunk by the time agents enter the market. Once in the market, agents bargain according to a standard protocol in the spirit of Rubinstein (1982): In each period, one agent is randomly selected to be the proposer. The proposer chooses whom to make an offer to as well as how to split the resulting surplus. The agent receiving the offer then decides whether to accept it—in which case she matches with the proposer (and both leave the market)—or reject it—in which case no match occurs in this period.

We characterize the type-symmetric Markov-perfect equilibria of this game for all sufficiently high discount factors, and we show that there is no holdup problem in any such equilibrium—in the sense that no investment deviation by any agent affects anyone else's payoffs, and hence each agent obtains the full returns of any unilateral investment deviation that she makes.[3] The intuition is roughly as follows: On the one hand, if one chooses to invest more than the other agents of her type, she can play off the agents on the other side of the market to obtain the full marginal returns from this deviation. On the other hand, if one chooses to invest less than the other agents of her type, the future entry of agents of her type allows the agents on the other side to ignore her without any payoff consequences, so she again obtains the full (negative) marginal returns from this deviation. In other words, even while engaging in decentralized non-cooperative bargaining in a market that may appear thin at every point in time, everyone is a price taker—and hence a residual claimant of the surplus created or lost by any unilateral investment deviation. As a result, everyone's private and social incentives to invest are perfectly aligned.

A natural benchmark for our work is the literature on the *assignment game* with ex ante

[3]In a type-symmetric Markov-perfect equilibrium, all agents of a given type follow the same strategy—which conditions only on the (current and expected future) matching opportunities in the market.

non-contractible investments.[4] There is a large literature—dating back at least to Becker (1975)—that investigates the holdup problem in this setting. An important message from this literature is that only under special circumstances can everyone fully appropriate the social value of their investments. To illustrate this point, assume that, once investments are sunk, a Walrasian outcome of the resulting matching market is selected. In such markets, there are typically many Walrasian equilibria, and these support a continuum of possible payoffs for each agent. Leonard (1983) (see also Demange 1982) shows that an agent fully appropriates the social value of her investments if and only if she receives her highest possible payoff among all the Walrasian equilibria. Hence, a natural way to guarantee investment efficiency is to give each agent her best possible Walrasian payoff. Unfortunately, however, this is usually impossible, since it requires the existence of a unique Walrasian equilibrium that pins down all prices—a situation that, as shown by Gretsky, Ostroy, and Zame (1999), is not generic in finite markets.[5]

This paper contributes to the literature investigating the conditions under which matching markets are competitive—in the sense that its members are price takers (e.g., Gretsky, Ostroy, and Zame 1999; Cole, Mailath, and Postlewaite 2001a). For example, in finite matching markets with unidimensional attributes and complementarities in these attributes, Cole, Mailath, and Postlewaite (2001a) provide a condition, called "doubly overlapping attributes", that guarantees that there is an essentially unique stable outcome, and that the associated prices continue to clear the market after any unilateral investment deviation. Under these conditions, agents are price takers—in the sense that no unilateral change in attributes affects the market prices—and, as a result, efficient non-contractible investments can be supported in equilibrium. We take a dynamic approach to address similar questions, and we find that essentially no restrictions on the nature of the investments and resulting matching surpluses are required to preclude holdup problems when agents are sufficiently patient.

In order to treat attribute choices as a non-cooperative game, Cole, Mailath, and Postlewaite (2001a) introduce a bargaining function that associates a particular stable outcome to each choice of attributes. They show that, for any efficient profile of investments, there is a bargaining function that—by appropriately selecting the stable outcome after any devia-

[4]The assignment game is a static two-sided one-to-one matching market with transferable utility. See for example Shapley and Shubik (1972).

[5]It is possible to simultaneously give everyone on one side of the market her maximum possible Walrasian equilibrium payoff, but this requires also giving everyone on the other side of the market her minimum possible Walrasian equilibrium payoff (Shapley and Shubik 1972). Nevertheless, if only one side of the market has investment opportunities, this selection can support efficient investments (e.g., Kranton and Minehart 2001; Hatfield, Kojima, and Kominers 2014; Felli and Roberts 2016).

tion from these investments—supports these investments in equilibrium. However, as they discuss, this approach allows a lot of freedom for choosing the bargaining function, and alternative choices are generally not consistent with efficient equilibrium investments. An advantage of our non-cooperative approach is that it essentially uniquely pins down bargaining outcomes as a function of investments: For any type-symmetric investment profile, the notion of subgame-perfect equilibrium uniquely pins down everyone's payoffs.

Makowski and Ostroy (1995) generalize the First Theorem of Welfare Economics by relaxing the price-taking and market-making assumptions: They consider a finite population model in which individuals choose occupations, and those occupations determine the goods that can be consumed. They show that a version of the First Theorem holds in their environment under two conditions. The first condition requires that everyone fully appropriates the social value of her actions, and the second condition eliminates coordination problems. In this paper, we show that the full appropriability condition is endogenously satisfied in our dynamic non-cooperative bargaining game in the limit as agents become arbitrarily patient.

Several non-contractual solutions to the holdup problem in bilateral settings have been proposed. For example, Gul (2001) shows how holdup need not be a problem for unobservable investments, and Che and Sákovics (2004) show that holdup need not be a problem when the investment and bargaining stages are intertwined.[6] In contrast, we focus on settings in which agents must sink their observable investments before bargaining in a matching market, and we show how intertemporal competition can eliminate the holdup problem.

Roadmap

The rest of this paper is organized as follows. In section 2, we illustrate the main ideas in the context of a simple example. In section 3, we describe the general model and, in section 4, we present and prove our main result. We relegate some details of the analysis in section 2 to Appendix A, and relatively standard results that we use to prove our main result to Appendix B and Appendix C.

[6] Che and Sákovics (2018) investigate the role of contracts in settings in which the investment and bargaining stages are intertwined. See Che and Sákovics (2008) for a brief overview of the literature on the holdup problem.

2 Example

In this section, we present a simple example that captures the essential ideas of this paper. We start, in subsection 2.1, by reviewing the standard holdup problem in the context of an investment and bargaining game featuring two buyers and two sellers.[7] Then, in subsection 2.2, we describe a homologous market with sequential entry, and we illustrate how, in this case, the holdup problem vanishes as agents become arbitrarily patient. For simplicity, in the version of this example featuring sequential entry, we assume that each agent that leaves the market is immediately replaced by an identical agent (or replica).[8]

Later on, we show the analogous *no holdup* result in a more general setting that allows (i) stochastic entry (a relaxation of the replica assumption), (ii) many types of buyers and sellers, and (iii) a rich investment technology including general and type-specific investments.

2.1 Holdup in a market without sequential entry

Let us describe a simple game featuring a standard investment holdup problem. There are two identical buyers, b_1 and b_2, and two identical sellers, s_1 and s_2, with a common discount factor δ. In the first period $t = 0$, they simultaneously make investments. They can choose to either *invest* or to *not invest*. Their investments shape their matching surpluses: When a buyer and a seller match in any period $t = 1, 2, \dots$, they generate

(1)

2	units of surplus	if *both* have invested,
1	unit of surplus	if *only one of them* has invested, and
0	units of surplus	if *none of them* has invested.

Not investing costs zero, and investing costs c, with $1/2 < c < 1$. Hence, efficiency requires that everyone invests if the discount factor δ is sufficiently close to 1.

We focus on the case in which investments at time 0 are not contractible. This requires specifying how the outcome (that is, who matches with whom and how the resulting surplus is shared) is determined as a function of the realized investments. We take a non-cooperative approach: Once the agents have sunk their investments, they bargain according to the fol-

[7]The idea that holdup arises in finite static markets goes through in larger markets as well as in unbalanced markets (i.e., markets with different amounts of buyers and sellers). See for example Gretsky, Ostroy, and Zame (1999) and Cole, Mailath, and Postlewaite (2001a) for general treatments of this idea.

[8]This simplifying assumption has been widely used in the dynamic matching and bargaining literature; see for example Rubinstein and Wolinsky (1985), Manea (2011), Nguyen (2015) and Polanski and Vega-Redondo (2018).

lowing standard protocol (e.g., Elliott and Nava 2019): In each period $t = 1, 2, \ldots$, one of the four agents is selected uniformly at random to be the proposer. If the selected agent has already matched in a previous period, no trade occurs in this period. Otherwise, the proposer chooses one agent on the other side of the market, and makes her a take-it-or-leave-it offer to share their gains from trade. The receiver of this offer then either accepts it—in which case the pair match with the agreed shares—or rejects it—in which case no trade occurs in this period.

This game features a standard *holdup problem*: Each agent pays the full costs of her investment at time $t = 0$, but the thinness of the market implies that she need not fully appropriate the resulting increase in surplus in the matching stage—limiting her incentives to invest efficiently. Indeed, focusing on (Markov) strategies that only condition on the surpluses that the agents that are yet to match can generate, we now argue that there does not exist any Markov-perfect equilibrium featuring efficient investments. For brevity, we focus on the case in which agents are arbitrarily patient.

Towards a contradiction, suppose that an efficient equilibrium exists. Given that the aggregate surplus is at most 4, at least one of the agents has a limit gross payoff that is bounded above by 1. Suppose, without loss of generality, that the limit gross equilibrium payoff of b_1 is bounded above by 1, and consider a deviation by b_1 to *not invest*. We show that b_1's limit gross payoff under this deviation is bounded below by $1/2$—i.e., this deviation reduces her limit gross payoff by at most half of the corresponding reduction in gross aggregate surplus. Since this deviation involves no investment costs, b_1's limit *net* payoff under this deviation is also bounded below by $1/2$, which is strictly higher than $1 - c$ (the upper bound on her limit equilibrium net payoff). Hence, this deviation is profitable.

The key observation driving the argument is that, when everyone but b_1 invests, b_2 does not delay in equilibrium.[9] Hence, b_1 can just wait until b_2 matches, and then share the remaining unit of surplus approximately equally with the remaining seller—as specified by the unique subgame perfect equilibrium at that point. As a result, her payoff is bounded below by $1/2$ in the limit as δ goes to 1. Intuitively, the deviator can hold out until her competitor leaves, at which point she faces a bilateral monopoly situation—where the surplus loss generated by her deviation is shared with another agent (while she pockets all the associated savings).

[9]While the fact that b_2 does not delay in equilibrium seems intuitive enough (it is difficult to imagine how her bargaining position can improve after b_1 leaves), proving this formally requires some care. We relegate the details of the argument to subsection A.1 in Appendix A.

2.2 No holdup in a market with sequential entry

Now consider a homologous market featuring *sequential entry*. In the first period $t = 0$, a continuum of identical buyers and a continuum of identical sellers simultaneously make (non-contractible) investments. As before, they can choose to either *invest* or to *not invest*, and their investments determine the surplus of each match—as specified by (1). Each agent that invests has to pay the investment cost c in the period in which she enters the market.[10] As in the market without sequential entry, when agents are sufficiently patient, it is efficient that everyone invests.

Once the agents have sunk their investments, they bargain according to the following standard protocol (e.g., Talamàs 2019b): In each period $t = 1, 2, \ldots$, there are two active buyers and two active sellers. In particular, in the first period $t = 1$, two buyers and two sellers are selected uniformly at random to be active and, every time a buyer and a seller trade, they leave the market, and a new buyer-seller pair is drawn uniformly at random (from those that are yet to become active) to replace them. Hence, in each period, both the bargaining protocol and the set of surpluses that the active agents can create are exactly as in a subgame that starts in period $t = 1$ of the benchmark setting without sequential entry described above.

We argue that, in stark contrast to the setting without sequential entry, holdup in this game need not be a problem. Indeed, focusing on Markov strategies that condition only on the profile of investments made at $t = 0$ and the investments of the active agents, we show that—when agents are sufficiently patient—there exists an efficient Markov-perfect equilibrium (in which everyone invests).[11] Intuitively, unlike in the case without entry, an agent that chooses to deviate from an efficient equilibrium (by not investing) cannot hold out until she faces a bilateral monopoly situation that allows her to share the surplus reduction generated by her deviation. In fact, we show that the price that such a deviator has to pay in order to match with an agent on the other side of the market is not affected by her deviation. As a result, she faces the full negative consequences of her deviation.

In Appendix C, we show that there exists a Markov-perfect equilibrium of the subgame that starts at $t = 1$ for any profile of investments made in period $t = 0$. Hence, in order to show that there exists an efficient Markov-perfect equilibrium, it is enough to:

[10] This guarantees that, in equilibrium, each agent chooses the investment profile that maximizes her payoffs conditional on entering the market (even if this happens with probability zero).

[11] Furthermore, when agents are arbitrarily patient, every type-symmetric Markov-perfect equilibrium is efficient; see subsection A.2 in Appendix A.

1. Describe Markov-perfect equilibrium strategy profiles in the subgames that start with (i) everyone that is yet to trade (active or inactive) having invested, and (ii) all but one of the active agents having invested.

2. Show that, under any strategy profile that specifies these actions in these subgames and that everyone invests in $t = 0$, any unilateral investment deviation from this strategy profile is unprofitable.

First, let us describe Markov-perfect equilibrium strategies for the subgames where everyone that is yet to trade (active or inactive) has invested: Each agent accepts every offer that gives her at least w, and each proposer offers w to an agent on the other side of the market, who accepts. Each agent must be indifferent between accepting and rejecting an offer that gives her w; that is,

$$w = \delta \left(\frac{1}{4} \underbrace{(2 - w)}_{\text{proposer's payoff}} + \frac{3}{4} \underbrace{w}_{\text{non-proposer's payoff}} \right), \text{ that is, } w = \frac{\delta}{2 - \delta} \to 1. \tag{2}$$

Second, let us describe Markov-perfect equilibrium strategies for every subgame in which all but one agent, who is active, has invested. As soon as the deviator leaves, switch to the strategy just described. While the deviator is active: Each non-deviator (i) offers w (defined by Equation 2) to some other non-deviator, who accepts with probability one, and (ii) accepts an offer if and only if it gives her at least w. The deviator (i) offers w to some non-deviator, who accepts with probability one, and (ii) accepts an offer if and only if it gives her at least w'. The deviator obtains $1 - w$ when she is the proposer, so her cutoff w' must satisfy

$$w' = \delta \left(\frac{1}{4} \underbrace{(1 - w)}_{\text{deviator's payoff when proposer}} + \frac{3}{4} \underbrace{w'}_{\text{deviator's payoff when non-proposer}} \right),$$

or, rearranging,

$$w' = w \frac{2 - 2\delta}{4 - 3\delta} = w \left(1 - \frac{2 - \delta}{4 - 3\delta} \right) = w - \frac{\delta}{4 - 3\delta} > w - 1.$$

Hence, this is indeed an equilibrium, since $2 - w > 1 - w'$ implies that the best the non-deviators can do is to obtain $2 - w$ when they are the proposers. We conclude that the deviator saves c but loses 1 in the limit as δ goes to 1. Hence, for all sufficiently high discount factors, her deviation is not profitable.

3 Model

There is a finite set I of types of agents, and a continuum of agents of each type. The type of an agent determines her investment opportunities and her resulting gains from trade, as specified below. All the agents have a common discount factor $0 \leq \delta < 1$, common knowledge of the game and perfect information about all the events preceding any of their decision nodes in the game.

3.1 Investment

In the first period $t = 0$, all the agents simultaneously choose their investments: Each agent of type i chooses an investment from a finite set $K_i \subset \mathbb{R}^{m_i}$, where $m_i \geq 1$. An agent of type i with investment profile $\boldsymbol{x}_i$ and an agent of type $j \neq i$ with investment profile $\boldsymbol{x}_j$ produce $y(\boldsymbol{x}_i, \boldsymbol{x}_j) > 0$ units of surplus when they match, and the costs of their investments are $c(\boldsymbol{x}_i)$ and $c(\boldsymbol{x}_j)$, respectively. An agent of type i with investment profile $\boldsymbol{x}_i$ generates $y(\boldsymbol{x}_i, \boldsymbol{x}_i) > 0$ in isolation (this can capture her outside options, for example). Each agent pays her investment cost in the period in which she enters the market.

Remark 3.1. *Given that the function y determines the surplus of each match only as a function of the investment profiles of its members, this formulation encodes all the heterogeneities among types via their investment opportunities. This can capture arbitrary heterogeneity among different types of agents. For example, suppose that there are two seller types, i' and i'', and two buyer types, j' and j'', and further that i' is a much better fit for type j' than j'' is, while i'' is a much better fit for type j'' than j' is. To capture this situation, we can simply take the surplus $y(\boldsymbol{x}_i, \boldsymbol{x}_j)$ associated with any investment profile $(\boldsymbol{x}_i, \boldsymbol{x}_j) \in (K_{i'} \times K_{j'}) \cup (K_{i''} \times K_{j''})$ to be high relative to the associated investment costs, and the surplus $y(\boldsymbol{x}_i, \boldsymbol{x}_j)$ associated with any investment profile $(\boldsymbol{x}_i, \boldsymbol{x}_j) \in (K_{i''} \times K_{j'}) \cup (K_{i'} \times K_{j''})$ to be low relative to the associated investment costs.*

Remark 3.2. *We assume that all investments are decided before any bargaining occurs for two reasons. First, this highlights that our mechanism does not rely on intertwining the investment and bargaining stages (as is the case in Che and Sákovics 2004, for example). Second, this substantially simplifies the analysis by allowing us to leverage existing results in the non-cooperative bargaining literature (e.g., Elliott and Nava 2019 and Talamàs 2019b).*

3.2 Non-cooperative bargaining

Once everyone chooses her investment in period $t = 0$, bargaining occurs in discrete periods $t = 1, 2, \ldots$. For each type i, there are $n_i \geq 2$ *bargaining slots*. In any given period, each slot of a given type can be occupied by one agent of that type, or be empty. We refer to the agents occupying the slots in any given period as the *active agents* in that period, and we denote the total number of slots by $n := \sum_{i \in I} n_i$.

In each period $t = 1, 2, \ldots$, one slot is selected uniformly at random (i.e., each slot is selected with probability $1/n$). If the slot is empty, no trade occurs in this period. Otherwise, its occupant becomes the *proposer*. The proposer a chooses an active agent b (which can be herself) and makes her a take-it-or-leave-it offer specifying a split of the surplus $y(\boldsymbol{x}_a, \boldsymbol{x}_b)$, where $\boldsymbol{x}_a$ and $\boldsymbol{x}_b$ denote agents a and b's investment profiles, respectively. The receiver of this offer can then *accept* or *reject*. If she accepts, then a and b exit the market with the agreed shares, vacating their respective bargaining slots. Otherwise no trade occurs (and no bargaining slots are vacated) in this period.

Remark 3.3. *The assumption that there are finitely many bargaining slots together with* Assumption 3.4 *below prevents the number of agents in the market from either growing without bound or shrinking to zero.*[12]

3.3 Stochastic entry

For each type i and each $s \leq n_i$, at the beginning of each period that starts with s empty bargaining slots of type i, a number $s' \leq s$ is drawn according to a stationary probability distribution q^i_s. Then, s' agents are drawn uniformly at random from those agents of type i that are yet to become active, and these are randomly assigned to different empty slots of type i. Assumption 3.4 substantially simplifies the analysis.

Assumption 3.4. *There are always at least two active agents of each type.*

Assuming that there is always at least *one* active agent of each type simplifies the analysis by guaranteeing that, when all the agents of the same type choose the same investments, payoffs are uniquely determined by our notion of Markov-perfect equilibrium (Proposition B.2). Assuming that there are always at least *two* active agents of each type further simplifies the analysis by guaranteeing that a deviating agent can play off the agents of

[12] Binmore and Herrero (1988) consider these limits in a related dynamic bargaining model.

other types, and that no unilateral investment deviation shrinks the relevant bargaining opportunities of the non-deviators.

Remark 3.5. *Assumption 3.4 holds under fairly mild conditions on the stochastic inflow process. It holds, for example, if there is at least one active agent of each type in period $t = 1$, and $q^i_{n_i-1}(0) = 0$ for each type i. It also holds under the classical replica framework of the literature on non-cooperative bargaining in stationary markets (e.g., Rubinstein and Wolinsky 1985 and 1990; Gale 1987; de Fraja and Sákovics 2001, Manea 2011, Lauermann 2013, Nguyen 2015, Polanski and Vega-Redondo 2018, Talamàs 2019b) as long as there are two or more agents of each type.*[13]

3.4 Histories, strategies and equilibrium

There are three kinds of *histories*. We denote by h_t a history of the game up to—but not including—time t. We denote by $(h_t; i)$ the history that consists of h_t followed by agent i being selected to be the proposer at time t. We denote by $(h_t; i \to j; s)$ the history that consists of $(h_t; i)$ followed by agent i offering a share s to agent j. A *strategy* σ_i for agent i specifies her investment and, for all possible histories h_t, the offer $\sigma_i(h_t; i)$ that she makes following the history $(h_t; i)$ and her response $\sigma_i(h_t; j \to i; s)$.

The strategy profile $\boldsymbol{\sigma}$ is a *type-symmetric Markov-perfect equilibrium* if it induces a Nash equilibrium in every subgame, all the agents of any given type follow the same strategy, and each agent a's bargaining strategy conditions only on (i) the investment profile, (ii) the set $\{y(\boldsymbol{x}_b, \boldsymbol{x}_c) \mid \text{agents } b, c \text{ active}\}$ of surpluses among the active agents, (iii) the set $\{y(\boldsymbol{x}_a, \boldsymbol{x}_b) \mid \text{agent } b \text{ active}\}$ of surpluses that she can create with the active agents, (iv) for each type i that is such that not all agents of type i yet to enter have chosen the same investment, the number of vacant slots of type i, and (v) the going proposal (in the case of a response).[14]

4 No holdup in equilibrium

Theorem 4.1 below shows that an investment profile $(\boldsymbol{x}_i)_{i \in I}$ can be implemented as a type-symmetric Markov-perfect equilibrium for all sufficiently high discount factors if and only if it is constrained efficient—in the sense that no agent, *taking others' payoffs as given and free*

[13]This framework assumes that each agent that leaves the market is immediately replaced by an identical agent.

[14]The number of vacant slots of type i is payoff relevant only when not all agents of type i yet to enter have chosen the same investment.

to choose whom to match with, has a profitable investment deviation. In particular, in the limit as agents become arbitrarily patient, in every type-symmetric Markov-perfect equilibrium, each agent's private benefit from any investment deviation coincides with its social benefit.

As background for this result, we note that, for each type-symmetric investment profile $\boldsymbol{x} := (\boldsymbol{x}_i)_{i \in I}$ and each type i, there exists $V_i(\boldsymbol{x}) > 0$ such that, in every subgame-perfect equilibrium of the subgame that starts at $t = 1$ with the investment profile $\boldsymbol{x}$, $V_i(\boldsymbol{x})$ is the expected equilibrium (gross) payoff at the beginning of each period of each agent of type i (Proposition B.2). We denote the limit of $V_i(\boldsymbol{x})$ as δ goes to 1 by $V_i^\star(\boldsymbol{x})$.[15]

Theorem 4.1. *A type-symmetric Markov-perfect equilibrium with investment profile $\boldsymbol{x} := (\boldsymbol{x}_i)_{i \in I}$ exists for all sufficiently high discount factors if and only if*

$$(3) \qquad \boldsymbol{x}_i \in \underset{\boldsymbol{z}_i \in K_i}{\mathit{argmax}} \left[\max \left(y(\boldsymbol{z}_i, \boldsymbol{z}_i), \max_{j \in I} \left[y(\boldsymbol{z}_i, \boldsymbol{x}_j) - V_j^\star(\boldsymbol{x}) \right] \right) - c(\boldsymbol{z}_i) \right] \textit{for each } i \textit{ in } I.$$

Proof. Necessity: Fix a type-symmetric Markov-perfect equilibrium $\boldsymbol{\sigma}$ with investment profile $(\boldsymbol{x}_i)_{i \in I}$. Let v_i and w_i denote the (gross) expected equilibrium payoff of each active agent of type i in a period in which she is and she is not the proposer, respectively. Given that $\boldsymbol{\sigma}$ is Markov perfect, each agent gets—when she is the proposer—the maximum amount that she can obtain while leaving the receiver indifferent between accepting and rejecting (unless she chooses to match with herself). Hence,

$$v_i = \max \left(y(\boldsymbol{x}_i, \boldsymbol{x}_i), \max_{j \in I} \left[y(\boldsymbol{x}_i, \boldsymbol{x}_j) - w_j \right] \right).$$

Given that each agent is selected to be the proposer with probability $1/n$ and that, in equilibrium, no agent is ever offered more than her expected equilibrium payoff, we have that $w_i = \delta \left(\frac{1}{n} v_i + \frac{n-1}{n} w_i \right)$. Rearranging gives

$$w_i = \chi v_i = \chi \max \left(y(\boldsymbol{x}_i, \boldsymbol{x}_i), \max_{j \in I} \left[y(\boldsymbol{x}_i, \boldsymbol{x}_j) - w_j \right] \right) \text{ where } \chi := \frac{\delta}{n - \delta(n-1)} \to 1 \text{ as } \delta \to 1.$$

Hence, it is enough to show that, for any investment deviation from the equilibrium $\boldsymbol{\sigma}$ by an agent d, and for any agent $a \neq d$ (of type i, say), a's expected equilibrium payoff $\hat{w}_a$ when rejecting an offer from d gets arbitrarily close to w_i as δ goes to 1. Indeed, given that the set of investments is finite, and that w_i converges to $V_i^\star(\boldsymbol{x})$ for each type i, when δ is sufficiently close to 1 each agent a must then choose her investment $\boldsymbol{z}_a$ to maximize

$$\max \left(y(\boldsymbol{z}_a, \boldsymbol{z}_a), \max_{j \in I} \left[y(\boldsymbol{z}_a, \boldsymbol{x}_j) - V_j^\star(\boldsymbol{x}) \right] \right) - c(\boldsymbol{z}_a).$$

[15] Talamàs (2019a) describes a simple algorithm that computes the profile $\boldsymbol{V}^\star(\boldsymbol{x})$ for every investment profile $\boldsymbol{x} := (\boldsymbol{x}_i)_{i \in I}$, and characterizes $\boldsymbol{V}^\star(\boldsymbol{x})$ in terms of the the classical Nash bargaining solution.

Suppose that an agent d of type k deviates from $\boldsymbol{\sigma}$ by investing $\boldsymbol{x}_d \neq \boldsymbol{x}_k$. Given Assumption 3.4, neither the set $\{y(\boldsymbol{x}_a, \boldsymbol{x}_b) \mid \text{agents } a, b \text{ active}\}$ of surpluses among the active agents nor the set $\{y(\boldsymbol{x}_d, \boldsymbol{x}_b) \mid \text{agent } b \text{ active}\}$ of surpluses that the deviator d can generate with the active agents change while the deviator d is active. Hence, given that $\boldsymbol{\sigma}$ is Markov perfect, for each agent a we can let $\hat{w}_a$ be her expected equilibrium payoff when rejecting an offer while d is active. Furthermore, when d is the proposer, she offers $\hat{w}_a$ to some agent a, who accepts with probability one.[16]

Fix an arbitrary type i, and let $a \neq d$ be an agent of type i such that there exists an agent $c \neq a$ with whom the deviator trades with positive probability in equilibrium (Assumption 3.4 ensures that we can find such an agent). We argue that $\hat{w}_a - w_i$ converges to 0 as δ goes to 1. Since $\boldsymbol{\sigma}$ is type symmetric, this implies that, for each agent b of type i, $\hat{w}_b - w_i$ also converges to 0 in the limit as δ goes to 1.

Let $\epsilon > 0$. First, note that $\hat{w}_a \geq w_i - \epsilon$ for all sufficiently high discount factors. This is because agent a can always wait for the deviator to leave, and—once this happens—her expected equilibrium payoff (when rejecting an offer) is w_i. We now argue that $\hat{w}_a \leq w_i + \epsilon$ for all sufficiently high discount factors. For contradiction, suppose otherwise. Given that, as we have just argued, for each type j and each agent b of type j other than the deviator, $\hat{w}_b \geq w_j - \epsilon$ for all sufficiently high discount factors, agent a must be making offers to the deviator for all sufficiently high discount factors. For each such discount factor δ, letting $\pi > 0$ be the probability that the deviator trades with someone other than a when a is not the proposer, we have that

$$\hat{w}_a = \delta \left[\frac{1}{n}(y(\boldsymbol{x}_a, \boldsymbol{x}_d) - \hat{w}_d) + \frac{n-1}{n}\left(\pi w_i + (1-\pi)\hat{w}_a\right) \right]$$

and, given that d can always make offers to a,

$$\hat{w}_d \geq \delta \left[\frac{1}{n}(y(\boldsymbol{x}_a, \boldsymbol{x}_d) - \hat{w}_a) + \frac{n-1}{n}\hat{w}_d \right].$$

If the weak inequality holds with equality, it is easy to check that $\hat{w}_a$ gets arbitrarily close to w_i as δ goes to 1, a contradiction. Otherwise, $\hat{w}_a$ is strictly smaller than w_i for all large enough δ, also a contradiction.

Sufficiency: Proposition C.1 shows that there exists a type-symmetric Markov perfect equilibrium in the subgame starting at $t = 1$ for every choice of agents' investments. Hence, for

[16]Note that agent d deviates at the investment stage only, so $\boldsymbol{\sigma}$ still governs her bargaining strategy. Given that $\boldsymbol{\sigma}$ is Markov, and that the environment is stationary from the point of view of the deviator d, she can obtain a strictly bigger amount when she is the proposer than when she is the receiver, so she leaves the market—by matching to herself or to someone else—with probability one when she is the proposer.

each investment profile $\boldsymbol{z}$, we can pick an equilibrium $\boldsymbol{\sigma}(\boldsymbol{z})$ of the subgame that starts at $t = 1$. Given any investment profile $(\boldsymbol{x}_i)_{i \in I}$, define a strategy profile as follows: All the agents of type i invest $\boldsymbol{x}_i$, and each agent's bargaining strategy given any investment profile $\boldsymbol{z}$ is as specified by $\boldsymbol{\sigma}(\boldsymbol{z})$. This strategy profile is a Markov-perfect equilibrium if no agent has incentives to deviate at the investment stage ($t = 0$), which—as argued in the necessity part of the proof—is guaranteed (for all sufficiently high discount factors) by condition (3). □

Remark 4.2. *The absence of holdup in equilibrium does not imply that every equilibrium involves efficient investments, nor that every efficient investment profile can be implemented in equilibrium. For example, coordination failures can sustain inefficient investment profiles in equilibrium for all sufficiently high discount factors. To see this, consider the example described in subsection 2.2, except that the surplus of any match is* 0 *unless both sides invest (in which case this surplus is* 2*). It is still efficient that everyone invests (for sufficiently high discount factors), but there is an (inefficient) equilibrium in which no one invests. Participation constraints can also prevent the existence of an equilibrium that implements an efficient type-symmetric investment profile. See for example Cole, Mailath, and Postlewaite (2001a, 2001b) and Nöldeke and Samuelson (2015) for an investigation of these sources of inefficiency in a competitive matching environment (that precludes holdup problems).*

Appendices

A Details omitted from section 2

A.1 Details omitted from subsection 2.1

We show formally the key observation from the example in subsection 2.1: When everyone but b_1 invests, b_2 does not delay in equilibrium. Towards a contradiction, suppose that, when b_2 is the proposer, she delays (makes an unacceptable proposal) with probability $\pi > 0$, she makes an acceptable offer to s_1 with probability $(1-\pi)\beta$, and she makes an acceptable offer to s_2 with probability $(1-\pi)(1-\beta)$.

First note that, since b_2 delays, no one else can delay. To see this, let w_i denote i's expected equilibrium gross payoff in any given period conditional on not trading in this period. It follows from $\min\{w_{b_2}+w_{s_1}, w_{b_2}+w_{s_2}\} \geq 2$ (which holds because b_2 delays) and $w_{b_1}+w_{s_2}+w_{b_2}+w_{s_1} < 3$ (which holds because the aggregate discounted surplus is below 3) that $w_{b_1}+w_{s_1} < 1$ and $w_{b_1}+w_{s_2} < 1$. Hence, in equilibrium, b_1 makes an acceptable offer with probability one to either s_1 or s_2 when she is the proposer. Moreover, by the same argument, the sellers both make acceptable offers to b_1 with probability one when they are the proposers.

Second, letting w be the quantity that a seller that is yet to match is indifferent between accepting and rejecting in a subgame in which b_1 has already matched, the fact that b_2 delays implies that $w_{s_1} > w$. We have that

$$w = \delta\left(\frac{1}{4}(2-w)+\frac{3}{4}w\right) \tag{4}$$

and that

$$w_{s_1} = \delta\left(\frac{1}{4}\underbrace{(1-w_{b_1})}_{s_1\text{ proposes}}+\frac{1}{4}\underbrace{\pi w_{s_1}+(1-\pi)(\beta w_{s_1}+(1-\beta)w')}_{b_2\text{ proposes}}+\frac{1}{4}\underbrace{(\kappa w_{s_1}+(1-\kappa)w)}_{b_1\text{ proposes}}+\frac{1}{4}\underbrace{w}_{s_2\text{ proposes}}\right) \tag{5}$$

where κ denotes the probability that b_1 makes an offer to s_1 in any period before anyone has matched when b_1 is the proposer, and w' is the quantity that a seller that is yet to match is indifferent between accepting and rejecting in a subgame in which b_2 has already matched.

Given that $2-w \geq 1 \geq 1-w_{b_1}$, that $w' < w$, and that $w_{s_1} = w_{s_2}$ unless $\beta = 1$ (because b_2 makes offers only to a seller with the lowest cutoff), the combination of Equation 4 and Equation 5 implies that $w_{s_1} \leq w$, a contradiction.

A.2 Details omitted from subsection 2.2

We show that—for all sufficiently high discount factors—every type-symmetric Markov-perfect equilibrium of the game described in subsection 2.2 is efficient. Suppose for contradiction that there exists a sequence $\mathcal{D}$ of discount factors converging to 1, such that, for each $\delta \in \mathcal{D}$, there exists a type-symmetric Markov-perfect equilibrium $\boldsymbol{\sigma}$ in which only the sellers invest. We show that, for all sufficiently high $\delta \in \mathcal{D}$, a buyer can profitably deviate by investing. A similar argument shows that—when agents are arbitrarily patient—there exists no type-symmetric Markov-perfect equilibrium in which only the buyers invest, or in which neither the buyers nor the sellers invest.

Let $\delta \in \mathcal{D}$ and consider the associated equilibrium $\boldsymbol{\sigma}$. On the equilibrium path, each agent's expected payoff when she rejects an offer satisfies

$$w = \delta \left(\frac{1}{4} \underbrace{(1-w)}_{\text{proposer's payoff}} + \frac{3}{4} \underbrace{w}_{\text{non-proposer's payoff}} \right), \text{ that is, } w = \frac{\delta}{4-2\delta} \to \frac{1}{2}. \tag{6}$$

Suppose that buyer b_1' deviates and invests, and consider a subgame in which b_1' is active. From the point of view of b_1', the environment is stationary. Hence, when she is the proposer, she makes offers that leave the receiver indifferent between accepting and rejecting, and which are accepted with probability one.[17]

Let s be a seller such that there exists a seller $s' \neq s$ with whom the deviator trades with positive probability, and let $\hat{w}_s$ be her expected equilibrium payoff when rejecting an offer from b_1'. We show that, in the limit as δ goes to 1, $\hat{w}_s$ converges to $1/2$. Given that $\boldsymbol{\sigma}$ is Markov perfect and specifies that each agent of the same type follows the same strategy, this implies that every other seller's expected equilibrium payoff when rejecting an offer from b_1' also converges to $1/2$, so when δ is sufficiently high this deviation is profitable (the deviator's net gain is $1-c>0$).

First, we argue that $\hat{w}_s$ is *bounded below* by $1/2$ in the limit as δ goes to 1. Given that the seller s can always wait for the deviator to leave (at which point she obtains $1/2$), for each $\epsilon > 0$, $\hat{w}_s$ is bounded below by $1/2-\epsilon$ for all high enough $\delta \in \mathcal{D}$. A similar argument shows that the expected equilibrium payoff $\hat{w}_b$ of each buyer $b \neq b_1'$ conditional on not being the proposer in a period in which b_1' is active is bounded below by $1/2$ in the limit as δ goes to 1.

Second, we argue that $\hat{w}_s$ is *bounded above* by $1/2$ in the limit as δ goes to 1. Suppose for contradiction that there exists $\epsilon > 0$ such that $\hat{w}_s \geq 1/2+\epsilon$ for all sufficiently high $\delta \in \mathcal{D}$.

[17] Note that we are considering an *investment* deviation from $\boldsymbol{\sigma}$, so the Markov-perfect equilibrium $\boldsymbol{\sigma}$ still governs the deviator's bargaining strategy.

Given that, as we have just argued, for each buyer $b \neq b_1'$, $\hat{w}_b$ is bounded below by $1/2$ in the limit as δ goes to 1, seller s must be making offers to the deviator b_1' for all sufficiently high $\delta \in \mathcal{D}$. For each such discount factor δ, letting $\pi > 0$ be the probability that the deviator trades with someone other than s when s is not the proposer, we have that

$$\hat{w}_s = \delta \left[\frac{1}{4}(2 - \hat{w}_{b_1'}) + \frac{3}{4}\left(\pi w + (1-\pi)\hat{w}_s\right) \right] \text{ and } \hat{w}_{b_1'} = \delta \left[\frac{1}{4}(2 - \hat{w}_s) + \frac{3}{4}\hat{w}_{b_1'} \right],$$

which implies that $\hat{w}_s$ converges to $1/2$ as δ goes to 1, a contradiction.

B Uniqueness of perfect equilibrium payoffs

Proposition B.2 shows that, as long as there is always at least one agent of each type active in the market, the notion of subgame-perfect equilibrium pins down the payoffs of all agents conditional on their (type-symmetric) investment strategies. Moreover, these payoffs are independent of the details of the process by which bargaining slots are filled. This is a slight generalization of the analogous result in Talamàs (2019b), where it is assumed that exactly one agent of each type is active in the market at each point in time. Proposition B.2 holds under the following assumption, which is weaker than Assumption 3.4.

Assumption B.1. *There is always at least one active agent of each type.*

Proposition B.2. *Fix an investment profile $\boldsymbol{x} := (\boldsymbol{x}_i)_{i \in I}$, and suppose that Assumption B.1 holds. For every type i, there exists a value $V_i(\boldsymbol{x}) > 0$ such that, in every subgame-perfect equilibrium with investment profile $\boldsymbol{x}$, the expected equilibrium payoff of each active agent of type i at the beginning of each period is $V_i(\boldsymbol{x})$.*

The proof of Proposition B.2 is identical to the corresponding result in Talamàs (2019b), which is itself similar to the proof of the analogous result in Manea (2017) in the context of a model with random matching (as opposed to the framework with strategic choice of partners that we focus on in this paper). For completeness, we provide this proof here. Proposition B.2 follows from Proposition B.3, since every subgame-perfect equilibrium of a game with perfect information (as the one we study) survives the process of iterated conditional dominance (Theorem 4.3 in Fudenberg and Tirole 1991).

Following Fudenberg and Tirole (1991, page 128), we define iterated conditional dominance on the class of multi-stage games with observed actions as follows.

Definition B.1. Action a_i^t available to some agent i at information set H_t is *conditionally dominated* if every strategy of agent i that assigns positive probability to action a_i^t in the information set H_t is strictly dominated. *Iterated conditional dominance* is the process that, at each round, deletes every conditionally-dominated action given the strategies that have survived all the previous rounds.

Fudenberg and Tirole (1991) show how iterated conditional dominance solves the alternating-offers bilateral model of Rubinstein (1982). Manea (2017) shows how iterated conditional dominance also solves a wide class of models similar to the one considered in this article. We prove Proposition B.3 using the techniques developed in Manea (2017).

Proposition B.3. *Fix an investment profile $(\boldsymbol{x}_i)_{i \in I}$. For every type i, there exists $w_i > 0$ such that, in every game in which Assumption B.1 holds, after the process of iterated conditional dominance, every agent of type i always accepts (rejects) an offer that gives her strictly more (less) than w_i.*

Proof. The proof consists of two steps. First, we define recursively two sequences $(m_i^k)_{i \in I}$ and $(M_i^k)_{i \in I}$, and show by induction on k that after every step s of iterated conditional dominance (see below for a formal definition of such a step), each agent of type i always rejects every offer that gives her strictly less than δm_i^s and always accepts every offer that gives her strictly more than δM_i^s. Second, we show that both sequences $(m_i^k)_{i \in I}$ and $(M_i^k)_{i \in I}$ converge to the same point $(w_i)_{i \in I}$.

We denote the surplus $y(\boldsymbol{x}_i, \boldsymbol{x}_j)$ that a buyer of type i and a seller of type j generate when they match by s_{ij}.

(i) Iterated Conditional Dominance Procedure

Let us start by reviewing how the process of iterated conditional dominance works in the present context. For simplicity, we break up the procedure into steps $0, 1, \ldots$, with each step containing three rounds.

Step 0.

Round 0a. Note that a strategy that ever accepts with positive probability a negative share is strictly dominated by the strategy *reject all offers and make only offers that give me a positive share.* These are all the actions that are eliminated in Round 0a. Hence, after this round *every agent of type i always rejects every offer that gives her strictly less than δm_i^0*, where

(7) $$m_i^0 := 0.$$

Round 0b. Given the actions that survive round 0a, *each agent of type i has an expected payoff (at the beginning of the period, before the proposer has been chosen) of at most M_i^0*, where

$$M_i^0 := \max_j \{s_{ij}\}. \tag{8}$$

because, by assumption, no agent of type j can ever offer any agent of type i a payoff higher than s_{ij}, and, by the actions eliminated in round 0a, no agent ever accepts a negative payoff. Hence, every strategy κ of an agent of type i that ever rejects with positive probability an offer a that gives her strictly more than δM_i^0 is strictly dominated by a similar strategy κ' that specifies *accept a with probability π* in every instance in which κ specifies *reject a with probability π.* These are all the actions that are eliminated in Round 0b; so after this round *every agent of type i always accepts every offer that gives her strictly more than δM_i^0.*

Round 0c. Given the actions that survive rounds 0a and 0b, every strategy κ of every agent of type i that ever makes an offer with positive probability that gives $y > \delta M_j^0$ to an agent of type j is strictly dominated by a similar strategy κ' that specifies *offer $y - \epsilon > \delta M_j^0$ to agent j with probability π* in every instance in which κ specifies *offer y to an agent of type j with probability π*, since every agent of type j must accept both y and $y - \epsilon$. These are all the actions that are eliminated in round 0c; after this round *no agent ever makes an offer giving $y > \delta M_j^0$ to any agent of type j.*

Proceeding inductively, imagine that, after step $s = k \in \mathbb{Z}_{\geq 0}$, we have concluded (as we have just done for the case $s = 0$) that every agent of type i:

1. rejects every offer that gives her strictly less than δm_i^s,
2. has an expected payoff (at the beginning of each period) of at most M_i^s,
3. accepts every offer that gives her strictly more than δM_i^s, and
4. does not make offers that give strictly more than δM_j^s to any agent of type j.

We now show that points (1) to (4) also hold at step $s = k + 1$.

<u>Step $k + 1$.</u>

We refer to strategies that assign positive probability only to actions that have survived all previous rounds of iterated conditional dominance as "surviving strategies."

Round (k+1)a. Given the surviving strategies, it is conditionally dominated for any agent of type i to ever accept an offer that gives her a surplus strictly lower than δm_i^{k+1}, where m_i^{k+1}

is defined as follows:

$$m_i^{k+1} := \frac{1}{n} \max \left(\max_{j \in N} (s_{ij} - \delta M_j^k), \delta m_i^k \right) + \frac{n-1}{n} \delta m_i^k \tag{9}$$

To see this, consider a period-t subgame where an agent of type i has to respond to an offer $x < \delta m_i^{k+1}$. We argue that, for sufficiently small $\epsilon > 0$, accepting this offer is conditionally dominated by the following plan of action—which is designed to give her a time-t expected payoff that approaches δm_i^{k+1} as ϵ goes to 0: *Reject all offers received at dates $t' \geq t$. When selected to be the proposer at time t', offer $\delta M_j^{k+t+1-t'} + \epsilon$ if $t' \in [t+1, t+k+1]$ and $\max_{j \in N} (s_{ij} - \delta M_j^{k+t+1-t'}) > \delta m_i^{k+t+1-t'}$, and make an unacceptable offer otherwise (e.g. offer a negative amount to some agent).* Note that since $t' \geq t+1$, we have that $k+t+1-t' \leq k$. Hence, by the induction hypothesis, all agents j accept the offer $\delta M_j^{k+t+1-t'} + \epsilon$ at period $t' \in [t+1, t+k+1]$. Moreover, note that Equation 9 can be written as

$$m_i^{k+1} = \begin{cases} \delta m_i^k & \text{if } \max_{j \in N} (s_{ij} - \delta M_j^k) \leq \delta m_i^{k+t+1-t'} \\ \frac{1}{n} \max_{j \in N} (s_{ij} - \delta M_j^k) + \frac{n-1}{n} \delta m_i^k & \text{otherwise} \end{cases} \tag{10}$$

and an analogous equation can be used to expand the term m_i^k in Equation 10, and then m_i^{k-1} in the resulting equation, and so on until reaching $m_i^0 = 0$. It is clear from the resulting formula for m_i^{k+1} that, under the surviving strategies, the strategy constructed above generates an expected period-t payoff for i of δm_i^{k+1} as $\epsilon \to 0$. Hence, letting $\epsilon > 0$ be sufficiently small, this strategy conditionally dominates accepting x in period t. These are the actions eliminated in round (k+1)a; after this round *no agent of type i ever accepts any offer that gives her a surplus lower than δm_i^{k+1}*.

Round (k+1)b. Given the surviving strategies, it is conditionally dominated for any agent of type i to reject an offer that gives her strictly more than δM_i^{k+1}, where M_i^{k+1} is defined by

$$M_i^{k+1} := \frac{1}{n} \max \left(\max_{j \in N} (s_{ij} - \delta m_j^k), \delta M_i^k \right) + \frac{n-1}{n} \delta M_i^k \tag{11}$$

To prove this, we show that for each agent of type i, all surviving strategies deliver expected payoffs of at most M_i^{k+1} at the beginning of period t. First, consider a period-t subgame where i is the proposer. Note that i cannot make an offer that generates an expected payoff greater than

$$\max \left(\max_{j \in N} (s_{ij} - \delta m_j^k), \delta M_i^k \right).$$

To see this note that, under the surviving strategies, all agents of type j reject all offers lower than δm_j^k, and when an agent of type j rejects an offer, every agent o type i can expect a period-$(t+1)$ payoff of at most M_i^k. Second, consider a period-t subgame where an agent of type i is not the proposer; under the surviving strategies, this agent can expect a period-t payoff of at most M_i^k. Therefore, *agent of type i has an expected payoff (at the beginning of each period) of at most M_i^{k+1}*. These are all the actions that are eliminated in round (k+1)b; after this round, *no agent ever offers strictly more than δM_j^{k+1} to any agent of type j.*

Round (k+1)c. Given the surviving strategies, every strategy κ of agent of type i that ever makes an offer that gives $y > \delta M_j^{k+1}$ to agent of type j is strictly dominated by a similar strategy κ' that specifies *offer $y - \epsilon > \delta M_j^{k+1}$ to agent of type j with probability* π in every instance in which κ specifies *offer y to agent of type j with probability* π, since every agent of type j must accept both y and $y - \epsilon$. These are all the actions that are eliminated in round (k+1)c; after this round *no agent ever makes an offer giving $y > \delta M_j^{k+1}$ to any agent of type j.*

(ii) The sequences $(m_i^k)_{i \in N}$ and $(M_i^k)_{i \in N}$ converge to the same limit.

First, we prove by induction on k that for all $i \in N$, the sequence $(m_i^k)_{k \geq 0}$ is increasing in k, the sequence $(M_i^k)_{k \geq 0}$ is decreasing in k, and $\max_{j \in N} (s_{ij}) \geq M_i^k \geq m_i^k \geq 0$ for all $k \geq 0$. This implies that both sequences $(m_i^k)_{i \in N}$ and $(M_i^k)_{i \in N}$ converge.

Note that $m_i^0 = 0$ and $M_i^0 := \max_j \{s_{i,j}\}$, and that Equation 9 and Equation 11 imply that $m_i^1 \geq 0$ and $M_i^1 \leq \max_j \{s_{i,j}\}$, so $m_i^1 \geq m_i^0$ and $M_i^1 \leq M_i^1$. Now suppose that for some $l \in \mathbb{N}$:

$$m_i^l \geq m_i^{l-1} \text{ and } M_i^l \leq M_i^{l-1}.$$

We show that

$$m_i^{l+1} \geq m_i^l \text{ and } M_i^{l+1} \leq M_i^l.$$

Note that, by the induction hypothesis, every summand in Equation 9 when $k = l + 1$ is smaller than when $k = l$, which implies that $m_i^{l+1} \leq m_i^l$. Similarly, every summand in Equation 11 when $k = l + 1$ is bigger than when $k = l$, which implies that $M_i^{l+1} \geq M_i^l$. Hence, the sequence $(m_i^k)_{k \geq 0}$ is increasing in k and the sequence $(M_i^k)_{k \geq 0}$ is decreasing in k, which, implies that

$$\max_{j \in N} (s_{ij}) \geq M_i^k \geq m_i^k \geq 0 \text{ for all } k \geq 0.$$

since $\max_{j \in N} (s_{ij}) = M_i^0 > m_i^0 = 0$.

Second, we show that the sequences $(m_i^k)_{i \in N}$ and $(M_i^k)_{i \in N}$ converge to the same limit. Let D^k be $\max_{i \in N}(M_i^k - m_i^k)$. We show that

$$D^k \leq \left(\max_{j \in N} \delta \right)^k D^0 = \left(\max_{j \in N} \delta \right)^k \max_{j,j' \in N} (s_{jj'})$$

for all $k \geq 0$; that is, that D^k converges to 0 as k grows large. Indeed,

$$\begin{aligned}
D^{k+1} &= \max_{i\in N} \; [M_i^{k+1} - m_i^{k+1}] \\
&= \max_{i\in N} \; \Big[\tfrac{1}{n}\max\Big(\max_{j\in N}(s_{ij} - \delta m_j^k), \delta M_i^k\Big) + (1-\tfrac{1}{n})\delta M_i^k \\
&\qquad -\tfrac{1}{n}\max\Big(\max_{j\in N}(s_{ij} - \delta M_j^k), \delta m_i^k\Big) + (1-\tfrac{1}{n})\delta m_i^k\Big] \\
&= \max_{i\in N} \; \Big[\tfrac{1}{n}\Big[\max\Big(\max_{j\in N}(s_{ij} - \delta m_j^k), \delta M_i^k\Big) - \max\Big(\max_{j\in N}(s_{ij} - \delta M_j^k), \delta m_i^k\Big)\Big] \\
&\qquad +(1-\tfrac{1}{n})\Big[\delta M_i^k - \delta m_i^k\Big]\Big] \\
&\leq \max_{i\in N} \; \Big[\tfrac{1}{n}\Big[\max\Big(s_{ij'} - \delta m_{j'}^k, \delta M_i^k\Big) - \max\Big(s_{ij'} - \delta M_{j'}^k, \delta m_i^k\Big)\Big] \\
&\qquad +(1-\tfrac{1}{n})\Big[\delta M_i^k - \delta m_i^k\Big]\Big] \\
&\leq \max_{i\in N} \; \Big[\tfrac{1}{n}\max\Big(\delta(M_{j'}^k - m_{j'}^k), \delta(M_i^k - m_i^k)\Big) + \tfrac{n-1}{n}\delta D^k\Big] \\
&\leq \max_{j\in N} \; \delta D^k
\end{aligned}$$

where j' in the first inequality is any element of $\underset{j\in N}{\operatorname{argmax}}(s_{i,j} - \delta M_j^k)$, and the second inequality is a consequence of Lemma B.4 below. □

Lemma B.4 (Manea 2017)**.** *For all* $w_1, w_2, w_3, w_4 \in \mathbb{R}$,

$$|max(w_1, w_2) - \max(w_3, w_4)| \leq \max(|w_1 - w_3|, |w_2 - w_4|).$$

C Existence of a type-symmetric Markov-perfect equilibrium

Proposition C.1 below is analogous to the Markov-perfect equilibrium existence proof in Elliott and Nava (2019).

Proposition C.1. *For every investment profile $\boldsymbol{x}$, there exists a strategy profile that is a type-symmetric Markov-perfect equilibrium of the subgame starting in period $t = 1$ with investment profile $\boldsymbol{x}$.*

Proof. Let the *kind* of an agent be determined by her type and her investment profile. Without loss of generality, we can assume that the investment sets $\{K_i\}_{i\in I}$ do not overlap, so we can identify the set of agent kinds by $K := \cup_{i\in I} K_i$, which is finite because each K_i is itself finite. Let m denote the number of elements of K. We abuse terminology by referring to $i \in K$ as "agent i." Let $\mathcal{K}$ be the finite set of all possible profiles of agents that can be active in the market at any given time. We characterize the Markov perfect equilibrium of the subgame that starts at $t = 1$ with any given investment profile, and then use it to show that such an equilibrium exists.

Consider a Markov-perfect-equilibrium strategy profile and its corresponding value function $V : \mathcal{K} \to \mathbb{R}^m$, where $V(K)$ gives each agent's expected equilibrium payoff in any period at the beginning of a period that starts with active agent set K (before any agents become active this period). Consider a subgame with active agent set $K \in \mathcal{K}$, and let s_{ij} denote the surplus that agents i and j generate when they match in this subgame. By Markov perfection, agent j accepts any offer that gives her strictly more than $\delta V_j(K)$, and rejects any offer that gives her strictly less than $\delta V_j(K)$. This implies that no one offers more than $\delta V_j(K)$ to any agent j. Therefore, a proposer i makes offers with positive probability only to j that maximizes her net payoff $s_{ij} - \delta V_j(K)$. Hence, when $i \in K$ is the proposer, the expected payoff of $k \in K\backslash\{i\}$ is

$$\sum_{j \in K\backslash\{i,k\}} \pi_{ij} \delta V_k(K\backslash\{i,j\}) + \left(1 - \sum_{j \in K\backslash\{i,k\}} \pi_{ij}\right) \delta V_k(K)$$

where π_{ij} denotes the probability that i and j agree to trade. When i is the proposer, if there exists $j \in K$ such that $\delta(V_i(K) + V_j(K)) < s_{ij}$, then she makes offers only to $j \in K$ for which $s_{ij} - \delta V_j(K)$ is maximum, and agreement obtains with probability one. Otherwise, she delays—in the sense that she makes offers that are not accepted in equilibrium. We denote the probability that $i \in K$ delays by π_{ii}. Thus, any agreement probability profile $\pi_i(K) \in \Delta(K)$—corresponding to the histories in which i is the proposer—that is consistent with the value function V must be in

$$\Pi^{i,K}(V) = \left\{ \pi_i \in \Delta(K) \;\middle|\; \begin{array}{lll} \pi_{ii} = 0 & \text{if} & \delta V_i < \max\limits_{j \in K\backslash\{i\}} \{s_{ij} - \delta V_j(K)\}, \\ \pi_{ik} = 0 & \text{if} & s_{ik} - \delta V_k(K) < \max\{\delta V_i(K), \max\limits_{j \in K\backslash\{i\}} \{s_{ij} - \delta V_j(K)\} \end{array} \right\}.$$

For any value function V, any $K \in \mathcal{K}$ and any agent $i \in K$, define $f^{i,K}(V) : K \to \mathbb{R}^m$ by

$$\begin{aligned} f_i^{i,K}(V) &= \pi_{ii} \delta V_i(K) + (1 - \pi_{ii}) \max\nolimits_{j \in K\backslash\{i\}} \{s_{ij} - \delta V_j(K)\} \\ f_k^{i,K}(V) &= (\pi_{ii} + \pi_{ik}) \delta V_k(K) + \sum_{j \in K\backslash\{i,k\}} \pi_{ij} \delta V_k(K\backslash\{i,j\}) \qquad \forall k \neq i, \end{aligned}$$

for any $\pi_i \in \Pi^{i,K}(V)$. That is, $f_i^{i,K}(V)$ gives the set of expected payoffs that are consistent with the value function V in any history in which active agent set is K and the proposer is agent i. Letting $\mathcal{V}$ denote the set of value functions $V : \mathcal{K} \to \mathbb{R}^m$, consider the correspondence $F : \mathcal{V} \to \mathcal{V}$ defined by

$$F(V)(K) = \frac{1}{n} \sum_{i \in K} f^{i,K}(V), \text{ for all value functions } V \text{ and all } K \in \mathcal{K}. \tag{12}$$

The value function V corresponds to a Markov-perfect equilibrium payoff profile if and only if $V \in F(V)$. So it is enough to show that the correspondence F has a fixed point. This follows from Kakutani's fixed point theorem (Kakutani 1941). Indeed, the domain $\mathcal{V}$ of F is a non-empty, compact and convex subset of an Euclidean space. Moreover, since, for any $K \in \mathcal{K}$ and any $i \in K$, the correspondence $\Pi^{i,K}$ is upper-hemicontinuous with non-empty convex images, so is the correspondence $f^{i,K}$, and hence so is F. □

References

ACEMOGLU, D. (1996): "A microfoundation for social increasing returns in human capital accumulation," *The Quarterly Journal of Economics*, 111, 779–804.

——— (1997): "Training and innovation in an imperfect labour market," *The Review of Economic Studies*, 64, 445–464.

ACEMOGLU, D. AND R. SHIMER (1999): "Holdups and efficiency with search frictions," *International Economic Review*, 40, 827–849.

BECKER, G. S. (1975): *Human capital: A theoretical and empirical analysis, with special reference to education*, NBER.

BINMORE, K. G. AND M. J. HERRERO (1988): "Matching and bargaining in dynamic markets," *The Review of Economic Studies*, 55, 17–31.

CHADE, H. AND J. EECKHOUT (2019): "Competing teams," *The Review of Economic Studies*.

CHE, Y.-K. AND J. SÁKOVICS (2004): "A dynamic theory of holdup," *Econometrica*, 72, 1063–1103.

——— (2008): "Hold-up problem," *The New Palgrave Dictionary of Economics: Volume 1–8*, 2778–2782.

——— (2018): "To trade, or not to trade, that is the question: New roles for incomplete contracts in dynamic settings," *Mimeo*.

CHIAPPORI, P.-A., M. C. DIAS, AND C. MEGHIR (2018): "The Marriage Market, Labor Supply, and Education Choice," *Journal of Political Economy*, 126, S26–S72.

CHIAPPORI, P.-A., M. IYIGUN, AND Y. WEISS (2009): "Investment in schooling and the marriage market," *The American Economic Review*, 99, 1689–1713.

CHIAPPORI, P.-A., B. SALANIÉ, AND Y. WEISS (2017): "Partner choice, investment in children, and the marital college premium," *The American Economic Review*, 107, 2109–67.

COLE, H., G. MAILATH, AND A. POSTLEWAITE (2001a): "Efficient non-contractible investments in finite economies," *The BE Journal of Theoretical Economics*, 1, 1–34.

——— (2001b): "Efficient non-contractible investments in large economies," *Journal of Economic Theory*, 101, 333–373.

DE FRAJA, G. AND J. SÁKOVICS (2001): "Walras retrouvé: Decentralized trading mechanisms and the competitive price," *Journal of Political Economy*, 109, 842–863.

DE MEZA, D. AND B. LOCKWOOD (2010): "Too much investment? A problem of endogenous outside options," *Games and Economic Behavior*, 69, 503–511.

DEMANGE, G. (1982): "Strategyproofness in the assignment market game," *Laboratoire d'Économétrie de l'École Polytechnique*.

DIZDAR, D. (2018): "Two-sided investment and matching with multidimensional cost types and attributes," *American Economic Journal: Microeconomics*, 10, 86–123.

EECKHOUT, J. AND P. KIRCHER (2018): "Assortative matching with large firms," *Econometrica*, 86, 85–132.

ELLIOTT, M. (2015): "Inefficiencies in networked markets," *American Economic Journal: Microeconomics*, 7, 43–82.

ELLIOTT, M. AND F. NAVA (2019): "Decentralized bargaining in matching markets: Efficient stationary equilibria and the core," *Theoretical Economics*, 14, 211–251.

FELLI, L. AND K. ROBERTS (2016): "Does Competition Solve the Hold-up Problem?" *Economica*, 83, 172–200.

FUDENBERG, D. AND J. TIROLE (1991): *Game theory*, MIT Press.

GALE, D. (1987): "Limit theorems for markets with sequential bargaining," *Journal of Economic Theory*, 43, 20–54.

GRETSKY, N. E., J. M. OSTROY, AND W. R. ZAME (1999): "Perfect competition in the continuous assignment model," *Journal of Economic Theory*, 88, 60–118.

GROSSMAN, S. J. AND O. D. HART (1986): "The costs and benefits of ownership: A theory of vertical and lateral integration," *Journal of Political Economy*, 94, 691–719.

GROUT, P. A. (1984): "Investment and wages in the absence of binding contracts: A Nash bargaining approach," *Econometrica*, 449–460.

GUL, F. (2001): "Unobservable Investment and the Hold-Up Problem," *Econometrica*, 69, 343–376.

HART, O. AND J. MOORE (1990): "Property Rights and the Nature of the Firm," *Journal of Political Economy*, 98, 1119–1158.

HATFIELD, J. W., F. KOJIMA, AND S. D. KOMINERS (2014): "Investment incentives in labor market matching," *American Economic Review P&P*, 104, 436–41.

HOSIOS, A. J. (1990): "On the efficiency of matching and related models of search and unemployment," *The Review of Economic Studies*, 57, 279–298.

KAKUTANI, S. (1941): "A generalization of Brouwer's fixed point theorem," *Duke Mathematical Journal*, 8, 457–459.

KLEIN, B., R. G. CRAWFORD, AND A. A. ALCHIAN (1978): "Vertical integration, appropriable rents, and the competitive contracting process," *The Journal of Law and Economics*, 21, 297–326.

KRANTON, R. E. AND D. F. MINEHART (2001): "A theory of buyer-seller networks," *The American Economic Review*, 91, 485–508.

LAUERMANN, S. (2013): "Dynamic matching and bargaining games: A general approach," *The American Economic Review*, 103, 663–689.

LEONARD, H. B. (1983): "Elicitation of honest preferences for the assignment of individuals to positions," *Journal of Political Economy*, 91, 461–479.

MAILATH, G. J., A. POSTLEWAITE, AND L. SAMUELSON (2013): "Pricing and investments in matching markets," *Theoretical Economics*, 8, 535–590.

——— (2017): "Premuneration values and investments in matching markets," *The Economic Journal*, 127, 2041–2065.

MAKOWSKI, L. AND J. M. OSTROY (1995): "Appropriation and efficiency: A revision of the first theorem of welfare economics," *The American Economic Review*, 808–827.

MANEA, M. (2011): "Bargaining in stationary networks," *The American Economic Review*, 101, 2042–2080.

——— (2017): "Bargaining in dynamic markets," *Games and Economic Behavior*, 104, 59–77.

NGUYEN, T. (2015): "Coalitional bargaining in networks," *Operations Research*, 63, 501–511.

NÖLDEKE, G. AND L. SAMUELSON (2015): "Investment and competitive matching," *Econometrica*, 83, 835–896.

PETERS, M. AND A. SIOW (2002): "Competing premarital investments," *Journal of Political Economy*, 110, 592–608.

POLANSKI, A. AND F. VEGA-REDONDO (2018): "Bargaining frictions in trading networks," *The BE Journal of Theoretical Economics*, 18.

RUBINSTEIN, A. (1982): "Perfect equilibrium in a bargaining model," *Econometrica*, 97–109.

RUBINSTEIN, A. AND A. WOLINSKY (1985): "Equilibrium in a market with sequential bargaining," *Econometrica*, 1133–1150.

——— (1990): "Decentralized trading, strategic behaviour and the Walrasian outcome," *The Review of Economic Studies*, 57, 63–78.

SHAPLEY, L. AND M. SHUBIK (1972): "The assignment game," *International Journal of Game Theory*, 1, 111–130.

TALAMÀS, E. (2019a): "Nash bargaining with endogenous outside options," *Mimeo*.

——— (2019b): "Price dispersion in stationary networked markets," *Games and Economic Behavior*, 115, 247–264.

TIROLE, J. (1986): "Procurement and renegotiation," *Journal of Political Economy*, 94, 235–259.

WILLIAMSON, O. E. (1975): *The Economic Institutions of Capitalism*, New York Free Press.