BOUNDS FOR BEST RESPONSE FUNCTIONS IN BINARY GAMES[1]

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ABSTRACT. This paper studies the identification of best response functions in binary games without making strong parametric assumptions about the payoffs. The best response function gives the utility maximizing response to a decision of the other players. This is analogous to the response function in the treatment-response literature, taking the decision of the other players as the treatment, except that the best response function has additional structure implied by the associated utility maximization problem. Further, the relationship between the data and the best response function is not the same as the relationship in the treatment-response literature between the data and the response function. We focus especially on the case of a complete information entry game with two firms. We also discuss the case of an entry game with many firms, non-entry games, and incomplete information. Our analysis of the entry game is based on the observation of realized entry decisions, which we then link to the best response functions under various assumptions including those concerning the level of rationality of the firms, including the assumption of Nash equilibrium play, the symmetry of the payoffs between firms, and whether mixed strategies are admitted.

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1. Introduction

In *Identification Problems in the Social Sciences*, Manski (1995, p. 110) studies the identification problems "that arise when observations of equilibrium outcomes are used to analyze social interactions." Most important from a historical perspective is the analysis of supply and demand, which Manski points out was called "the" identification problem by Fisher. We focus here on the link between data and response functions in simple games. Many of the identification issues that arise in supply and demand, a social interaction, also arise in the analysis of games, a different social interaction, "when observations of equilibrium outcomes of games" are used to identify players' best response functions.[1] This paper contributes to that area.

We study the problem of the identification of best response functions in binary games. The best response function gives the utility maximizing response to decisions of the other players. This is analogous to the response function in the treatment-response literature, taking the decisions of the other firms as the treatment. The motivating example throughout the paper is an entry game, and especially a two firm entry game with complete information. Economists and particularly the econometrics and industrial organization literatures have routinely used entry game models and other similar models to learn about strategic interaction. These games model the profit of a firm as depending on the entry decisions of the other firms, with the consequence that there is strategic interaction between the firms. In applied work the model is usually parametric and is based on the underlying economic situation that is being studied. See Bresnahan and Reiss (1991a), Berry (1992), Mazzeo (2002),

[1]The analysis in Manski (1995) deals with the case in which the best response functions have a parametric structure that depends on observed covariates, and in which equilibrium is assumed. We make different assumptions.

Tamer (2003), Seim (2006), Beresteanu, Molchanov, and Molinari (2009), Ciliberto and Tamer (2009), Grieco (2009), Aradillas-Lopez (2010), and Bajari, Hong, and Ryan (2010) among many others. In these models problems arise due to the presence in the underlying game of multiple equilibria and mixed strategies, among other things, which complicate the inferential question since they add nuisance parameters that need to be accommodated. See for example Tamer (2003) for more on this. We consider this problem without making parametric assumptions. We give a more complete comparison of our results with the results of the literature in section 4, after reporting our results.

The objects of interest in this paper are the best response functions. This paper uses the convention that for the entry game entering is action 1 and not entering is action 0. Then the best response functions in the two firm entry game are the functions $v^i(t) : \{0,1\} \to \{0,1\}$ for firm $i \in \{1,2\}$. The best response function is a function of the entry decision of the other firm, and gives the utility maximizing entry decision in response to that entry decision.[2] Our analysis of best response functions is motivated by their potential for use in policy analysis. In particular, the best response functions are the relevant objects if a planner is considering regulating the entry decision of one firm and is interested in the reaction of the other firm. The econometrician does not observe the best response functions, but is interested in learning about the best response functions based on the observation of realized entry decisions. The realized entry decisions (the data) result in the probabilities $(P(1,1), P(1,0), P(0,1), P(0,0))$, where $P(y_1, y_2)$ is the probability that firm 1 has realized entry decision y_1 and firm 2 has realized entry decision y_2.

[2]We make a mild assumption on the payoffs that guarantees that this utility maximization problem has a unique solution, so the best response function is well-defined.

Despite the analogy between the best response function and the response function, it turns out that the relationship between the best response function and the data is qualitatively different than the relationship between the response function and the data in the treatment-response literature. In the standard treatment-response literature if treatment t is realized and $\upsilon(\cdot)$ is the response function, then the observed outcome must be $\upsilon(t)$. Without further assumptions, this effectively exhausts the information in the data. That the response is observed only at the realized treatment is the selection problem. This basic model implies in particular that under reasonable regularity conditions like discreteness of the treatments even without any assumptions, something non-trivial can be learned about the distribution of response functions from the data alone. See the worst-case bounds in Manski (1995).

The analogous relationship between the data and the best response function does not hold in our setting. First, without further assumptions on the behavior of the firms the data need not be informative about the best response functions at all. For example, the data could be realizations of arbitrary entry decisions, completely unrelated to the utility maximization problem associated with the best response function. In order to account for this we use a game theory model. This provides us with a useful structuring of the data and makes additional assumptions more transparent. Second, even with the game structure the data does not have the same relationship to the best response function as it does in the treatment-response literature. This is because the best response function concerns the utility maximization problem when one firm is allowed to best respond to the decision of the other firm. In the data this assumption cannot usually be justified, if firms can make decisions simultaneously.[3] For example, it could be that $\upsilon^1(1) = 0 = \upsilon^2(1)$, despite observing

[3]By simultaneously we mean in the game theory sense of without knowledge of the other firms' decisions.

that both firms enter the market, if the firms are playing a mixed strategy. Thus, we do not observe data that is necessarily disciplined by the best response function in the way that the data is disciplined to be realizations of the response function in the treatment-response literature.[4] In particular, this implies that the data can be completely uninformative under weak assumptions, as in our Lemma 2.2. This model can be viewed as a particularly severe, but useful, relaxing of the stable unit treatment value assumption (SUTVA), since the "treatment" of one firm is the "outcome" of the second firm.[5] We then add various plausible assumptions which allow us to draw sharper inferences about the best response functions. I particular, we exploit the identification power of different levels of rationality, and Nash equilibrium play.

This identification problem is related to the question of nonparametric identification in a simultaneous equations model, which has a long history in econometrics. See for example the recent work of Matzkin (2008) and references therein. The defining difference is that in our model we consider a problem with multiple decision makers where the effect of strategic interaction (like implications of Nash equilibrium play) result in possibly multiple predicted outcomes. So, methods developed for nonparametric identification of triangular systems, or other simultaneous systems, though important, are not directly applicable to our setup.

We focus on deriving results in the case of a two firm entry game. However, our method of analysis can be applied to other games. For example, we consider the case of a many firm

[4]Assumptions on the timing of the treatments (e.g., one firm observes the decision of the other firm in the data) can place the problem closer to the usual treatment-response model. However, timing assumptions will typically not be attractive as they are unlikely to hold in the game actually being played by the firms, and in particular assume away the fact that decisions might be made simultaneously, an essential feature we are trying to capture. In addition, timing assumptions can be more complicated since they can involve dynamic considerations.

[5]Heuristically, suppose that there is a treatment in an elaborated model that is common knowledge, and is modeled to directly affect the entry decision of firm 1. Then, for example, if that treatment causes firm 1 to enter, because of the strategic interaction among firms, it has an effect on firm 2, an apparent violation of SUTVA with respect to the treatment in the elaborated model.

entry game in section 3. Further, our method of analysis can be applied to different sets of assumptions than we consider in this paper. In particular, in the conclusions we show that without the assumption of the entry game payoff structure, and no assumption to replace it, much less can be learned. We consider only identification in this paper; we do not consider estimation because the estimation problems are basically standard. We start with the setup and then we provide our main results.

2. Identification of best responses in an entry game

We consider in this section what can be learned from data on entry in a two firm entry game with complete information. Two firms simultaneously decide whether to enter a market. The realized entry decision of firm i is y_i. By convention $y_i = 1$ if firm i enters the market and $y_i = 0$ if firm i does not enter the market. If mixed strategies are admitted there is not an invertible mapping from the realized entry decision to the strategy of the firm. The payoff to firm i when the entry decisions are (y_1, y_2) is $\pi^i(y_1, y_2)$. These payoff functions are common knowledge among the firms in a market, but unobserved by the econometrician. The entry game structure imposes that the payoffs are such that each firm gets 0 payoff if it does not enter the market. Thus, $\pi^1(0, y_2) = 0 = \pi^2(y_1, 0)$. The game is summarized in Table 1 below.

Table 1. Entry game with general payoffs

	$y_2 = 0$	$y_2 = 1$
$y_1 = 0$	$0, 0$	$0, \pi^2(0, 1)$
$y_1 = 1$	$\pi^1(1, 0), 0$	$\pi^1(1, 1), \pi^2(1, 1)$

The object of interest is the best response function of firm 1 to an entry decision of firm 2, and the best response function of firm 2 to an entry decision of firm 1. The best response of firm i when the entry decision of firm $-i$ is t_{-i} is $\upsilon^i(t_{-i})$. The argument t_{-i} of the best response function refers to an entry decision conjectured by the econometrician, not a realized entry decision observed in the data. This paper assumes that the payoffs are in general position, which means that firm i is never indifferent between entering and not entering in response to an entry decision of firm $-i$. This is equivalent to $\pi^1(1,1) \neq 0$, $\pi^1(1,0) \neq 0$, $\pi^2(1,1) \neq 0$ and $\pi^2(0,1) \neq 0$. This implies that the best response functions in this game are:

$$\upsilon^1(t_2) = 1[t_2 = 1]1[\pi^1(1,1) > 0] + 1[t_2 = 0]1[\pi^1(1,0) > 0]$$

and

$$\upsilon^2(t_1) = 1[t_1 = 1]1[\pi^2(1,1) > 0] + 1[t_1 = 0]1[\pi^2(0,1) > 0]$$

Since the payoff functions are random from the perspective of the econometrician, the best response functions are random from the perspective of the econometrician.

2.1. **Objects of interest.** The objects of interest are the best response probabilities:

$$P(\upsilon^1(t_2) = 1) = P(1[t_2 = 1]1[\pi^1(1,1) > 0] + 1[t_2 = 0]1[\pi^1(1,0) > 0] = 1)$$

and

$$P(\upsilon^2(t_1) = 1) = P(1[t_1 = 1]1[\pi^2(1,1) > 0] + 1[t_1 = 0]1[\pi^2(0,1) > 0] = 1)$$

For example, from the perspective of the econometrician $P(\upsilon^1(t_2) = 1)$ is the probability that firm 1 would enter the market if firm 2 were regulated to have entry decision t_2, and firm 1 were allowed to re-optimize its entry decision. This counterfactual random variable is not observed since the observed data does not come from markets in which the entry decision of one firm is known to the other firm. The identification analysis asks what can be learned about the distribution of these best response functions given observations of realized entry decisions. The paper answers this question under various assumptions. Especially, we derive our identification results under assumptions concerning the level of rationality of the firms, including the assumption of Nash equilibrium play. The next section elaborates on that.

2.2. **Behavioral restrictions: levels of rationality and Nash equilibrium.** This paper entertains different assumptions about how firms behave in these markets, and especially about "how rational" they are. This will affect what we are able to learn using data from these markets. In particular, note that if we make no assumptions on the behavior of firms there is no necessary relationship between the data and the utility maximization problem associated with the best response functions.

We use the notion of levels of rationality implicit in the definition of rationalizability introduced by Bernheim (1984) and Pearce (1984).[6] The level of rationality of a firm can be interpreted as a measure of "how rational" that firm is. The levels of rationality start at level 0 rationality. Every strategy is level 0 rational; equivalently, every firm exhibits 0 levels of rationality. A strategy that is a best response to some level 0 strategy of the other firm is level 1 rational; equivalently, a firm that plays such a strategy exhibits 1 level of rationality.[7] In general the levels of rationality are defined recursively such that a strategy

[6]This was also used by Aradillas-Lopez and Tamer (2008) to examine the identification power of equilibrium is parametric setups.

[7]It is important to note here that a strategy, or a firm, can exhibit many different levels of rationality. In particular, level 1 rationality is necessarily also level 0 rationality, and in general level k' rationality is necessarily also level k rationality for $k' \geq k$.

that is a best response to some level k rational strategy of the other firm is level $k+1$ rational. An interpretation of the levels of rationality is that firm i makes a conjecture about the strategy of firm $-i$, and best responds to that conjecture. The sophistication of that conjecture determines the level of rationality. Adapting slightly the words of Fudenberg and Tirole (1991, p. 49), firm 1 can reason like: "I'm playing strategy σ_1 because I think firm 2 is using σ_2, which is a reasonable belief because I would play it if I were firm 2 and I thought firm 1 were using σ_1'." This reflects the reasoning of firm 1 exhibiting 2 levels of rationality. Additional levels of this sort of reasoning increase the level of rationality.

More formally, the set of all strategies for firm i are collected in the set $\mathcal{R}^i(0,\pi) = \Delta^1$. A strategy of firm i is a best response to a conjecture of firm i if, given the distribution over entry decisions implied for firm $-i$ by that conjecture, the strategy of firm i maximizes the expected payoff to firm i. The levels of rationality are then defined recursively from $\mathcal{R}^i(0,\pi)$. Strategies of firm i that are best responses against some conjecture of the strategy of firm $-i$ that is in $\mathcal{R}^{-i}(k,\pi)$ are collected in $\mathcal{R}^i(k+1,\pi)$. That is, for $k \geq 0$, $\mathcal{R}^i(k+1,\pi) = \{\sigma^i \in \Delta^1 : \exists \sigma^{-i} \in \mathcal{R}^{-i}(k,\pi) \text{ s.t. } E_{\sigma^i,\sigma^{-i}}\pi^i(y_1,y_2) \geq E_{\sigma^{i'},\sigma^{-i}}\pi^i(y_1,y_2) \text{ for all } \sigma^{i'} \in \Delta^1\}$. Equivalently, the set $\mathcal{R}^i(k+1,\pi)$ is the set of best responses to $\mathcal{R}^{-i}(k,\pi)$.

A firm i that uses a strategy that is in $\mathcal{R}^i(k,\pi)$ is said to exhibit k levels of rationality; this can be written as $\mathcal{R}k$. Similarly, the strategies in $\mathcal{R}^i(k,\pi)$ are said to exhibit k levels of rationality. The set of strategies for firm i that are consistent with Nash equilibrium play are collected in the set $\mathcal{N}^i(\pi)$. Therefore, the set $\mathcal{N}^i(\pi)$ is the set of strategies such that there is a strategy in $\mathcal{N}^{-i}(\pi)$ that together comprise a Nash equilibrium. The collection of all Nash equilibrium strategy pairs for the two firms is the set $\mathcal{N}(\pi)$. A market that uses a strategy that is in $\mathcal{N}(\pi)$ is said to exhibit Nash equilibrium play. The level of rationality exhibited by one firm is unrelated to the strategy of the other firm in the market, but Nash equilibrium requires coordination in strategies across firms in the market.

The following lemma collects some standard facts about these solution concepts. The first claim in this lemma establishes that a strategy that exhibits k' levels of rationality also exhibits k levels of rationality when $k' \geq k$. The second claim establishes that a Nash equilibrium strategy exhibits k levels of rationality for any k. Finally, the third claim establishes that there are strategies that exhibit k levels of rationality for every k, but that are not Nash equilibrium strategies. The proof is standard and so is omitted.

Lemma 2.1. *If $k' \geq k$ then $\mathcal{R}^i(k', \pi) \subseteq \mathcal{R}^i(k, \pi)$. For any k, $\mathcal{N}^i(\pi) \subseteq \mathcal{R}^i(k, \pi)$. There are payoffs π in general position such that there is a strategy σ^i that satisfies $\sigma^i \in \mathcal{R}^i(k, \pi)$ for all k but $\sigma^i \notin \mathcal{N}^i(\pi)$.*

In appendix A, we show that this definition of level of rationality is equivalent to the one used by Pearce (1984) to characterize rationalizability. Next, we will provide identification results for the best response probabilities.

2.3. **Definition of the identified set.** We assume throughout that we observe a population of realized entry decisions (y_1, y_2). The uncertainty of the econometrician is specified through a probability space $(\Omega, \mathcal{F}, P)$. We assume without further consideration that the entry decisions and payoffs are measurable with respect to this probability space. Also, in the proofs establishing sharpness of the identified sets we also use the fact that we are allowed to construct, for any measurable set $B \in \mathcal{F}$ with positive probability, a finite measurable partition $\{C_k\}$ of B of any cardinality, with arbitrary conditional probabilities $P(C_k|B)$, other than satisfying $\sum_k P(C_k|B) = 1$. This is basically a continuity assumption on the probability measure, and guarantees that there is sufficient richness of the probability space to avoid complications about what probabilities can be achieved from measurable sets. This condition is satisfied in particular by Lebesgue measure.

As noted in the introduction, at least two problems complicate the relationship between the realized entry decisions and the underlying best response functions. These are the presence of multiple equilibria or multiple strategies that are rational to a firm, and the presence of (non-pure) mixed strategies. Both complicate the relationship since they imply that for given payoffs there may be more than one possible realized entry decision.

The identification problem asks what can be learned about the best response functions given knowledge of $P(y_1, y_2)$. We define the joint identified set for the best response probabilities below.

Definition 2.1 (Sharp identified set)**.** *Suppose that the econometrician maintains some set of assumptions about the entry game and the data. The sharp joint identified set for*

$$(\upsilon_{11}, \upsilon_{10}, \upsilon_{21}, \upsilon_{20}) = \big(P(\upsilon^1(1) = 1), = P(\upsilon^1(0) = 1), P(\upsilon^2(1) = 1), P(\upsilon^2(0) = 1)\big)$$

is the set Θ_I of values $(\upsilon_{11}, \upsilon_{10}, \upsilon_{21}, \upsilon_{20})$ such that for each $(\upsilon_{11}, \upsilon_{10}, \upsilon_{21}, \upsilon_{20}) \in \Theta_I$, there are realized entry decisions $y_1(\omega)$ and $y_2(\omega)$ and payoffs $\pi(\omega)$ for each realization of the uncertainty such that: (i) the realized entry decisions have probability distribution the same as the observed probability distribution $P(y_1, y_2)$, (ii) the payoffs $\pi(\omega)$ are consistent with the assumptions, (iii) the realized entry decisions $y_1(\omega)$ and $y_2(\omega)$ could be observed as an outcome of the game given the payoffs $\pi(\omega)$ and the assumptions, and (iv) the payoffs are consistent with the values of $(\upsilon_{11}, \upsilon_{10}, \upsilon_{21}, \upsilon_{20})$.

This defines the sharp joint identified set to be the set of $(\upsilon_{11}, \upsilon_{10}, \upsilon_{21}, \upsilon_{20})$ that can be rationalized by an underlying entry game, $\{y_1(\omega), y_2(\omega), \pi(\omega)\}_{\omega\in\Omega}$, consistent with the data and the assumptions. The first condition is an extremely minimal consistency condition that requires that the rationalization of the data has the same distribution of realized entry decisions as does the data. The second condition requires that the payoffs be consistent with the assumptions, and the third condition requires the same of the realized entry decisions as a function of the payoffs. Finally, the fourth condition is the link between the rationalization of the data and the objects of interest, and requires that for any value of $(\upsilon_{11}, \upsilon_{10}, \upsilon_{21}, \upsilon_{20})$ in the sharp identified set, indeed these payoffs considered in the other conditions imply that value of $(\upsilon_{11}, \upsilon_{10}, \upsilon_{21}, \upsilon_{20})$.

It might be reasonable to add to the definition of the sharp identified set the following additional conditions.

Definition 2.2 (Sharp identified set, additional conditions). *Additionally there are strategies* $\sigma_1(\omega)$ *and* $\sigma_2(\omega)$ *such that: (v)* $\sigma_1(\omega)$ *and* $\sigma_2(\omega)$ *are consistent with the payoffs* $\pi(\omega)$ *and the assumptions, (vi)* $y_1(\omega)$ *and* $y_2(\omega)$ *could be observed as an outcome of the game given* $\sigma_1(\omega)$ *and* $\sigma_2(\omega)$*, and (vii) the distribution of realized entry decisions implied by* σ_1 *and* σ_2 *is the same as the observed probability distribution* $P(y_1, y_2)$*.*

The first two of these additional conditions are implicit in Definition 2.1; the new condition is condition (vii). This requires a consistency between the distribution of entry decisions according to the strategies used to rationalize the data and the observed entry decisions. For example, if the rationalization has that for each realization of the uncertainty the firms use mixed strategies such that both enter the market with probability p, then condition (vii) requires that both firms enters the market with probability p in the data.

This might be a reasonable condition to impose if the econometrician is certain that the realized entry decisions are independent draws from the strategies, but might not be otherwise. This issue relates to deep questions about what it means for a firm to use a mixed strategy and how firms actually decide what action to take given their mixed strategy. The view that a mixed strategy reflects the fact that firms deliberately randomize their action is taken by von Neumann and Morgenstern (1944). This does not necessarily imply, however, that realized entry decisions should be assumed to be independent draws. First, it could be that the way that firms draw from the strategies is somehow correlated across markets. Perhaps firm i decides to enter or not enter by using the randomization from sunspots. This would cause correlation between the entry decisions of firm i across markets, but in each market the marginal strategy would be the same and an equilibrium, as long as firm $-i$ does not observe this sunspot. This would violate condition (vii). Second, the sense of a mixed strategy equilibrium is now increasingly interpreted to be an equilibrium in beliefs (e.g., Harsanyi (1973), Aumann (1987)), rather than an equilibrium in which the firms deliberately randomize their action. This relates to the difficulty with mixed strategy equilibrium that, in the words of Aumann (1987, p. 15), "the reason a player must randomize in equilibrium is only to keep others from deviating; for himself, randomizing is unnecessary" since the player is indifferent between all the actions in the support of its mixed strategy (and indeed possibly actions off the support of its mixed strategy). This sense of a mixed strategy equilibrium does not imply condition (vii). Consequently, in the spirit of worst case bounds it seems more reasonable to take condition (vii) as an additional assumption to maintain, and not part of the basic definition of the sharp joint identified set. We derive below the bounds on the objects of interest when (vii) is imposed and when it is not.

Overall, the identifying power of the additional conditions in Definition 2.2 is surprisingly limited, but is not nothing. If only pure strategies are used to rationalize the data condition

(vii) has no additional identifying power since then it is implied by Definition 2.1. The additional condition (vii) can tighten the identified set with mixed strategies as follows. Suppose that we assume that there is Nash equilibrium play and that in the data at least one joint entry decision has probability zero. Under condition (vii) this requires us to conclude that (for probability one of the uncertainty) both firms use pure strategies. This is because if in a market either firm uses a (non-pure) mixed strategy, under the assumption that payoffs are in general position, both firms use a (non-pure) mixed strategy. This would imply observing all joint entry decisions with positive probability under condition (vii). Then this implies, for example, that when $(1,1)$ is observed, $(1,1)$ is a pure strategy Nash equilibrium. This implies in turn that $\pi^1(1,1) > 0$ and $\pi^2(1,1) > 0$. If we could not conclude that $(1,1)$ is observed from a pure strategy Nash equilibrium, it could be from a mixed strategy Nash equilibrium in which monopoly profits are positive but duopoly profits are negative. This argument applies to the apparently non-generic case that at least one joint entry decision has probability zero. The proof of Lemma 2.2 shows that, at least without assumptions beyond the assumption of Nash equilibrium play, as long as all joint entry decisions are observed with positive probability, it cannot be ruled out that all realized entry decisions are the outcome of mixed strategy Nash equilibrium play consistent with the additional conditions in Definition 2.2.

Consequently we use the definition of the sharp joint identified set in Definition 2.1, and note some changes under the addition of the conditions in Definition 2.2.

2.4. **Identification of best response functions.** Our first assumption formalizes the assumption that payoffs are in general position. This assumption allows us to avoid dealing with non-generic cases in which the firms are indifferent between entering and not entering, even if the entry decision of the other firm were known.

Assumption 2.1. *Let the following hold:*

$$\pi^1(1,1) \neq 0; \quad \pi^1(1,0) \neq 0; \quad \pi^2(1,1) \neq 0; \quad \textit{and} \quad \pi^2(0,1) \neq 0$$

Under only this assumption we find the following negative result about the identification of the best response probabilities.

Lemma 2.2. *Let Assumption 2.1 hold. Assume further that there is Nash equilibrium play in each market. The following holds:*

(1) *The sharp identified sets for* $P(\upsilon^1(1) = 1)$, $P(\upsilon^1(0) = 1)$, $P(\upsilon^2(1) = 1)$, *and* $P(\upsilon^2(0) = 1)$ *are* $[0, 1]$.

(2) *Let the additional conditions in Definition 2.2 hold.*

- *If all four joint entry decisions have positive probability, then the sharp identified sets for* $P(\upsilon^1(1) = 1)$, $P(\upsilon^1(0) = 1)$, $P(\upsilon^2(1) = 1)$, *and* $P(\upsilon^2(0) = 1)$ *remain* $[0, 1]$.
- *Suppose that at least one of the joint entry decisions has zero probability. Then with probability one the firms use pure strategies, and the sharp identified sets are* $P(\upsilon^1(1) = 1) = [P(1,1), 1 - P(0,1)]$, $P(\upsilon^1(0) = 1) = [P(1,0), 1 - P(0,0)]$, $P(\upsilon^2(1) = 1) = [P(1,1), 1 - P(1,0)]$, *and* $P(\upsilon^2(0) = 1) = [P(0,1), 1 - P(0,0)]$.

Proof. (1): Consider payoff functions π_1 and π_2 such that in π_1 both firms have positive monopoly profits and negative duopoly profits, and in π_2 both firms have negative monopoly profits and positive duopoly profits. For either payoff function, since for firm i the monopoly payoff is on the opposite side of zero from the duopoly payoff, there is a (non-pure) mixed strategy of firm $-i$ that gives i payoff 0 to entering. When firm $-i$ enters with that probability, firm i is indifferent between entering and not entering. Thus, there is a (non-pure) mixed strategy Nash equilibrium. That mixed strategy is $\sigma_1 = \frac{\pi^2(0,1)}{\pi^2(0,1) - \pi^2(1,1)}$ and $\sigma_2 = \frac{\pi^1(1,0)}{\pi^1(1,0) - \pi^1(1,1)}$. This implies that for any realization of the uncertainty ω, since $\pi(\omega)$ can be specified to be either π_1 or π_2, the realized entry decisions $y_1(\omega)$ and $y_2(\omega)$ can be specified to take any of the four logically possible combinations of values.

In particular, let $p \in [0,1]$ be given. Take any set $B \in \mathcal{F}$ such that $P(B) = p$. Specify the payoff function to be π_1 on B and π_2 on B^C. Let $A_{1,1}, A_{1,0}, A_{0,1}, A_{0,0} \in \mathcal{F}$ be a partition of Ω such that $P(A_{1,1}) = P((1,1))$, $P(A_{1,0}) = P((1,0))$, $P(A_{0,1}) = P((0,1))$ and $P(A_{0,0}) = P((0,0))$. Specify that the realized entry decisions (y_1, y_2) are $(1,1)$ on $A_{1,1}$, $(1,0)$ on $A_{1,0}$, $(0,1)$ on $A_{0,1}$, and $(0,0)$ on $A_{0,0}$. These payoffs and realized entry decisions satisfy all of the consistency requirements in Definition 2.1. And, $P(\pi^1(1,1) > 0) = P(\pi^2(1,1) > 0) = 1 - p$ and $P(\pi^1(1,0) > 0) = P(\pi^2(0,1) > 0) = p$. This gives the claim, since p is arbitrary.

(2): First suppose that all four joint entry decisions have positive probability. Then it is possible to find four pairs of (non-pure) mixed strategies (σ_1^s, σ_2^s) with $0 < \sigma_1^s, \sigma_2^s < 1$ for $s = 1,2,3,4$ such that there are probabilities p^s such that

$$(P(1,1), P(1,0), P(0,1), P(0,0)) = \sum_s p^s \left(\sigma_1^s \sigma_2^s, \sigma_1^s(1-\sigma_2^s), (1-\sigma_1^s)\sigma_2^s, (1-\sigma_1^s)(1-\sigma_2^s)\right).$$

This can be accomplished as follows. It is an obvious result that any point in the 3-simplex is a convex combination of the vertices, which are the standard basis for $\mathbb{R}^4$. That is, any point in the 3-simplex can be written as a convex combination of $(1,0,0,0)$, $(0,1,0,0)$, $(0,0,1,0)$, and $(0,0,0,1)$. Now consider perturbing these vertices slightly towards the interior of the 3-simplex. Any point on the interior of the 3-simplex can be written as a combination of the perturbed vertices as long as the perturbed vertices are sufficiently close to the basis vertices. And it is possible to get these perturbed vertices with the functional form $(\sigma_1^s\sigma_2^s, \sigma_1^s(1 - \sigma_2^s), (1-\sigma_1^s)\sigma_2^s, (1-\sigma_1^s)(1-\sigma_2^s))$ with $0 < \sigma_1^s, \sigma_2^s < 1$. For example, $(1,0,0,0)$ can be approximately arbitrarily well on the interior of the 3-simplex by taking $\sigma_1^s \approx 1$ and $\sigma_2^s \approx 1$.

Let $A_1, A_2, A_3, A_4 \in \mathcal{F}$ be a partition of Ω such that such that $P(A_s) = p^s$ for $s = 1,2,3,4$. Specify that the mixed strategy Nash equilibrium on A_s is (σ_1^s, σ_2^s) for $s = 1,2,3,4$. Further, specify that the payoffs on A_s are $\pi^1(1,1) = \frac{\sigma_2^s \pi^1(1,0)}{\sigma_2^s - 1}$ and $\pi^2(1,1) = \frac{\sigma_1^s \pi^2(0,1)}{\sigma_1^s - 1}$ for $s = 1,2,3,4$. Then let $A_{s,1,1}, A_{s,1,0}, A_{s,0,1}, A_{s,0,0}$ be a partition of A_s for $s = 1,2,3,4$. Specify that $P(A_{s,1,1}) = p^s\sigma_1^s\sigma_2^s$, $P(A_{s,1,0}) = p^s\sigma_1^s(1-\sigma_2^s)$, $P(A_{s,0,1}) = p^s(1-\sigma_1^s)\sigma_2^s$, and $P(A_{s,0,0}) = p^s(1-\sigma_1^s)(1-\sigma_2^s)$. Specify that the realized entry decision on $A_{s,1,1}$ is $(1,1)$, on $A_{s,1,0}$ is $(1,0)$, on $A_{s,0,1}$ is $(0,1)$, and on $A_{s,0,0}$ is $(0,0)$. This satisfies all of the conditions in Definition 2.1 and the additional conditions in Definition 2.2, and the sign of any given payoff is arbitrary, establishing sharpness of the bounds.

Now suppose that fewer than all four of the joint entry decisions have positive probability. Under Assumption 2.1 it cannot be a Nash equilibrium for one firm to use a pure strategy and the other firm to use a (non-pure) mixed strategy. This is because the firm that uses a mixed strategy must be indifferent, given the pure strategy of the other firm, between entering and not entering. Suppose the other firm enters. Then this means that duopoly profits are zero to the firm using a mixed strategy, but Assumption 2.1 rules this out. Similarly if the other firm does not enter this means that monopoly profits are zero to the firm using a mixed strategy, but Assumption 2.1 rules this out.

If (with non-zero probability) there were markets in Nash equilibrium using (non-pure) mixed strategies, under the additional conditions in Definition 2.2 all four joint entry decisions would be observed with positive probability. Therefore, if fewer than all four joint entry decisions are observed with positive probability, it must be because probability zero of markets are using mixed strategies. That means that each (with probability one) realized entry decision must be from a pure strategy Nash equilibrium.

So suppose that $(0,0)$ is a pure strategy Nash equilibrium. Then this implies that $\pi^1(1,0) < 0$ and $\pi^2(0,1) < 0$. Suppose that $(0,1)$ is a pure strategy Nash equilibrium. This implies that $\pi^1(1,1) < 0$ and $\pi^2(0,1) > 0$. Suppose that $(1,0)$ is a pure strategy Nash equilibrium. This implies that $\pi^1(1,0) > 0$ and $\pi^2(1,1) < 0$. Suppose that $(1,1)$ is a pure strategy Nash equilibrium. This implies that $\pi^1(1,1) > 0$ and $\pi^2(1,1) > 0$. The other payoffs are unrestricted; any specification of the other payoffs is consistent with that pure strategy Nash equilibrium.

Then by the law of total probability and the fact that we conclude with probability one that realized entry decisions are from a pure strategy Nash equilibrium,

$$
\begin{aligned}
P(v^1(1) = 1) &= P(\pi^1(1,1) \geq 0) \\
&= P(\pi^1(1,1) \geq 0|(1,1))P((1,1)) + P(\pi^1(1,1) \geq 0|(1,0))P((1,0)) \\
&+ P(\pi^1(1,1) \geq 0|(0,1))P((0,1)) + P(\pi^1(1,1) \geq 0|(0,0))P((0,0)) \\
&= P((1,1)) + P(\pi^1(1,1) \geq 0|(1,0))P((1,0)) + P(\pi^1(1,1) \geq 0|(0,0))P((0,0))
\end{aligned}
$$

$$
\begin{aligned}
P(v^1(0)=1) &= P(\pi^1(1,0)\geq 0) \\
&= P(\pi^1(1,0)\geq 0|(1,1))P((1,1)) + P(\pi^1(1,0)\geq 0|(1,0))P((1,0)) \\
&+ P(\pi^1(1,0)\geq 0|(0,1))P((0,1)) + P(\pi^1(1,0)\geq 0|(0,0))P((0,0)) \\
&= P(\pi^1(1,0)\geq 0|(1,1))P((1,1)) + P((1,0)) + P(\pi^1(1,0)\geq 0|(0,1))P((0,1))
\end{aligned}
$$

$$
\begin{aligned}
P(v^2(1)=1) &= P(\pi^2(1,1)\geq 0) \\
&= P(\pi^2(1,1)\geq 0|(1,1))P((1,1)) + P(\pi^2(1,1)\geq 0|(1,0))P((1,0)) \\
&+ P(\pi^2(1,1)\geq 0|(0,1))P((0,1)) + P(\pi^2(1,1)\geq 0|(0,0))P((0,0)) \\
&= P((1,1)) + P(\pi^2(1,1)\geq 0|(0,1))P((0,1)) + P(\pi^2(1,1)\geq 0|(0,0))P((0,0))
\end{aligned}
$$

$$
\begin{aligned}
P(v^2(0)=1) &= P(\pi^2(0,1)\geq 0) \\
&= P(\pi^2(0,1)\geq 0|(1,1))P((1,1)) + P(\pi^2(0,1)\geq 0|(1,0))P((1,0)) \\
&+ P(\pi^2(0,1)\geq 0|(0,1))P((0,1)) + P(\pi^2(0,1)\geq 0|(0,0))P((0,0)) \\
&= P(\pi^2(0,1)\geq 0|(1,1))P((1,1)) + P(\pi^2(0,1)\geq 0|(1,0))P((1,0)) + P((0,1))
\end{aligned}
$$

The claimed bounds obtain after replacing the remaining conditional probability statements with probabilities in $[0,1]$. It remains only to show that the bounds are sharp.

Let $A_{1,1}, A_{1,0}, A_{0,1}, A_{0,0} \in \mathcal{F}$ be a partition of Ω such that $P(A_{1,1}) = P((1,1))$, $P(A_{1,0}) = P((1,0))$, $P(A_{0,1}) = P((0,1))$ and $P(A_{0,0}) = P((0,0))$. Specify that the realized entry decisions (y_1, y_2) are $(1,1)$ on $A_{1,1}$, $(1,0)$ on $A_{1,0}$, $(0,1)$ on $A_{0,1}$, and $(0,0)$ on $A_{0,0}$. Also specify the pure strategy Nash equilibrium on each of the A sets is the realized entry decision. Finally, on each of the A sets specify that payoffs satisfy the conditions established above for that pure strategy Nash equilibrium. The unrestricted payoffs can be positive or negative, and this satisfies all of the conditions in Definition 2.1 and the additional conditions in Definition 2.2, establishing sharpness of the bounds. □

This is an interesting negative result that shows, when assuming only Nash equilibrium play and not imposing the additional conditions in Definition 2.2, that the implications of the game are too weak to provide any restrictions on any given best response probability. Further, this result shows that when assuming Nash equilibrium play and imposing the additional conditions in Definition 2.2, except for the apparently non-generic case that at least one joint entry decision occurs with zero probability, the implications of the game are too weak to provide any restrictions on any given best response probability. Notice that the proof relies on admitting (non-pure) mixed strategies and the result will not hold if we rule out (non-pure) mixed strategies.[8] Consequently, for this model to provide more information about the best response functions than is already logically implied we need to add assumptions.

2.5. **Adding more assumptions.** The next assumption we consider, monotonicity, is one that is natural in these settings. This paper assumes that monopoly payoffs are weakly greater than duopoly payoffs.

Assumption 2.2. *Let the following hold:*

$$\pi^1(1,0) \geq \pi^1(1,1) \quad \textit{and} \quad \pi^2(0,1) \geq \pi^2(1,1)$$

This assumption is maintained throughout the rest of the paper because it is plausible, and because of the observation in Lemma 2.2 that without a monotonicity assumption there are severe limits on what can be learned about the best response functions. Note that the monotonicity assumption implies that if $\upsilon^i(1) = 1$ then $\upsilon^i(0) = 1$, since if a firm would enter

[8]For example, consider $P(\pi^1(1,1) \geq 0|(1,1))$. If (non-pure) mixed strategies are not admitted, it must be that $(1,1)$ is a pure strategy Nash equilibrium, and so $\pi^1(1,1) > 0$. Therefore we can conclude (not necessarily sharply) that $P(\pi^1(1,1) \geq 0) \geq P(1,1)$ if we rule out (non-pure) mixed strategies.

the market when the other firm is known to enter the market, it would enter the market when the other firm is known to not enter the market. As a condition on the (best) response function, this is the monotone treatment response assumption of Manski (1997). Moreover, the monotonicity assumption implies the existence of a pure strategy Nash equilibrium, as the following lemma establishes. The proof is in appendix A.

Lemma 2.3. *If Assumption 2.2 holds, then there is a pure strategy Nash equilibrium.*

The main result in this section characterizes the sharp joint identified set for the best response probabilities under this set of basic assumptions. The proof serves as a basis for the proofs of many of the rest of the results.

Theorem 2.1. *Let Assumptions 2.1 and 2.2 hold. Assume further that each firm exhibits at least* 2 *levels of rationality. The following holds:*

(1) *The sharp joint identified set for*

$$(\upsilon_{11}, \upsilon_{10}, \upsilon_{21}, \upsilon_{20}) = \big(P(\upsilon^1(1) = 1), P(\upsilon^1(0) = 1), P(\upsilon^2(1) = 1), P(\upsilon^2(0) = 1)\big)$$

is

$$\mathcal{S} = \left\{ \begin{pmatrix} pP(y_1 = 1, y_2 = 1) + qP(y_1 = 1, y_2 = 0) \\ P(y_1 = 1) + sP(y_1 = 0, y_2 = 1) + tP(y_1 = 0, y_2 = 0) \\ pP(y_1 = 1, y_2 = 1) + rP(y_1 = 0, y_2 = 1) \\ P(y_2 = 1) + uP(y_1 = 1, y_2 = 0) + tP(y_1 = 0, y_2 = 0) \end{pmatrix}' : \quad \textit{where} \quad p, q, r, s, t, u \in [0, 1] \right\}$$

We would also obtain this same sharp joint identified set even if we assume that there is Nash equilibrium play in each market.

(2) *The set $\mathcal{S}$ above is also the sharp joint identified set under the additional conditions in Definition 2.2. It remains sharp under the assumption that firms exhibit at least k levels of rationality for some $k \geq 2$.*

(3) *Assume that there is Nash equilibrium play in each market, and let the additional conditions in Definition 2.2 hold. The sharp joint identified set $\mathcal{S}'$ is as follows: If all four joint entry decisions have positive probability, then any point in $\mathcal{S}$ with $p, q, r \in [0, 1)$ and $s, t, u \in (0, 1]$ is also in $\mathcal{S}'$ (so, in particular, $cl(\mathcal{S}') = \mathcal{S}$); if at least one of the joint entry decisions has probability zero, then with probability one the firms use pure strategies, and $\mathcal{S}'$ is equal to $\mathcal{S}$ with $p = 1$ and $t = 0$.*

Corollary 2.1. *Let Assumptions 2.1 and 2.2 hold. Assume further that each firm exhibits at least 1 level of rationality. The sharp marginal identified set for υ_{11} is $[0, P(y_1 = 1)]$, for υ_{10} is $[P(y_1 = 1), 1]$, for υ_{21} is $[0, P(y_2 = 1)]$ and for υ_{20} is $[P(y_2 = 1), 1]$. The same bounds hold if we assume that there is Nash equilibrium play in each market. These are also the sharp marginal identified sets under the additional conditions in Definition 2.2.*

Proof of Theorem 2.1. By the law of total probability, where it is understood that $P(B|A)P(A) = 0$ if $P(A) = 0$, it holds that

$$\begin{aligned} P(\upsilon^1(1) = 1) &= P(\pi^1(1,1) \geq 0) \\ &= P(\pi^1(1,1) \geq 0|(1,1))P((1,1)) + P(\pi^1(1,1) \geq 0|(1,0))P((1,0)) \\ &+ P(\pi^1(1,1) \geq 0|(0,1))P((0,1)) + P(\pi^1(1,1) \geq 0|(0,0))P((0,0)) \end{aligned}$$

$$\begin{aligned} P(\upsilon^1(0) = 1) &= P(\pi^1(1,0) \geq 0) \\ &= P(\pi^1(1,0) \geq 0|(1,1))P((1,1)) + P(\pi^1(1,0) \geq 0|(1,0))P((1,0)) \\ &+ P(\pi^1(1,0) \geq 0|(0,1))P((0,1)) + P(\pi^1(1,0) \geq 0|(0,0))P((0,0)) \end{aligned}$$

$$\begin{aligned} P(\upsilon^2(1) = 1) &= P(\pi^2(1,1) \geq 0) \\ &= P(\pi^2(1,1) \geq 0|(1,1))P((1,1)) + P(\pi^2(1,1) \geq 0|(1,0))P((1,0)) \\ &+ P(\pi^2(1,1) \geq 0|(0,1))P((0,1)) + P(\pi^2(1,1) \geq 0|(0,0))P((0,0)) \end{aligned}$$

$$
\begin{aligned}
P(v^2(0)=1) = & P(\pi^2(0,1) \geq 0) \\
& = P(\pi^2(0,1) \geq 0|(1,1))P((1,1)) + P(\pi^2(0,1) \geq 0|(1,0))P((1,0)) \\
& + P(\pi^2(0,1) \geq 0|(0,1))P((0,1)) + P(\pi^2(0,1) \geq 0|(0,0))P((0,0))
\end{aligned}
$$

By Lemma 2.1, and the assumption that for each realization of the uncertainty each firm exhibits at least 2 levels of rationality, or there is Nash equilibrium play, we have that each firm is $\mathcal{R}2$, and also $\mathcal{R}1$.

Consider the implications of the assumption that for each realization of the uncertainty each firm is $\mathcal{R}1$. Firm 1 can enter the market with positive probability if and only if there is a strategy of firm 2 that enters the market with probability σ such that $\sigma\pi^1(1,1) + (1-\sigma)\pi^1(1,0) \geq 0$. By monotonicity and general position, this implies that $\pi^1(1,0) > 0$. Therefore, if $y_1 = 1$, then $\pi^1(1,0) > 0$. Similarly, if $y_2 = 1$, then $\pi^2(0,1) > 0$. Firm 1 can not enter the market with positive probability if and only if there is a strategy of firm 2 that enters the market with probability σ such that $\sigma\pi^1(1,1) + (1-\sigma)\pi^1(1,0) \leq 0$. By monotonicity and general position, this implies that $\pi^1(1,1) < 0$. Therefore, if $y_1 = 0$, then $\pi^1(1,1) < 0$. Similarly, if $y_2 = 0$, then $\pi^2(1,1) < 0$. Therefore, under $\mathcal{R}1$, the expressions above simplify to

$$P(v^1(1)=1) = P(\pi^1(1,1) \geq 0|(1,1))P((1,1)) + P(\pi^1(1,1) \geq 0|(1,0))P((1,0))$$

$$P(v^1(0)=1) = P(y_1=1) + P(\pi^1(1,0) \geq 0|(0,1))P((0,1)) + P(\pi^1(1,0) \geq 0|(0,0))P((0,0))$$

$$P(v^2(1)=1) = P(\pi^2(1,1) \geq 0|(1,1))P((1,1)) + P(\pi^2(1,1) \geq 0|(0,1))P((0,1))$$

$$P(v^2(0)=1) = P(y_2=1) + P(\pi^2(0,1) \geq 0|(1,0))P((1,0)) + P(\pi^2(0,1) \geq 0|(0,0))P((0,0))$$

This intermediate derivation is useful to avoid repetition when proving the Corollary about the sharp marginal identified sets. The assumption that each firm is $\mathcal{R}2$ adds restrictions across these expressions, since probabilities conditional on the same realized entry decision appear in multiple expressions.

Consider the implications of the fact that for each realization of the uncertainty each firm is $\mathcal{R}2$, for a given realization of the uncertainty. Under $\mathcal{R}2$ it must be, if $(1,1)$ is the realized entry decision, that either ($\pi^1(1,1) > 0$ and $\pi^2(1,1) > 0$) or ($\pi^1(1,1) < 0$ and $\pi^2(1,1) < 0$). Otherwise, suppose one firm has positive duopoly payoffs and the other firm has negative duopoly payoffs. By monotonicity, the firm with positive duopoly payoffs gets positive payoff to entering the market no matter what the other firm does. Thus, the only $\mathcal{R}1$ strategy for that firm is to enter the market. Thus, since the other firm has negative duopoly payoffs, its only $\mathcal{R}2$ strategy is to not enter the market. This would contradict observing the entry decision $(1,1)$.

Also under $\mathcal{R}2$ it must be, if $(0,0)$ is the realized entry decisions, that either ($\pi^1(1,0) > 0$ and $\pi^2(0,1) > 0$) or ($\pi^1(1,0) < 0$ and $\pi^2(0,1) < 0$). Otherwise, suppose one firm has positive monopoly payoffs and the other firm has negative monopoly payoffs. By monotonicity, the firm with negative monopoly payoffs gets negative payoff to entering the market no matter what the other firm does. Thus, the only $\mathcal{R}1$ strategy for that firm is to not enter the market. Thus, since the other firm has positive monopoly payoffs, its only $\mathcal{R}2$ strategy is to enter the market. This would contradict observing the entry decision $(0,0)$.

Therefore under $\mathcal{R}2$ the expressions above further simplify to the following, where $p = P(\pi^1(1,1) \geq 0|(1,1)) = P(\pi^2(1,1) \geq 0|(1,1))$ and $t = P(\pi^1(1,0) \geq 0|(0,0)) = P(\pi^2(0,1) \geq 0|(0,0))$.

$$P(\upsilon^1(1) = 1) = pP((1,1)) + P(\pi^1(1,1) \geq 0|(1,0))P((1,0))$$

$$P(\upsilon^1(0) = 1) = P(y_1 = 1) + P(\pi^1(1,0) \geq 0|(0,1))P((0,1)) + tP((0,0))$$

$$P(\upsilon^2(1) = 1) = pP((1,1)) + P(\pi^2(1,1) \geq 0|(0,1))P((0,1))$$

$$P(\upsilon^2(0) = 1) = P(y_2 = 1) + P(\pi^2(0,1) \geq 0|(1,0))P((1,0)) + tP((0,0))$$

The claimed bounds obtain after replacing the remaining conditional probability statements with probabilities in $[0,1]$. It remains only to show that these bounds are sharp. It is enough to show that the bounds are sharp under the assumption that for each realization of the uncertainty there is Nash equilibrium play, since by Lemma 2.1 this implies the firms are also $\mathcal{R}k$ for any k.

It is consistent with Nash equilibrium play to observe $(1,1)$ with either ($\pi^1(1,1) > 0$ and $\pi^2(1,1) > 0$) or ($\pi^1(1,1) < 0$ and $\pi^2(1,1) < 0$). If $\pi^1(1,1) > 0$ and $\pi^2(1,1) > 0$ then it is a pure strategy Nash equilibrium for both firms to enter the market. If $\pi^1(1,1) < 0$ and $\pi^2(1,1) < 0$, as long as $\pi^1(1,0) > 0$ and $\pi^2(0,1) > 0$, there is a (non-pure) mixed strategy Nash equilibrium, so that $(1,1)$ could be the realized entry decisions.

In addition, $(1,1)$ could be the realized entry decision with $\pi^1(1,1) < 0$ and $\pi^2(1,1) < 0$ if the econometrician assumes that the firms are $\mathcal{R}k$ for any k, but that there is not necessarily Nash equilibrium play, from pure strategies. This holds because when $\pi^1(1,1) < 0$, $\pi^2(1,1) < 0$, $\pi^1(1,0) > 0$, and $\pi^2(0,1) > 0$, then entering the market is $\mathcal{R}k$ for every k. This is because, for either firm, entering is a best response to a conjecture that the other firm does not enter, and not entering is a best response to a conjecture that the other firm enters. This is useful in establishing the later corollaries when mixed strategies are not admitted.

Let $A_{1,1}, A_{1,0}, A_{0,1}, A_{0,0} \in \mathcal{F}$ be a partition of Ω such that $P(A_{1,1}) = P((1,1))$, $P(A_{1,0}) = P((1,0))$, $P(A_{0,1}) = P((0,1))$ and $P(A_{0,0}) = P((0,0))$.

Let the realized entry decisions on $A_{1,1}$ be $(1,1)$. For any $p \in [0,1]$, let $B \in \mathcal{F}$ be such that $B \subset A_{1,1}$ and $P(B|A_{1,1}) = p$. On B specify that the payoffs are such that $\pi^1(1,1) > 0$ and $\pi^2(1,1) > 0$, and on $A_{1,1} \cap B^C$ specify that the payoffs are such that $\pi^1(1,1) < 0$ and $\pi^2(1,1) < 0$. Thus, the sharp identified set for $P(\pi^1(1,1) \geq 0|(1,1)) = P(\pi^2(1,1) \geq 0|(1,1))$ is $[0,1]$.

Further, it is consistent with Nash equilibrium play to observe $(0,0)$ with either ($\pi^1(1,0) > 0$ and $\pi^2(0,1) > 0$) or ($\pi^1(1,0) < 0$ and $\pi^2(0,1) < 0$). If $\pi^1(1,0) < 0$ and $\pi^2(0,1) < 0$ then it is a pure strategy Nash equilibrium for both firms to not enter the market. If $\pi^1(1,0) > 0$ and $\pi^2(0,1) > 0$, as long as $\pi^1(1,1) < 0$ and $\pi^2(1,1) < 0$, there is a (non-pure) mixed strategy Nash equilibrium, so that $(0,0)$ could be the realized entry decision.

In addition, $(0,0)$ could be the realized entry decision with $\pi^1(1,0) > 0$ and $\pi^2(0,1) > 0$ if the econometrician assumes that the firms are $\mathcal{R}k$ for any k, but that there is not necessarily Nash equilibrium play, from pure strategies. This holds because when $\pi^1(1,1) < 0$, $\pi^2(1,1) < 0$, $\pi^1(1,0) > 0$, and $\pi^2(0,1) > 0$, then not entering the market is $\mathcal{R}k$ for every k. This is because, for either firm, entering is a best response to a conjecture that the other firm does not enter, and not entering is a best response to a conjecture that the other firm enters. This is useful in establishing the corollaries when mixed strategies are not admitted.

As before, this implies that the sharp identified set for $P(\pi^1(1,0) \geq 0|(0,0)) = P(\pi^2(0,1) \geq 0|(0,0))$ is $[0,1]$. Let the realized entry decisions on $A_{0,0}$ be $(0,0)$. For any $t \in [0,1]$, let $B \in \mathcal{F}$ be such that $B \subset A_{0,0}$ and $P(B|A_{0,0}) = t$. On B specify that the payoffs are such that $\pi^1(1,0) > 0$ and $\pi^2(0,1) > 0$, and on $A_{0,0} \cap B^C$ specify that the payoffs are such that $\pi^1(1,0) < 0$ and $\pi^2(0,1) < 0$. Thus, the sharp identified set for $P(\pi^1(1,0) \geq 0|(0,0)) = P(\pi^2(0,1) \geq 0|(0,0))$ is $[0,1]$.

It remains to show that Nash equilibrium play places no restriction on the remaining conditional probabilities. The first step is to show that $P(\pi^1(1,1) \geq 0|(1,0)), P(\pi^2(0,1) \geq 0|(1,0))$ can take on any value in $[0,1] \times [0,1]$. It is enough to show that $(1,0)$ can be the realized entry decisions under Nash equilibrium play for any of the four possible joint signs of $\pi^1(1,1)$ and $\pi^2(0,1)$.

If $\pi^1(1,0) > 0$, then as long as $\pi^2(1,1) < 0$, it is a pure strategy Nash equilibrium for firm 1 to enter and firm 2 to not enter. Since $\pi^1(1,0) > 0$ firm 1 has no profitable deviation. Since $\pi^2(1,1) < 0$ firm 2 has no profitable deviation. The fact that $\pi^1(1,0) > 0$ implies nothing about the sign of $\pi^1(1,1)$. Further, $\pi^2(1,1) < 0$ implies nothing about the sign of $\pi^2(0,1)$. This establishes that $(1,0)$ can be the realized entry decisions under pure strategy Nash equilibrium play when $\pi^1(1,1) > 0$, $\pi^2(0,1) > 0$ and when $\pi^1(1,1) > 0$, $\pi^2(0,1) < 0$ and when $\pi^1(1,1) < 0$ and $\pi^2(0,1) > 0$ and when $\pi^1(1,1) < 0$, $\pi^2(0,1) < 0$.

Let the realized entry decisions on $A_{1,0}$ be $(1,0)$. Also for any $p_1, p_2, p_3, p_4 \in [0,1]$ such that $\sum p_k = 1$, let $B_1, B_2, B_3, B_4 \in \mathcal{F}$ be a partition of $A_{1,0}$ and $P(B_k|A_{1,0}) = p_k$. On B_1 specify that the payoffs are such that $\pi^1(1,1) > 0$ and $\pi^2(0,1) > 0$, on B_2 specify that the payoffs are such $\pi^1(1,1) > 0$ and $\pi^2(0,1) < 0$, on B_3 specify that the payoffs are such that $\pi^1(1,1) < 0$ and $\pi^2(0,1) > 0$, and on B_4 specify that the payoffs are such that $\pi^1(1,1) < 0$ and $\pi^2(0,1) < 0$. This implies that $P(\pi^1(1,1) \geq 0|(1,0)) = p_1 + p_2$ and $P(\pi^2(0,1) \geq 0|(1,0)) = p_1 + p_3$. For any $q, u \in [0,1]$ it is possible to specify p_k such that $P(\pi^1(1,1) \geq 0|(1,0)) = q$ and $P(\pi^2(0,1) \geq 0|(1,0)) = u$. If $q \geq u$, then specify $p_1 = u$, $p_2 = q - u$, $p_3 = 0$, and $p_4 = 1 - q$. If $u > q$, then specify $p_1 = q$, $p_2 = 0$, $p_3 = u - q$, and $p_4 = 1 - u$. Thus, the sharp joint identified set for $P(\pi^1(1,1) \geq 0|(1,0))$ and $P(\pi^2(0,1) \geq 0|(1,0))$ is $[0,1] \times [0,1]$. By exchanging firm 1 with firm 2 in this analysis, this establishes also that the sharp joint identified set for $P(\pi^1(1,0) \geq 0|(0,1))$ and $P(\pi^2(1,1) \geq 0|(0,1))$ is $[0,1] \times [0,1]$.

Thus, using these specifications of the payoffs and realized entry decisions, the claimed bounds are sharp.

Under the assumption that each firm exhibits at least 2 levels of rationality, but if there is not necessarily Nash equilibrium play in each market, this rationalization of the data can use only pure strategies, so this is also the sharp joint identified set under the additional conditions in Definition 2.2.

Now consider the identified set under the additional conditions in Definition 2.2 when the econometrician is willing to maintain that there is Nash equilibrium play in each market.

Suppose that all four joint entry decisions have positive probability. Obviously the sharp joint identified set with the additional conditions in Definition 2.2 is contained in the sharp joint identified set under only Definition 2.1. Further, the sharp joint identified set under only Definition 2.1 is closed.[9] Consequently, the closure of the sharp joint identified set with the additional conditions in Definition 2.2 is contained in the sharp joint identified set under only Definition 2.1. Therefore, showing that all points where $p, q, r \in [0, 1)$ and $s, t, u \in (0, 1]$ are in the the sharp joint identified set with the additional conditions in Definition 2.2 establishes the claim.

So consider such a point. Let $A_1, A_2, A_3, A_4, A_5, A_6, A_7, A_8 \in \mathcal{F}$ be a partition of Ω.

Specify that $\varepsilon_1 = (1 - p)P(1, 1)$ and $\varepsilon_4 = tP(0, 0)$. Let $0 < \delta_2 < \min\{1 - q, u\}$ and $\varepsilon_2 = \delta_2 P(1, 0)$ and $0 < \delta_3 < \min\{1 - r, s\}$ and $\varepsilon_3 = \delta_3 P(0, 1)$. Since $p, q, r \in [0, 1)$ and $s, t, u \in (0, 1]$ and all four joint entry decisions have positive probability, the min in the definition of δ_2 and δ_3 are strictly positive, and all ε variables are strictly positive. Let $\varepsilon = \varepsilon_1 + \varepsilon_2 + \varepsilon_3 + \varepsilon_4$. As in the proof of Lemma 2.2, it is possible to find four pairs of (non-pure) mixed strategies (σ_1^s, σ_2^s) with $0 < \sigma_1^s, \sigma_2^s < 1$ for $s = 1, 2, 3, 4$ such that there are probabilities p^s such that

$$\frac{1}{\varepsilon}(\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4) = \sum_s p^s \left(\sigma_1^s \sigma_2^s, \sigma_1^s(1 - \sigma_2^s), (1 - \sigma_1^s)\sigma_2^s, (1 - \sigma_1^s)(1 - \sigma_2^s)\right).$$

Specify that the mixed strategy Nash equilibrium on A_s is (σ_1^s, σ_2^s) for $s = 1, 2, 3, 4$. Then let $A_{s,1,1}, A_{s,1,0}, A_{s,0,1}, A_{s,0,0}$ be a partition of A_s for $s = 1, 2, 3, 4$. Specify that $P(A_{s,1,1}) = \varepsilon p^s \sigma_1^s \sigma_2^s$, $P(A_{s,1,0}) = \varepsilon p^s \sigma_1^s (1 - \sigma_2^s)$, $P(A_{s,0,1}) = \varepsilon p^s (1 - \sigma_1^s)\sigma_2^s$, and $P(A_{s,0,0}) = \varepsilon p^s (1 - \sigma_1^s)(1 - \sigma_2^s)$. Specify that the realized entry decision on $A_{s,1,1}$ is $(1, 1)$, on $A_{s,1,0}$ is $(1, 0)$, on $A_{s,0,1}$ is $(0, 1)$, and on $A_{s,0,0}$ is $(0, 0)$. On A_1-A_4 specify that $\pi^1(1, 1) < 0$, $\pi^2(1, 1) < 0$, $\pi^1(1, 0) > 0$ and $\pi^2(0, 1) > 0$.

[9] Consider any sequence of points in the identified set that converges in $\mathbb{R}^4$. Then consider the associated sequence p, q, r, s, t, u. Since this sequence is in $[0, 1]^6$, there is a convergent subsequence of p, q, r, s, t, u. Then along this subsequence, clearly the sequence of points in the identified set converges to a point in the identified set. But, since the original sequence of points in the identified set converges, the limit of that sequence is the same as the limit of this subsequence. So the original sequence converges to a point in the identified set. So the identified set is closed.

On A_5 specify that the realized entry decision is $(1,1)$, on A_6 specify that it is $(1,0)$, on A_7 specify that it is $(0,1)$, and on A_8 specify that it is $(0,0)$. Further on each of these sets that the strategy is the corresponding pure strategy Nash equilibrium. And on A_5 specify that $\pi^1(1,1) > 0$, $\pi^2(1,1) > 0$, $\pi^1(1,0) > 0$, and $\pi^2(0,1) > 0$. On A_6 specify that $\pi^2(1,1) < 0$ and $\pi^1(1,0) > 0$. On A_7 specify that $\pi^1(1,1) < 0$ and $\pi^2(0,1) > 0$. On A_8 specify that $\pi^1(1,1) < 0$, $\pi^2(1,1) < 0$, $\pi^1(1,0) < 0$, and $\pi^2(0,1) < 0$. All other payoffs are unrestricted. Specify that $P(A_5) = pP(1,1)$, $P(A_6) = (1-\delta_2)P(1,0)$, $P(A_7) = (1-\delta_3)P(0,1)$, and $P(A_8) = (1-t)P(0,0)$. This satisfies all the additional conditions in Definition 2.2 and achieves the claimed point in the identified set.

Suppose that strictly fewer than all four joint entry decisions have positive probability. Then as in the proof of Lemma 2.2 we conclude that with probability one the firms use pure strategies. This adds to the conditions derived already under $\mathcal{R}2$ that if $(1,1)$ is the realized entry decision, then $\pi^1(1,1) > 0$ and $\pi^2(1,1) > 0$. And if $(0,0)$ is the realized entry decision, then $\pi^1(1,0) < 0$ and $\pi^2(0,1) < 0$. Consequently, $p = 1$ and $t = 0$. Otherwise, sharpness follows from the same arguments. □

Proof of Corollary. The proof of the Theorem establishes the expressions for the marginal identified set using the law of total probability under only $\mathcal{R}1$, so the validity of the bounds follows. The sharpness follows from the arguments used to establish the sharpness of the joint identified set in the Theorem. Under the additional conditions in Definition 2.2, Theorem 2.1 shows every point is in the marginal identified set, except the upper bound on $P(\nu^1(1) = 1)$ and $P(\nu^2(1) = 1)$ and the lower bound on $P(\nu^1(0) = 1)$ and $P(\nu^2(0) = 1)$. But these points can be achieved by rationalizing the data using only pure strategy Nash equilibria. □

Remark 2.1. *Theorem 2.1 establishes that, under the assumptions maintained, there is never point identification of all of the best response probabilities. This is evident from the sharp marginal identified sets in the Corollary. If $P(y_1 = 1) > 0$, then $P(\upsilon^1(1) = 1)$ is not point identified, and if $P(y_1 = 1) = 0$, then $P(\upsilon^1(0) = 0)$ is only identified trivially to be in* $[0,1]$.

Remark 2.2. *The Theorem above establishes that the assumption of Nash equilibrium play has the same identification power on the best response probabilities as does the much less controversial assumption that each firm exhibits at least* 2 *levels of rationality, at least without* *the additional conditions in Definition 2.2. With the additional conditions in Definition 2.2, as long as all joint entry decisions have positive probability, the Theorem establishes that the assumptions of Nash equilibrium play effectively has the same identification power as the assumption that each firm exhibits at least* 2 *levels of rationality. Aradillas-Lopez and Tamer (2008) study the identifying power of assuming some level of rationality and assuming Nash equilibrium play, for games with parametric assumptions. Their objects of interest are parameters that describe the payoff function. They show that the identified set for their object of interest under the assumption of Nash equilibrium play can be tighter than under the assumption of* 1 *level of rationality.*

That firms exhibit at least 2 *levels of rationality is logically a less controversial assumption than the assumption of Nash equilibrium play, by Lemma 2.1. Moreover, the Theorem establishes that the potentially controversial assumptions that there is always Nash equilibrium play, or that all firms exhibit the same number of levels of rationality, have the same identification power as the less controversial assumption that each firm exhibits at least* 2 *levels of rationality. This is interesting since this does not require that every firm's level of rationality is the same. This implies that, unless the econometrician is willing to maintain assumptions involving more than just the rationality of the firms, there is no need to assume Nash equilibrium play for the purposes of identification of the best response probabilities. This narrows the scope of disagreements between different econometricians studying the same population, as long as it is uncontroversial that all firms exhibit at least* 2 *levels of rationality.*

Remark 2.3. *The bounds in Theorem 2.1 allow for* $P(\upsilon^1(1) = 1) = P(\upsilon^1(0) = 1)$ *and for* $P(\upsilon^2(1) = 1) = P(\upsilon^2(0) = 1)$ *by setting* $p = q = r = 1$ *and* $s = t = u = 0$*. This is important since this implies that it is not possible to rule out that there is no non-trivial strategic interaction, in probability. This has potentially important policy implications. Consider from before the planner who is considering regulating the decision of one of the firms and is concerned about the reaction of the other firm, the unregulated firm. This result establishes that the planner cannot rule out that, in probability, the unregulated firm will react the same way to either decision of the regulated firm.*

However, this will only *happen at the extreme points of the identified set where* $p = q = r = 1$ *and* $s = t = u = 0$ *as long as all joint entry decisions have positive probability. Consider, for example, the condition that* $p = 1$*. From the proof of the Theorem, this means that* $P(\pi^1(1,1) \geq 0|(1,1)) = 1$*, which the econometrician may consider to be implausible, for example if Nash equilibrium outcomes from (non-pure) mixed strategies are "likely." This is because with mixed strategies it is possible to observe the realized entry decision* $(1,1)$ *but not have* $\pi^1(1,1) \geq 0$*. So, although theoretically possible, it might be considered practically or economically implausible, with the addition of* a priori *information, that the model allows for* $P(\upsilon^1(1) = 1) = P(\upsilon^1(0) = 1)$ *and for* $P(\upsilon^2(1) = 1) = P(\upsilon^2(0) = 1)$*. In addition, in the next sections we make further assumptions that will rule out this equality.*

If we are interested in the marginal identified set for the best response probabilities, the Corollary shows that similar results are possible under the weaker assumption that firms exhibit at least 1 level of rationality.

Next, we make further plausible assumptions on the entry game to help narrow the bounds.

2.6. **Further assumptions about the entry game.** In particular, it can be reasonable to assume that payoffs are symmetric, in the weak sense that duopoly payoffs have the same sign for both firms and monopoly payoffs have the same sign for both firms. This is, of course, implied by the assumption that payoffs are symmetric in the usual sense that $\pi^1(1,1) = \pi^2(1,1)$ and $\pi^1(1,0) = \pi^2(0,1)$.

Assumption 2.3. *Let the following hold:* $\pi^1(1,1) > 0$ *if and only if* $\pi^2(1,1) > 0$ *and* $\pi^1(1,0) > 0$ *if and only if* $\pi^2(0,1) > 0$*.*

Corollary 2.2. *Let Assumptions 2.1, 2.2, and 2.3 hold. Assume further that each firm exhibits at least* 2 *levels of rationality. The following holds:*

(1) *The sharp joint identified set for*

$$(\upsilon_{11}, \upsilon_{10}, \upsilon_{21}, \upsilon_{20}) = \big(P(\upsilon^1(1) = 1), P(\upsilon^1(0) = 1), P(\upsilon^2(1) = 1), P(\upsilon^2(0) = 1)\big)$$

is

$$\mathcal{T} = \{(pP(1,1), P(y_1 = 1) + P(0,1) + tP(0,0), pP(1,1), P(y_2 = 1) + P(1,0) + tP(0,0)) : \quad p, t \in [0,1]\}$$

We would also obtain this same sharp joint identified set even if we assume that there is Nash equilibrium play in each market.

(2) *The set $\mathcal{T}$ above is also the sharp joint identified set under the additional conditions in Definition 2.2. It remains sharp under the assumption that firms exhibit at least k levels of rationality for some $k \geq 2$.*

(3) *Assume that there is Nash equilibrium play in each market, and let the additional conditions in Definition 2.2 hold. The sharp joint identified set $\mathcal{T}'$ is as follows: If all four joint entry decisions have positive probability, then any point in $\mathcal{T}$ with $p \in [0,1)$ and $t \in (0,1]$ is also in $\mathcal{T}'$ (so, in particular, $cl(\mathcal{T}') = \mathcal{T}$); if at least one of the joint entry decisions has probability zero, then with probability one the firms use pure strategies, and $\mathcal{T}'$ is equal to $\mathcal{T}$ with $p = 1$ and $t = 0$ (i.e., see Assumption 2.4).*

Proof. Since this Corollary only adds assumptions to the Theorem, the same arguments establishing the bounds (but not necessarily sharpness) are valid here. The additional restriction implied by Assumption 2.3 is that $P(\pi^1(1,1) \geq 0|(1,0)) = 0 = P(\pi^2(1,1) \geq 0|(0,1))$ and $P(\pi^1(1,0) \geq 0|(0,1)) = 1 = P(\pi^2(0,1) \geq 0|(1,0))$.

If $(1,0)$ is the realized entry decision, then by the assumption that each firm is $\mathcal{R}1$, as in the proof of the Theorem, it must be that, since $y_2 = 0$, $\pi^2(1,1) < 0$, and therefore by symmetry, $\pi^1(1,1) < 0$. Thus, $P(\pi^1(1,1) \geq 0|(1,0)) = 0$. Similarly, $P(\pi^2(1,1) \geq 0|(0,1)) = 0$. This implies that $q = 0 = r$ in the statement of the identified set.

If $(0,1)$ is the realized entry decision, then by the assumption that each firm is $\mathcal{R}1$, as in the proof of the Theorem, it must be that, since $y_2 = 1$, $\pi^2(0,1) > 0$, and therefore by symmetry,

$\pi^1(1,0) > 0$. Thus, $P(\pi^1(1,0) \geq 0|(0,1)) = 1$. Similarly, $P(\pi^2(0,1) \geq 0|(1,0)) = 1$. This implies that $s = 1 = u$ in the statement of the identified set.

Otherwise, the same arguments as before still establish sharpness. □

Next, we derive the sharp joint identified set when we restrict attention to pure strategies.

Assumption 2.4. *Let the following hold: firms use only pure strategies.*

Corollary 2.3. *Let Assumptions 2.1, 2.2, 2.3, and 2.4 hold. Assume further that each firm exhibits at least* 2 *levels of rationality. The sharp joint identified set, even if the econometrician is willing to assume that firms exhibit at least k levels of rationality for some $k \geq 2$, for*

$$(\upsilon_{11}, \upsilon_{10}, \upsilon_{21}, \upsilon_{20}) = \big(P(\upsilon^1(1) = 1), P(\upsilon^1(0) = 1), P(\upsilon^2(1) = 1), P(\upsilon^2(0) = 1)\big)$$

is

$$\big\{(pP(1,1), P(y_1 = 1) + P(0,1) + tP(0,0), pP(1,1), P(y_2 = 1) + P(1,0) + tP(0,0)) : p, t \in [0,1]\big\}.$$

This is also the sharp joint identified set under the additional conditions in Definition 2.2.

Proof. Since this Corollary only adds assumptions to the Theorem, the same arguments establishing the bounds (but not necessarily sharpness) are valid here. However, as long as the econometrician allows that each firm exhibits 2 levels of rationality, but that there is not necessarily Nash equilibrium play, the sharpness proof still holds.

This is the sharp joint identified set under the additional conditions in Definition 2.2 because only pure strategies are used. □

Note that the sharp joint identified set in this Corollary, under the additional assumption that each firm uses a pure strategy, is the same as in the previous Corollary under the same other conditions. Thus, under the maintained assumptions that payoffs are in general position, that payoffs are monotonic, that payoffs are symmetric, and that firms exhibit at least 2 levels of rationality, the assumption that each firm uses a pure strategy has no empirical content. In the proof of the result, it is evident that establishing sharpness depends on the possibility that markets are not in Nash equilibrium play. Therefore, finally, there is a result in which the assumption of Nash equilibrium play is substantively stronger than the assumption that firms exhibit at least some minimal number of levels of rationality.

Corollary 2.4. *Let Assumptions 2.1, 2.2, 2.3, and 2.4 hold. Assume further that in each market there is Nash equilibrium play. The best responses are point identified:*

$$\begin{aligned}(\upsilon_{11}, \upsilon_{10}, \upsilon_{21}, \upsilon_{20}) &= \big(P(\upsilon^1(1) = 1), P(\upsilon^1(0) = 1), P(\upsilon^2(1) = 1), P(\upsilon^2(0) = 1)\big) \\ &= (P(1,1), P(y_1 = 1) + P(0,1), P(1,1), P(y_2 = 1) + P(1,0))\end{aligned}$$

This is also the sharp joint identified set under the additional conditions in Definition 2.2.

Proof. Since this Corollary only adds assumptions to the Theorem, the same arguments establishing the bounds (but not necessarily sharpness) are valid here. The additional restriction implied by Assumption 2.4 is that $P(\pi^1(1,1) \geq 0|(1,1)) = 1 = P(\pi^2(1,1) \geq 0|(1,1))$ and $P(\pi^1(1,0) \geq 0|(0,0)) = 0 = P(\pi^2(0,1) \geq 0|(0,0))$. This is because when the firms use pure strategies, the realized entry decisions are also the strategies. A market can have Nash equilibrium play with both firms entering if and only if the duopoly payoffs of both firms is positive. Similarly, a market can have Nash equilibrium play with both firms not entering if and only if the monopoly payoffs of both firms is negative. Thus, $p = 1$ and $t = 0$.

This is the sharp joint identified set under the additional conditions in Definition 2.2 because only pure strategies are used. □

This is an interesting model which implies that under pure strategy Nash equilibrium play and other assumptions the best response probabilities are point identified.

3. Identification of best responses with more than two firms

This section generalizes the identification above to entry games with $N > 2$ players. As before, the realized entry decision of firm i is y_i. The payoff to firm i when the realized entry decisions are $(y_1, y_2, \ldots, y_N)$ is $\pi^i(y_1, y_2, \ldots, y_N)$. The entry game structure imposes that the payoffs to the firms are such that each firm gets 0 payoff if it does not enter the market. Thus, $\pi^i(y_1, \ldots, y_{i-1}, 0, y_{i+1}, \ldots, y_N) = 0$. The best response function for firm i for when the econometrician conjectured entry decision of firms $-i$ is t_{-i} is $v^i(t_{-i})$. Thus, $v^i(t_{-i}) = 1[\pi^i(t_1, t_2, \ldots, t_{i-1}, 1, t_{i+1}, \ldots, t_N) > 0]$. In this section, the marginal identification of the best response probabilities when there are more than two firms is studied.

The notion of levels of rationality is the extension of that used already to the case of more than two firms. The strategies of firm i that are a best response to a conjecture that firms other than i use a strategy profile that is in $\prod_{j \neq i} \mathcal{R}^j(k, \pi)$ are collected in $\mathcal{R}^i(k+1, \pi)$. That is, for $k \geq 0$, $\mathcal{R}^i(k+1, \pi) = \{\sigma^i \in \Delta^1 : \exists \sigma^{-i} \in \prod_{j \neq i} \mathcal{R}^j(k, \pi) \text{ s.t. } E_{\sigma^i, \sigma^{-i}} \pi^i(y_1, y_2, \ldots, y_N) \geq E_{\sigma^{i'}, \sigma^{-i}} \pi^i(y_1, y_2, \ldots, y_N) \text{ for all } \sigma^{i'} \in \Delta^1\}$. Equivalently, the set $\mathcal{R}^i(k+1, \pi)$ is the set of best responses to $\prod_{j \neq i} \mathcal{R}^j(k, \pi)$. In fact, the proof in this section shows the result holds under an even weaker "correlated levels of rationality" assumption, which allows that conjectures are not necessarily the product of independent strategies across firms (i.e., correlated rationalizability from Brandenburger and Dekel (1987) or Tan and da Costa Werlang (1988)). See footnote 10.

This analysis requires modifying the assumptions slightly to the case of multiple firms. The assumption of general position now imposes that the payoffs to any firm i, if it enters the market, is not 0.

Assumption 3.1. *Let the following hold: for all firms i, and any entry decisions y_{-i},*

$$\pi^i(y_1, y_2, \ldots, y_{i-1}, 1, y_{i+1}, \ldots, y_N) \neq 0.$$

The assumption of the monotonicity of the payoffs now imposes that the payoffs to firm i, when firm i enters, depends only on the number of other firms that enter, and is weakly decreasing in the number of other firms that enter.

Assumption 3.2. *Let the following hold: for all firms i,*

$$\pi^i(y_1, y_2, \ldots, y_{i-1}, 1, y_{i+1}, \ldots, y_N) = \pi^i(\sum_{j \neq i} y_j),$$

and if $N - 1 \geq M \geq M' \geq 0$ then $\pi^i(M') \geq \pi^i(M)$.

Equivalently, for all realizations of the uncertainty, if y_{-i} and y'_{-i} are realized entry decisions in which there are weakly more entrants in y'_{-i}, it holds that

$$\pi^i(y_1, y_2, \ldots, y_{i-1}, 1, y_{i+1}, \ldots, y_N) \geq \pi^i(y'_1, y'_2, \ldots, y'_{i-1}, 1, y'_{i+1}, \ldots, y'_N).$$

Similar results can also be derived under other notions of monotonicity. For example, it could be that the payoffs depend on the identities of the other firms that enter, or on the numbers of firms that enter in each of a few known classes of firms. The results would be similar to the ones reported here, but the proofs would likely be more complicated. The next theorem derives the sharp marginal identified sets for the best response probabilities in the many firms case.

Theorem 3.1. *Let Assumptions 3.1 and 3.2 hold. Assume further that each firm exhibits at least 1 level of rationality. The sharp marginal identified set for $v_{i1} = P(v^i(1, 1, \ldots, 1) = 1)$*

is $[0, P(y_i = 1)]$ *and for* $\upsilon_{i0} = P(\upsilon^i(0,0,\ldots,0) = 1)$ *is* $[P(y_i = 1), 1]$. *The sharp marginal identified set for* $\upsilon_{im} = P(\upsilon^i(t_{-i}) = 1)$ *where* $t_{-i} \neq (1,1,\ldots,1)$ *and* $t_{-i} \neq (0,0,\ldots,0)$ *is* $[0,1]$. *These identified sets remain sharp even if the econometrician assumes that there is Nash equilibrium play in each market.*

Proof. Use the notation that $(1,1,\ldots,1) = 1$ and $(0,0,\ldots,0) = 0$. By the law of total probability, where it is understood that $P(B|A)P(A) = 0$ if $P(A) = 0$, it holds that

$$\begin{aligned}
P(\upsilon^i(1) = 1) &= P(\pi^i(1) \geq 0) \\
&= P(\pi^i(1) \geq 0 | y_i = 1)P(y_i = 1) + P(\pi^i(1) \geq 0 | y_i = 0)P(y_i = 0) \\
P(\upsilon^i(0) = 1) &= P(\pi^i(0,0,\ldots,1,\ldots,0) \geq 0) \\
&= P(\pi^i(0,0,\ldots,1,\ldots,0) \geq 0 | y_i = 1)P(y_i = 1) \\
&+ P(\pi^i(0,0,\ldots,1,\ldots,0) \geq 0 | y_i = 0)P(y_i = 0)
\end{aligned}$$

where in $\pi^i(0,0,\ldots,1,\ldots,0)$ the 1 corresponds to the entry decision of firm i.

Consider the implications of the assumption that for each realization of the uncertainty each firm is $\mathcal{R}1$, for a given realization of the uncertainty. Firm i can enter the market with positive probability if and only if there are strategies of firms $-i$ that enter the market with probabilities σ_{-i} such that $\int \pi^i(y_1, y_2, \ldots, y_{i-1}, 1, y_{i+1}, \ldots, y_N) d\sigma_{-i}(y_1, y_2, \ldots, y_N) \geq 0$.[10] By monotonicity, $\pi^i(0,0,\ldots,1,\ldots,0)$ is weakly greater than every term in this integral, so by general position, this implies that $\pi^i(0,0,\ldots,1,\ldots,0) > 0$. Therefore, if $y_i = 1$, then $\pi^i(0,0,\ldots,1,\ldots,0) > 0$. Firm i can not enter the market with positive probability if and only if there are strategies of firms $-i$ that enters the market with probability σ_{-i} such that $\int \pi^i(y_1, y_2, \ldots, y_{i-1}, 1, y_{i+1}, \ldots, y_N) d\sigma_{-i}(y_1, y_2, \ldots, y_N) \leq 0$. By monotonicity, $\pi^i(1)$ is weakly less than every term in this integral, so by general position, this implies that $\pi^i(1) < 0$.

[10] It is here where "correlated levels of rationality" is seen to be enough, since we do not use that $d\sigma_{-i}(y_1, y_2, \ldots, y_N)$ is the product of the marginal distributions.

Therefore, if $y_i = 0$, then $\pi^i(1) < 0$. Therefore, under $\mathcal{R}1$ the expressions above simplify to

$$P(\upsilon^i(1) = 1) = P(\pi^i(1) \geq 0) = P(\pi^i(1) \geq 0|y_i = 1)P(y_i = 1)$$

$$P(\upsilon^i(0) = 1) = P(\pi^i(0, 0, \ldots, 1, \ldots, 0) \geq 0)$$

$$= P(y_i = 1) + P(\pi^i(0, 0, \ldots, 1, \ldots, 0) \geq 0|y_i = 0)P(y_i = 0)$$

The claimed bounds obtain after replacing the remaining conditional probability statements with probabilities in $[0, 1]$. It remains only to show that these bounds are sharp. It is enough to show that the bounds are sharp under the assumption that for each realization of the uncertainty there is Nash equilibrium play since this implies the firms are also $\mathcal{R}k$ for any k.

It is consistent with Nash equilibrium play to observe $y_i = 1$ with either $\pi^i(1) > 0$ or $\pi^i(1) < 0$. First, suppose that $\pi^i(1) > 0$. Consider some realized entry decision y with $y_i = 1$. Specify the payoffs to all of the firms j other than i as follows. If j enters, specify that its payoff when it enters at the realized entry decisions for $-j$ is positive. If j does not enter, specify that its payoff when it enters at the realized entry decisions for $-j$ is negative. Then there is a pure strategy Nash equilibrium in which the firms use the strategy the same as the corresponding decision in y. Second, suppose that $\pi^i(1) < 0$. Consider some realized entry decision y with $y_i = 1$.

Suppose first that $y \neq 1$, so at least one firm other than i does not enter the market. Specify the payoffs to all of the firms j as follows. If j enters, specify that its payoff when it enters at the realized entry decisions for $-j$ is positive. If j does not enter, specify that its payoff when it enters at the realized entry decisions for $-j$ is negative. Then there is a pure strategy Nash equilibrium in which the firms use the strategy the same as the corresponding decision in y. Since at least one firm other than i does not enter the market, it is consistent with the assumptions that even though i gets positive payoff when the observed firms enter, it has $\pi^i(1) < 0$.

Suppose otherwise that $y = 1$. Specify the payoffs to all of the firms other than i and some $j \neq i$ to be always positive, and specify that their pure strategy is to enter the market. Specify the payoffs to i and j to be such that, given that all other firms enter the market, the payoff to i (j) when j (i) enters the market is negative, but the payoff when j (i) does not enter the market is positive. Thus, the decision of firm j is determinative to firm i about whether to enter, given the decisions of the other firms to enter. Therefore, firms i and j can each play a mixed strategy such that they are both indifferent between entering and not entering the market. Therefore, these strategies, together with the pure strategies of the other firms, comprises a Nash equilibrium. These strategies are consistent with the realized entry decision $y = 1$.

Let all the realized entry decisions such that $y_i = 1$ be listed in $x_1, \ldots, x_K$, and let x_{K+1} collect all of the the realized entry decisions with $y_i = 0$. Let a partition $A_k \in \mathcal{F}$ of Ω satisfy $P(A_k) = P(x_k)$ for $k = 1, 2, \ldots, K+1$. For $k = 1, 2, \ldots, K$, set the realized entry decision on A_k to be x_k. For any $k = 1, 2, \ldots, K$, for any $p \in [0, 1]$, let $B_k \in \mathcal{F}$ be such that $B_k \subset A_k$ and $P(B_k|A_k) = p$. On B_k specify that the payoffs are such that $\pi^1(1) > 0$, and on $A_k \cap B_k^C$ specify that the payoffs are such that $\pi^1(1) < 0$. Thus, the sharp identified set for $P(\pi^i(1) \geq 0|y_i = 1)$ is $[0, 1]$.

Similarly, it is consistent with Nash equilibrium play to observe $y_i = 0$ with either $\pi^i(0, 0, \ldots, 1, \ldots, 0) > 0$ or $\pi^i(0, 0, \ldots, 1, \ldots, 0) < 0$.

This establishes the claimed sharp identified sets.

Now consider the sharp identified set for $\upsilon_{im} = P(\upsilon^i(t_{-i}) = 1)$ where $t_{-i} \neq (1, 1, \ldots, 1)$ and $t_{-i} \neq (0, 0, \ldots, 0)$. By definition, $P(\upsilon^i(t_{-i}) = 1) = P(\pi^i(t_1, t_2, \ldots, t_{i-1}, 1, t_{i+1}, \ldots, t_N) > 0)$.

Any realized entry decision is consistent, under the assumption of Nash equilibrium play, with $\pi^i(t_1, t_2, \ldots, t_{i-1}, 1, t_{i+1}, \ldots, t_N) > 0$ and with $\pi^i(t_1, t_2, \ldots, t_{i-1}, 1, t_{i+1}, \ldots, t_N) < 0$. For $\pi^i(t_1, t_2, \ldots, t_{i-1}, 1, t_{i+1}, \ldots, t_N) > 0$, suppose that, for each firm, monopoly profits from entering are positive (call this π_M, which is the same for all firms) and profits to any other arrangement of entry decisions are negative (call this π_N, which is the same for all firms and for all arrangements of entry decisions other than monopoly). Suppose that all other firms enter the market with the same probability σ, so that no other firms enter the market with probability $\tau = (1 - \sigma)^{N-1}$. Then the profit to firm i from entering the market is $\tau\pi_M + (1 - \tau)\pi_N$. Since $\pi_M > 0$ and $\pi_N < 0$ there is some probability τ such that the payoff to firm i from entering the market is 0. Therefore, all firms using the mixed strategy σ is a mixed strategy Nash equilibrium. Similarly, for $\pi^i(t_1, t_2, \ldots, t_{i-1}, 1, t_{i+1}, \ldots, t_N) < 0$, suppose that, for each firm, profits from entering to sharing the market with every firm is negative (call this π_F, which is the same for all firms) and profits to any other arrangement of entry decisions are positive (call this π_N, which is the same for all firms and for all arrangements of entry decisions other than sharing the market with every firm). By similar arguments, there is a mixed strategy Nash equilibrium with all firms using the mixed strategy σ.

Let all the realized entry decisions be listed in $x_1, \ldots, x_K$. Let a partition $A_k \in \mathcal{F}$ of Ω satisfy $P(A_k) = P(x_k)$. For $k = 1, 2, \ldots, K$, set the realized entry decision on A_k to be x_k. For any $k = 1, 2, \ldots, K$, for any $p \in [0, 1]$, let $B_k \in \mathcal{F}$ be such that $B_k \subset A_k$ and $P(B_k|A_k) = p$. On B_k specify that the payoffs are some payoffs such that $\pi^i(t_1, t_2, \ldots, t_{i-1}, 1, t_{i+1}, \ldots, t_N) > 0$, and on $A_k \cap B_k^C$ specify that the payoffs are some payoffs such that $\pi^i(t_1, t_2, \ldots, t_{i-1}, 1, t_{i+1}, \ldots, t_N) < 0$. Thus, the sharp identified set for $P(\pi^i(t_1, t_2, \ldots, t_{i-1}, 1, t_{i+1}, \ldots, t_N) > 0)$ is $[0, 1]$. □

The sharp marginal identified set for the best response to all other firms entering, or not entering, is effectively the same as that in the case of two firms. Nothing is learned about the best response to any other conjectured entry decisions of the other firms, in the sense that the sharp marginal identified sets are $[0, 1]$, which is already logically implied without data. We do not address the question of the sharp joint identified set in the case of $N > 2$ firms. Extrapolating from the result for two firms, we should expect that assuming that firms exhibit at least 2 levels of rationality would tighten the joint identified set.

In fact, it is reasonable to suspect that assuming even more than 2 levels of rationality would further tighten the joint identified set when there are $N > 2$ firms. We provide an example of this in appendix B.

4. Related literature

As noted in the introduction, there is an important literature on entry games in econometrics and industrial organization. Some important early papers using game theory models in an econometric model of entry are Bresnahan and Reiss (1990), Bresnahan and Reiss (1991b) and Berry (1992). These papers are focused on achieving identification by assuming a linear functional form for profits that depends on observed covariates, by assuming that firms are engaged in Nash equilibrium play, and by other "coherency" conditions that deal with the multiple outcome problem of multiple equilibria and mixed strategies. The focus of these papers, and papers that follow them, is to identify the profit function. See Reiss (1996) for an early review of these papers. Tamer (2003) shows how to deal with the multiple outcome problem, but still assumes a linear functional form for profits and that firms are engaged in Nash equilibrium play. Aradillas-Lopez and Tamer (2008) consider the identification power of replacing the assumption of Nash equilibrium play with levels of rationality. Again, the focus of these papers is to identify the profit function. See reviews of this and related literatures in Berry and Reiss (2007) and Berry and Tamer (2007).

This paper has avoided parametric assumptions and explored the identification power of different behavioral assumptions. Moreover, this paper has not imposed any sort of "coherency" condition. The focus of this paper is to identify the best response function. Identification of the best response function is related to but not the same as identification of the profit functions. For example, it could be that the profit function is not point identified, but that the identified bounds on $\pi^1(1,1)$ are such that the sign of $\pi^1(1,1)$ is point identified, in which case $P(\upsilon^1(1) = 1) = P(\pi^1(1,1) \geq 0)$ is point identified. As has been discussed, in some policy situations the best response function is the quantity of interest, rather than the profit functions directly.

We also consider the inferential impact of admitting mixed strategies in detail (see also Bajari, Hong, and Ryan (2010)). We show that admitting or not admitting mixed strategies can affect the identified bounds when also assuming Nash equilibrium behavior, and other assumptions. We also show that the interpretation of mixed strategy can make a difference:

the identifying power of the von Neumann and Morgenstern (1944) interpretation of a mixed strategy Nash equilibrium with independence across markets is different than the identifying power of the Harsanyi (1973)-Aumann (1987) interpretation or without independence across markets.

5. Conclusions and further discussion

This paper has studied the identification of best response functions in an entry game, without parametric assumptions on the payoffs. This paper has done so under varied assumptions on the rationality of the firms, the symmetry of the payoffs between firms, and whether mixed strategies are admitted. In general there is not point identification, and identification under the assumption of Nash equilibrium is the same as the identification under an assumption of a minimal number of levels of rationality. More specifically, the identification power of Nash equilibrium compared to just a sufficient number of levels of rationality varies depending on the other assumptions made. For example, assuming Nash equilibrium play does not seem to matter under just Assumptions 2.1 - 2.3, but substantially adds identifying power under all Assumptions 2.1 - 2.4. See Corollaries 2.3 and 2.4. Although the paper does not discuss estimation, estimation is standard. Especially, estimation of the marginal identified sets is straightforward, as they are intervals with endpoints characterized by moment conditions.

We conclude by remarking on four important issues, and especially on extending these results to other settings. As a whole these remarks show that since our method of analysis applies to many other settings, this paper sheds further light on the general question, originally asked by Manski (1995) as described in the introduction, of the identification of best response functions in games. Our results show that the link between the data, or realized outcomes, and the underlying best response functions is not simple. In addition, our results shed further light on the inferential questions in general discrete outcome models with "peer effects," and more generally in situations in which treatments are outcomes for other units.

Relationship to more parametric models: There have been many identification results in the literature for discrete games, but mostly with distributional assumptions on the payoffs. In those papers there are still issues with multiple equilibria, mixed strategies, and, in the case of incomplete information, how to deal with the information structure. The results in this paper show that with only minimal assumptions the discrete game setup does not contain empirical content; note that key to this is allowing firms to play mixed strategies. This is in contrast to the usual treatment effect literature in which the bounds, though sometimes wide, are typically non-trivial, and thus informative. The size of the identified set will depend on the assumptions that one brings to bear. For example, it is plausible that some values in the identified set require extreme forms of behavior by the firms, which might be ruled out by explicit assumption, or ruled out by implication by some parametric models. See for example the discussion in Remark 2.2 above.

Non-entry games: The assumption of the entry game payoff structure does, in general, entail a loss of generality, compared to an arbitrary game with two players and two actions. The assumption of the entry game payoff structure is the assumption that the payoff to a firm is zero if that firm does not enter, for either entry decision of the other firm. The results are different if we allow payoffs for a non-entering firm to be nonzero when the other firm enters. In particular, the marginal identified sets for best response probabilities in a two player and two action game can be $[0, 1]$ without the entry game payoff structure. For example, consider the game with two specifications of possible payoffs given in Table 2.

TABLE 2. Non-entry game: (a) first game, (b) second game

(a)

	$y_2 = 0$	$y_2 = 1$
$y_1 = 0$	$0, 0$	$-3, -1$
$y_1 = 1$	$-1, -3$	$-2, -2$

(b)

	$y_2 = 0$	$y_2 = 1$
$y_1 = 0$	$0, 0$	$-1, 1$
$y_1 = 1$	$1, -1$	$-2, -2$

The first specification of payoffs in game 1 has a mixed strategy Nash equilibrium in which both firms choose action 1 with probability $\frac{1}{2}$, and the second specification of payoffs in game 2 has a mixed strategy Nash equilibrium in which both firms choose action 1 with probability $\frac{1}{2}$. Thus, any realized outcome is consistent with either payoff structure. This implies that the sharp marginal identified set for $P(\pi^1(1, 0) \geq 0)$, and thus for the best response of firm 1 to firm 2 playing 0, is $[0, 1]$ under the assumptions of payoffs in general position, monotonicity in the sense that given that firm i chooses either action it has higher payoff when firm $-i$ chooses action 0, that payoffs are symmetric, and that there is Nash equilibrium play.

We provide an example of such a non-entry game to illustrate this game has economic relevance. The payoffs might obtain if the game modeled a contest to research and adopt a new technology; especially, consider adopting a new technology to mean replacing an old product with a new product in the market. The decision to research and adopt is action 1 in the normal form. Suppose that the cost of researching and adopting the new technology is 8. Suppose that, for each firm, the profits gross of research costs in the market when neither firm adopts the new technology is 3. If both firms adopt the new technology the profits gross of research costs in the market is 9, for each firm. Thus, if neither firm adopts the new technology, both firms get payoffs 3, and if both adopt, both firms get payoff 1. The difference in the two specifications of payoffs concerns what happens if only one firm adopts. In the first specification, suppose that if only one firm adopts the new technology, the firm that adopts gets profits gross of research costs 10 and the other firm gets 0. Thus, the firm that adopts gets payoff 2 and the firm that does not adopt gets payoff 0. In the second specification, suppose that if only one firm adopts the new technology, the profits gross of research costs to the firm that adopts gets 12 and the other firm gets 2. Thus, the firm that adopts gets payoff 4 and the firm that does not adopt gets payoff 2. The payoffs in the normal forms are normalized, by subtracting 3, so that the payoff to both firms when both firms play 0, is 0.

It is also true that the entry game payoff structure is without loss of generality in the sense that the set of Nash equilibria is exactly the same for the two games given in normal form in Table 3. However, the monotonicity assumption, or its generalization to a non-entry game, can be satisfied in the first game but violated in the second game. Indeed, it can happen that $0 > \pi^1(0,1)$ and $\pi^1(1,0) > \pi^1(1,1)$, so that monotonicity is satisfied in the first game for firm 1 but that monotonicity is violated in the second game if $\pi^1(0,1)$ is sufficiently negative. This shows that payoff normalization needs to be without loss of generality not only with respect to the solution concept, but also the assumptions maintained by the econometrician. We focus in this paper on the important special case of an entry game, but note again that our method of analysis can be applied to study other games.

TABLE 3. Non-entry game with general payoffs: (a) first game, (b) second game

	$y_2 = 0$	$y_2 = 1$
$y_1 = 0$	$0, 0$	$\pi^1(0,1), \pi^2(0,1)$
$y_1 = 1$	$\pi^1(1,0), \pi^2(1,0)$	$\pi^1(1,1), \pi^2(1,1)$

(a)

	$y_2 = 0$	$y_2 = 1$
$y_1 = 0$	$0, 0$	$0, \pi^2(0,1)$
$y_1 = 1$	$\pi^1(1,0), 0$	$\pi^1(1,1) - \pi^1(0,1), \pi^2(1,1) - \pi^2(1,0)$

(b)

Incomplete information: We have studied an entry game with complete information, under various additional assumption. A similar analysis holds if we study an entry game with incomplete information, where complete information cannot be ruled out. An example of those games are ones in which firms do not know the profit function of other firms, possibly with heterogeneity unobserved by the econometrician.

In some cases, the results derived for a game with complete information translate immediately to a game with incomplete information. There are two issues in translating the results from a game with complete information to a game with incomplete information. The first issue is establishing that the restrictions implying the bounds still hold with incomplete information. Depending on the exact solution concepts entertained, this is likely to hold, for example, if the restrictions require only that the firms know their own payoffs, as in exhibiting 1 level of rationality. The second issue is establishing that the bounds are still sharp. Again, depending on the exact solution concepts entertained, this is likely to hold, for example, if complete information cannot be ruled out. In the case that the econometrician assumes there to be a non-trivial information structure, our analysis need not hold. An approach to this in parametric setups is in the work of Aradillas-Lopez and Tamer (2008). This is only a sketch of the ideas involved in the analysis of incomplete information, and extensions of our work to cover incomplete information with a non-trivial information structure is left for future work.

Testable restrictions: For each combination of assumptions entertained the joint identified set is never empty, and in that sense these assumptions cannot be ruled out for any observables. In particular, from Corollary 2.4 it is never possible to rule out that firms are engaged in Nash equilibrium play, monopoly payoffs are weakly greater than duopoly payoffs, payoffs are weakly symmetric across firms, and firms use pure strategies. For any data there is a simple game theoretic rationalization of the data satisfying these assumptions. For realizations of the uncertainty such that $(1,1)$ is the realized entry decision, set both firms to have positive monopoly and duopoly payoffs; such that $(1,0)$ or $(0,1)$ is the realized entry decision, set both firms to have positive monopoly but negative duopoly payoffs; and such that $(0,0)$ is the realized entry decision, set both firms to have negative monopoly and duopoly payoffs. In all cases, the assumptions are satisfied, and the realized entry decision is a pure strategy Nash equilibrium outcome for those payoffs.

It is possible to consider assumptions that result in an identified set that can be empty for some datasets, in which case the assumptions are rejected for that data. For example, the additional assumption that firm 1 has negative monopoly payoffs for probability p of the uncertainty is rejected for those values of the observables for which firm 1 enters the market with probability greater than $1-p$. This is because, under level 1 rationality and monotonicity, whenever firm 1 has negative monopoly profits, it must not enter the market. Alternatively, suppose some additional parametric assumption is made about the payoffs. This parametric assumption will imply some possibly complicated additional structure of the identified sets. If these additional assumptions result in the identified set being empty for some datasets, and the other assumptions are considered maintained, the parametric assumptions are rejected.

Appendix A. Results about solution concepts

Proof of Lemma 2.3. If $(0,0)$ is a Nash equilibrium, the claim is established. So assume that $(0,0)$ is not a Nash equilibrium. This requires that at least one firm has a profitable deviation, so this requires either that $\pi^2(0,1) > 0$ or $\pi^1(1,0) > 0$.

Suppose that $\pi^2(0,1) > 0$. If $(0,1)$ is not a Nash equilibrium it must be because firm 1 has a profitable deviation, so $\pi^1(1,1) > 0$. If $(1,1)$ is not a Nash equilibrium, it must be because firm 2 has a profitable deviation, so $\pi^2(1,1) < 0$. This implies that $(1,0)$ is a Nash equilibrium, since by monotonicity $\pi^1(1,0) \geq \pi^1(1,1) > 0$ and $\pi^2(1,1) < 0$.

By exchanging firm 1 with firm 2, this establishes also the existence of a pure strategy Nash equilibrium if $\pi^1(1,0) > 0$. □

The definition of levels of rationality used in this paper may appear to be not the same as that introduced by Bernheim (1984) and especially Pearce (1984), but the following lemma establishes the equivalency. In all the definitions considered in this lemma, a strategy in $\mathcal{R}^i(k+1,\pi)$ is a best response, in some sense, against some conjecture in $\mathcal{R}^{-i}(k,\pi)$. The first definition in this lemma is the one used in this paper. According to this definition, the strategies that are a best response to some conjecture in $\mathcal{R}^{-i}(k,\pi)$, compared to any other strategy of firm i, are collected in $\mathcal{R}^i(k+1,\pi)$. The second definition in this lemma seems to be another reasonable definition of levels of rationality. According to this definition, the strategies in $\mathcal{R}^i(k,\pi)$ that are a best response to some conjecture in $\mathcal{R}^{-i}(k,\pi)$, compared to any strategy of firm i in $\mathcal{R}^i(k,\pi)$, are collected in $\mathcal{R}^i(k+1,\pi)$. The third definition in this lemma is the one used by Pearce (1984, Definition 1), and also in Fudenberg and Tirole (1991). It is the same as the second definition, except that conjectures can be mixtures of strategies in $\mathcal{R}^{-i}(k,\pi)$. In a two player, binary game the definitions are equivalent.

Lemma A.1. *The sets recursively defined as: for each firm i, $\mathcal{R}^i(0,\pi) = \Delta^1$ and for $k \geq 0$,*

(i)

$$\mathcal{R}^i_1(k+1,\pi) =$$

$$\{\sigma^i \in \Delta^1 : \exists \sigma^{-i} \in \mathcal{R}^{-i}_1(k,\pi) \text{ s.t. } E_{\sigma^1,\sigma^2}\pi^i(y_1,y_2) \geq E_{\sigma^{i'},\sigma^{-i}}\pi^i(y_1,y_2) \text{ for all } \sigma^{i'} \in \Delta^1\}$$

(ii)

$$\mathcal{R}_2^i(k+1,\pi) =$$

$$\{\sigma^i \in \mathcal{R}_2^i(k,\pi) : \exists \sigma^{-i} \in \mathcal{R}_2^{-i}(k,\pi) \text{ s.t. } E_{\sigma^1,\sigma^2}\pi^i(y_1,y_2) \geq E_{\sigma^{i'},\sigma^{-i}}\pi^i(y_1,y_2) \text{ for all } \sigma^{i'} \in \mathcal{R}_2^i(k,\pi)\}$$

(iii)

$$\mathcal{R}_3^i(k+1,\pi) =$$

$$\{\sigma^i \in \mathcal{R}_3^i(k,\pi) : \exists \sigma^{-i} \in co(\mathcal{R}_3^{-i}(k,\pi)) \text{ s.t. } E_{\sigma^1,\sigma^2}\pi^i(y_1,y_2) \geq E_{\sigma^{i'},\sigma^{-i}}\pi^i(y_1,y_2) \text{ for all } \sigma^{i'} \in \mathcal{R}_3^i(k,\pi)\}$$

are the same for all k, for each firm i.

Proof. The proof that the sets defined by (i) and (ii) are equivalent is established by induction on k. It is immediate from the definitions that $\mathcal{R}_1^i(1,\pi) = \mathcal{R}_2^i(1,\pi)$. So, for some $k \geq 1$, assume that for all $k' \leq k$, $\mathcal{R}_1^i(k',\pi) = \mathcal{R}_2^i(k',\pi)$, in order to prove that $\mathcal{R}_1^i(k+1,\pi) = \mathcal{R}_2^i(k+1,\pi)$.

Suppose that $\sigma^i \in \mathcal{R}_1^i(k+1,\pi)$. Then, by Lemma 2.1, $\sigma^i \in \mathcal{R}_1^i(k,\pi)$, and thus by the induction hypothesis $\sigma^i \in \mathcal{R}_2^i(k,\pi)$. Thus, using the same conjecture as rationalizes σ^i for $\mathcal{R}_1^i$, $\sigma^i \in \mathcal{R}_2^i(k+1,\pi)$. Therefore, $\mathcal{R}_1^i(k+1,\pi) \subseteq \mathcal{R}_2^i(k+1,\pi)$.

Suppose that $\sigma^i \in \mathcal{R}_2^i(k+1,\pi)$. Then, $\sigma^i \in \mathcal{R}_2^i(k,\pi) = \mathcal{R}_1^i(k,\pi)$, so by Lemma 2.1, $\sigma^i \in \Delta^1$. Consider the conjecture $\sigma^{-i} \in \mathcal{R}_2^{-i}(k,\pi)$ against which σ^i is a best response, compared to the strategies of firm i in $\mathcal{R}_2^i(k,\pi)$. It holds that $\sigma^i \in \mathcal{R}_1^i(k+1,\pi)$ as long as σ^i is also a best response against σ^{-i}, compared to all strategies of firm i. Let $\sigma^{i'}$ be the utility maximizing response to σ^{-i} compared to all strategies of firm i. This exists since there are only finitely many actions. If it happens that $\sigma^{i'} \in R_2^i(k,\pi)$ then σ^i is a best response compared to all strategies of firm i, since it is a best response compared to the strategies of firm i in $\mathcal{R}_2^i(k,\pi)$. Otherwise, $\sigma^{i'} \in \Delta^1 \backslash R_2^i(k,\pi)$. Since $\sigma^{-i} \in \mathcal{R}_2^{-i}(k,\pi)$, also $\sigma^{-i} \in \mathcal{R}_2^{-i}(k-1,\pi)$, by Lemma 2.1 and the induction hypothesis. Thus, σ^{-i} is an admissible conjecture for $\mathcal{R}_2^i(k,\pi)$, and therefore is a admissible conjecture for $\mathcal{R}_1^i(k,\pi)$, by the induction hypothesis. Therefore, based on the conjecture σ^{-i}, $\sigma^{i'} \in \mathcal{R}_1^i(k,\pi)$, and therefore $\sigma^{i'} \in \mathcal{R}_2^i(k,\pi)$, by the induction hypothesis. This contradicts that $\sigma^{i'} \in \Delta^1 \backslash \mathcal{R}_2^i(k,\pi)$. Therefore, indeed σ^i is a best response against σ^{-i}, compared to all strategies of firm i, so $\sigma^i \in \mathcal{R}_1^i(k+1,\pi)$. Therefore, $\mathcal{R}_2^i(k+1,\pi) \subseteq \mathcal{R}_1^i(k+1,\pi)$.

The proof that the sets defined by (ii) and (iii) are equivalent is established by induction on k. It is immediate from the definitions that $\mathcal{R}_2^i(1,\pi) = \mathcal{R}_3^i(1,\pi)$. Moreover, $\mathcal{R}_3^i(1,\pi)$ is convex. If $\mathcal{R}_3^i(1,\pi)$ contains any mixed strategy, there is a conjecture against which entering and not entering gives the same payoff to firm i, and therefore $\mathcal{R}_3^i(1,\pi) = \Delta^1$, so is convex. If $\mathcal{R}_3^i(1,\pi)$ contains only one pure strategy, it is convex. So suppose that $\mathcal{R}_3^i(1,\pi)$ contains both pure strategies. Then there is a conjecture σ_1^{-i} against which entering gives non-negative payoff, and a conjecture σ_2^{-i} against which entering gives non-positive payoff. Thus, if $i=1$, identifying a strategy with the probability of entering, $\sigma_1^{-i}\pi^i(1,1)+(1-\sigma_1^{-i})\pi^i(1,0) \geq 0$ and $\sigma_2^{-i}\pi^i(1,1)+(1-\sigma_2^{-i})\pi^i(1,0) \leq 0$, and similarly if $i=2$, $\sigma_1^{-i}\pi^i(1,1)+(1-\sigma_1^{-i})\pi^i(0,1) \geq 0$ and $\sigma_2^{-i}\pi^i(1,1)+(1-\sigma_2^{-i})\pi^i(0,1) \leq 0$. Therefore there must be a conjecture σ^{-i} in the convex hull of σ_1^{-i} and σ_2^{-i} against which entering gives zero payoff, against which each mixed strategy is a best response, so, since $\mathcal{R}_3^{-i}(0,\pi)$ is convex and therefore includes σ^{-i}, $\mathcal{R}_3^i(1,\pi) = \Delta^1$ is convex. So, for some $k \geq 1$, assume that $\mathcal{R}_2^i(k,\pi) = \mathcal{R}_3^i(k,\pi)$ and $\mathcal{R}_3^i(k,\pi)$ is convex, in order to prove that $\mathcal{R}_2^i(k+1,\pi) = \mathcal{R}_3^i(k+1,\pi)$ and $\mathcal{R}_3^i(k+1,\pi)$ is convex.

If $\sigma^i \in \mathcal{R}_2^i(k+1,\pi)$, then there is a conjecture $\sigma^{-i} \in \mathcal{R}_2^{-i}(k,\pi)$ against which σ^i is a best response compared to all strategies of firm i in $\mathcal{R}_2^i(k,\pi)$. The conjecture σ^{-i} is also an element of $co(\mathcal{R}_3^{-i}(k,\pi))$, and therefore $\sigma^i \in \mathcal{R}_3^i(k+1,\pi)$. Therefore, $\mathcal{R}_2^i(k+1,\pi) \subseteq \mathcal{R}_3^i(k+1,\pi)$.

If $\sigma^i \in \mathcal{R}_3^i(k+1,\pi)$, then there is a conjecture $\sigma^{-i} \in co(\mathcal{R}_3^{-i}(k,\pi))$ against which σ^i is a best response compared to all strategies of firm i in $\mathcal{R}_3^i(k,\pi)$. Since $\mathcal{R}_3^{-i}(k,\pi)$ is convex, also $\sigma^{-i} \in \mathcal{R}_3^i(k,\pi) = \mathcal{R}_2^i(k,\pi)$. Thus, $\sigma^i \in \mathcal{R}_2^i(k+1,\pi)$. Therefore, $\mathcal{R}_3^i(k+1,\pi) \subseteq \mathcal{R}_2^i(k+1,\pi)$. The convexity of $\mathcal{R}_3^i(k+1,\pi)$ follows by the same arguments as the convexity of $\mathcal{R}_3^i(1,\pi)$, since $\mathcal{R}_3^i(k,\pi)$ is convex by the induction hypothesis. □

Appendix B. Identification under $\mathcal{R}k$, $k > 2$, with $N > 2$ firms

For example, suppose that there are 3 firms. We consider two possible arrangements of firm profits. The profits are as follows in the first arrangement. Firm 1 profits from entering are positive for any entry decisions of the other firms. Firm 2 profits from entering are positive if it has a monopoly, but are negative otherwise. Firm 3 profits from entering are positive as long as it does not share the market with both of the other two firms. The level 0 strategies for all firms is Δ^1. The level k strategy for firm 1, for all $k \geq 1$, is to enter, since it gets positive profits no matter what the other firms do. The level 1 strategies for firm 2 includes entering and not entering. If neither of the other firms enter, firm 2 entering is the best response, and otherwise if either firm enters, firm 2 not entering is the best response. Also any mixture is a level 1 strategy. Similarly, the level 1 strategies for firm 3 includes entering and not entering. If neither of the other firms enter, firm 3 entering is the best response, and if both other firms enter, firm 3 not entering is the best response. Also any mixture is a level 1 strategy. The level 2 strategy of firm 2 is to not enter, because the only level 1 strategy of firm 1 is to enter. Consequently the only admissible conjecture at level 2 to firm 2 has firm 1 entering, against which not entering is the best response for firm 2. The same is true for any level greater than 2. The level 2 strategies of firm 3 includes entering and not entering. If only firm 1 enters, firm 3 entering is the best response, and if both other firms enter, firm 3 not entering is the best response. Also any mixture is a level 2 strategy. Finally, the level 3 strategy of firm 3 is to enter the market. This is because the only admissible conjecture at level 3 to firm 3 is for firm 1 to enter and firm 2 to not enter, against which entering is the best response for firm 3.

The profits are as follows in the second arrangement, which only changes profits for firm 2. Firm 1 profits from entering are positive for any entry decisions of the other firms. Firm 2 profits from entering are positive as long as it does not share the market with both of the other two firms. Firm 3 profits from entering are positive as long as it does not share the market with both of the other two firms. The level 0 strategies for all firms is Δ^1. The level k strategy for firm 1, for all $k \geq 1$, is to enter, since it gets positive profits no matter what the other firms do. The level 1 strategies for firm 2 includes entering and not entering. If neither of the other firms enter, firm 2 entering is the best response, and otherwise if both other firms enter, firm 2 not entering is the best response. Also any mixture is a level 1 strategy. Similarly, the level 1 strategies for firm 3 includes entering and not entering. Also any mixture is a level 1 strategy. The level 2 strategy of firm 2 includes entering and not entering. If only firm 1 enters, firm 2 entering is the best response, and if both other firms enter, firm 2 not entering is the best response. Also any mixture is a level 2 strategy. Similarly, the level 2 strategies of firm 3 includes entering and not entering. Also any mixture is a level 2 strategy. Finally, the level 3 strategy of firm 2 includes entering and not entering. If only firm 1 enters, firm 2 entering is the best response, and if both other firms enter, firm 2 not entering is the best response. Also any mixture is a level 3 strategy. Similarly, the level 3 strategy of firm 3 includes entering and not entering. Also any mixture is a level 3 strategy.

We can summarize this in the following table.

Table 4. Strategies and levels of rationality: (a) firm 1, (b) firm 2, (c) firm 3

Level	payoffs 1	payoffs 2
0	Δ^1	Δ^1
1	1	1
2	1	1
3	1	1

(a)

Level	payoffs 1	payoffs 2
0	Δ^1	Δ^1
1	Δ^1	Δ^1
2	0	Δ^1
3	0	Δ^1

(b)

Level	payoffs 1	payoffs 2
0	Δ^1	Δ^1
1	Δ^1	Δ^1
2	Δ^1	Δ^1
3	1	Δ^1

(c)

Under the first arrangement of payoffs, the entry decision $(1, 0, 0)$ could be observed under level 2 rationality, but could not be observed under level 3 rationality. But, under the second arrangement of payoffs, the entry decision $(1, 0, 0)$ could be observed under level 3 rationality. This means that payoffs that can be ruled out based on realized entry decision $(1, 0, 0)$ under level 3 rationality are different than under level 2 rationality. Consequently, there seems to be scope for additional levels of rationality beyond 2 to have identifying power for the joint identified set when $N > 2$.

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