ONLINE APPENDIX: OPTIMAL STRATIFICATION IN RANDOMIZED EXPERIMENTS

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By Definition 2 we have

$$var\left(\overline{D}\right) = \left(\frac{2}{N}\right)^2 \left[\sum_{k=1}^{\frac{N}{2}} var(D_k) + \sum_{h\neq k} cov(D_k, D_h)\right] \tag{1}$$

Next, we find expressions for each component of the sum in equation 1

$$\begin{aligned} var(D_k) = {} & E(\theta_i^2) - \theta^2 + 2E(r(X_i)^2) - 2E(r(X_{2k-1})r(X_{2k})) \\ & + 2E\left(\theta_i[r(X_i) + \epsilon_i]\right) - E(\theta_{2k-1}[r(X_{2k}) + \epsilon_{2k}]) - E(\theta_{2k}[r(X_{2k-1}) + \epsilon_{2k-1}]) \\ & + 2E(\epsilon_i^2) \end{aligned} \tag{2}$$

$$cov(D_k, D_h) = \frac{1}{4}[E(\theta_{2k-1}\theta_{2h-1}) + E(\theta_{2k-1}\theta_{2h}) + E(\theta_{2k}\theta_{2h-1}) + E(\theta_{2k}\theta_{2h})] - \theta^2 \tag{3}$$

These give

$$\begin{aligned} \frac{var(\overline{D})}{\left(\frac{2}{N}\right)^2} = {} & \frac{N}{2}\left[E(\theta_i^2) - \theta^2 + 2E(r(X_i)^2) + 2E\left(\theta_i Y_i(0)\right) + 2E(\epsilon_i^2)\right] \\ & - \sum_{k=1}^{\frac{N}{2}} [2E(r(X_{2k-1})r(X_{2k})) + E(\theta_{2k-1}Y_{2k}(0)) + E(\theta_{2k}Y_{2k-1}(0))] \\ & + \sum_{h\neq k} \frac{1}{4}\left[E(\theta_{2k-1}\theta_{2h-1}) + E(\theta_{2k-1}\theta_{2h}) + E(\theta_{2k}\theta_{2h-1}) + E(\theta_{2k}\theta_{2h})\right] \\ & - \frac{N}{2}\left(\frac{N}{2} - 1\right)\theta^2 \end{aligned} \tag{4}$$

0.1. **Derivation of equation 2.**

$$
\begin{aligned}
var(D_k) &= E(\theta_i^2) - \theta^2 + 2E(r(X_i)^2) - 2E(r(X_{2k-1})r(X_{2k})) \\
&\quad + 2E\left(\theta_i[r(X_i) + \epsilon_i]\right) - E(\theta_{2k-1}[r(X_{2k}) + \epsilon_{2k}]) - E(\theta_{2k}[r(X_{2k-1}) + \epsilon_{2k-1}]) \\
&\quad + 2E(\epsilon_i^2)
\end{aligned} \tag{5}
$$

proof: Let $\beta_1 = E[Y_i(0)]$ and $\beta_0 = E[Y_i(1)]$.

$$
\begin{aligned}
E(D_k) = E[E(D_k|T_{2k-1})] &= E(T_{2k-1}[E(Y_{2k-1}(1)|T_{2k-1}) - E(Y_{2k}(0)|T_{2k-1})] \\
&\quad +(1 - T_{2k-1})[E(Y_{2k}(1) - Y_{2k-1}(0)|T_{2k-1})])
\end{aligned}
$$

since $\{Y_i(1), Y_i(0)\} \perp\!\!\!\perp T_j \forall i, j$

$$
\begin{aligned}
E(Y_{2k-1}(1)|T_{2k-1}) &= \beta_1 \\
E(Y_{2k}(0)|T_{2k-1}) &= \beta_0 \\
E(Y_{2k}(1)|T_{2k-1}) &= \beta_1 \\
E(Y_{2k-1}(0)|T_{2k-1}) &= \beta_0
\end{aligned}
$$

$$
E[E(D_k|T_{2k-1})] = E(T_{2k-1}[\beta_1 - \beta_0] + (1 - T_{2k-1})[\beta_1 - \beta_0]) = \theta \tag{6}
$$

$$
E(D_k^2) = E(A^2 + B^2 + 2AB) = E(A^2) + E(B^2) + 2E(AB)
$$

where

$$
\begin{aligned}
A &= T_{2k-1}[Y_{2k-1}(1) - Y_{2k}(0)] \\
B &= (1 - T_{2k-1})[Y_{2k}(1) - Y_{2k-1}(0)]
\end{aligned}
$$

$$
\begin{aligned}
E(A^2) &= E(T_{2k-1}^2[Y_{2k-1}(1) - Y_{2k}(0)]^2) \\
&= E(E(T_{2k-1}^2[Y_{2k-1}(1) - Y_{2k}(0)]^2|T_{2k-1})) \\
&= E(E(T_{2k-1}^2[Y_{2k-1}(1) - Y_{2k}(0)]^2)) \quad \text{since } \{Y_i(1), Y_i(0)\} \perp\!\!\!\perp T_j \\
&= E(T_{2k-1}^2)E([Y_{2k-1}(1) - Y_{2k}(0)]^2) \\
&= \frac{1}{2}[E(Y_i(1)^2) + E(Y_i(0)^2) - 2E(Y_{2k-1}(1)Y_{2k}(0))]
\end{aligned}
$$

$$
\begin{aligned}
E(B^2) &= E((1 - T_{2k-1})^2)E([Y_{2k}(1) - Y_{2k-1}(0)]^2) \\
&= \frac{1}{2}[E(Y_i(1)^2) + E(Y_i(0)^2) - 2E(Y_{2k}(1)Y_{2k-1}(0))]
\end{aligned}
$$

$$
\begin{aligned}
E(AB) &= E(T_{2k-1}[Y_{2k-1}(1) - Y_{2k}(0)](1 - T_{2k-1})[Y_{2k}(1) - Y_{2k-1}(0)]) \\
&= E(T_{2k-1}(1 - T_{2k-1}))E([Y_{2k-1}(1) - Y_{2k}(0)][Y_{2k}(1) - Y_{2k-1}(0)]) \\
&= 0 \quad \text{since } E(T_{2k-1}(1 - T_{2k-1})) = 0
\end{aligned}
$$

So

$$
\begin{aligned}
E(D_k^2) &= E(A^2) + E(B^2) + 2E(AB) \\
&= \frac{1}{2}[E(Y_i(1)^2) + E(Y_i(0)^2) - 2E(Y_{2k-1}(1)Y_{2k}(0))] \\
&\quad + \frac{1}{2}[E(Y_i(1)^2) + E(Y_i(0)^2) - 2E(Y_{2k}(1)Y_{2k-1}(0))] \\
&= E(Y_i(1)^2) + E(Y_i(0)^2) - E(Y_{2k-1}(1)Y_{2k}(0)) - E(Y_{2k}(1)Y_{2k-1}(0))
\end{aligned}
$$

So

$$Var(D_k) = E(Y_i(1)^2) + E(Y_i(0)^2) - E(Y_{2k-1}(1)Y_{2k}(0)) - E(Y_{2k}(1)Y_{2k-1}(0)) - \theta^2$$

Now we use

$$\begin{aligned} Y_i(0) &= r(X_i) + \epsilon_i \\ Y_i(1) &= \theta_i + r(X_i) + \epsilon_i \end{aligned}$$

$$\begin{aligned} E(Y_i(1)^2) &= E[(\theta_i + r(X_i) + \epsilon_i)^2] \\ &= E(\theta_i^2) + E[r(X_i)^2] + E(\epsilon_i^2) + 2E(\theta_i r(X_i)) + 2E(\theta_i\epsilon_i) \quad \text{since } E(r(X_i)\epsilon_i) = 0 \end{aligned}$$

$$\begin{aligned} E(Y_i(0)^2) &= E([r(X_i) + \epsilon_i]^2) \\ &= E(r(X_i)^2) + E(\epsilon_i^2) \end{aligned}$$

$$\begin{aligned} E(Y_{2k-1}(1)Y_{2k}(0)) &= E([\theta_{2k-1} + r(X_{2k-1}) + \epsilon_{2k-1}][r(X_{2k}) + \epsilon_{2k}]) \\ &= E(\theta_{2k-1}[r(X_{2k}) + \epsilon_{2k}] + r(X_{2k-1})[r(X_{2k}) + \epsilon_{2k}] + \epsilon_{2k-1}[r(X_{2k}) + \epsilon_{2k}]) \\ &= E(\theta_{2k-1}[r(X_{2k}) + \epsilon_{2k}]) + E(r(X_{2k-1})r(X_{2k})) \quad \text{since} \quad E(\epsilon_i r(X_j)) = 0 \quad \forall i \neq j. \end{aligned}$$

Similarly,

$$E(Y_{2k}(1)Y_{2k-1}(0)) = E(\theta_{2k}[r(X_{2k-1}) + \epsilon_{2k-1}]) + E(r(X_{2k})r(X_{2k-1}))$$

so

$$\begin{aligned} var(D_k) &= E(\theta_i^2) + E(r(X_i)^2) + E(\epsilon_i^2) + 2E(\theta_i r(X_i)) + 2E(\theta_i\epsilon_i) \\ &\quad + E(r(X_i)^2) + E(\epsilon_i^2) \\ &\quad - E(\theta_{2k-1}[r(X_{2k} + \epsilon_{2k})]) - E(r(X_{2k})r(X_{2k-1})) \\ &\quad - E(\theta_{2k}[r(X_{2k-1}) + \epsilon_{2k-1}]) - E(r(X_{2k})r(X_{2k-1})) - \theta^2 \\ &= E(\theta_i^2) - \theta^2 + 2E(r(X_i)^2) - 2E(r(X_{2k-1})r(X_{2k})) \\ &\quad + 2E(\theta_i[r(x_i) + \epsilon_i]) - E(\theta_{2k-1}[r(X_{2k}) + \epsilon_{2k}]) - E(\theta_{2k}[r(X_{2k-1}) + \epsilon_{2k-1}]) + 2E(\epsilon_i^2) \end{aligned}$$

0.2. **Derivation of equation 3.** Let $k \neq h$

$$cov(D_k, D_h) = \frac{1}{4}[E(\theta_{2k-1}\theta_{2h-1}) + E(\theta_{2k-1}\theta_{2h}) + E(\theta_{2k}\theta_{2h-1}) + E(\theta_{2k}\theta_{2h})] - \theta^2 \tag{7}$$

proof: Let $k \neq h$

$$\begin{aligned} cov(D_k, D_h) &= E(D_k D_h) - E(D_k)E(D_h) \\ &= E(D_k D_h) - \theta^2 \end{aligned}$$

$$\begin{aligned} E(D_k D_h) &= E([A_k + B_k][A_h + B_h]) \\ &= E(A_k A_h) + E(A_k B_h) + E(B_k A_h) + E(B_k B_h) \end{aligned}$$

where

$$\begin{aligned} A_k &= T_{2k-1}[Y_{2k-1}(1) - Y_{2k}(0)] \\ B_k &= (1 - T_{2k-1})[Y_{2k}(1) - Y_{2k-1}(0)] \end{aligned}$$

$$\begin{aligned} E(A_k A_h) &= E(T_{2k-1}[Y_{2k-1}(1) - Y_{2k}(0)]T_{2h-1}[Y_{2h-1}(1) - Y_{2h}(0)]) \\ &= \frac{1}{4}E([Y_{2k-1}(1) - Y_{2k}(0)][Y_{2h-1}(1) - Y_{2h}(0)]) \end{aligned}$$

$$\begin{aligned} E(A_k B_h) &= E(T_{2k-1}[Y_{2k-1}(1) - Y_{2k}(0)](1 - T_{2h-1})[Y_{2h}(1) - Y_{2h-1}(0)]) \\ &= \frac{1}{4} E([Y_{2k-1}(1) - Y_{2k}(0)][Y_{2h}(1) - Y_{2h-1}(0)]) \end{aligned}$$

$$E(B_k A_h) = \frac{1}{4} E([Y_{2k}(1) - Y_{2k-1}(0)][Y_{2h-1}(1) - Y_{2h}(0)])$$

$$E(B_k B_h) = \frac{1}{4} E([Y_{2k}(1) - Y_{2k-1}(0)][Y_{2h}(1) - Y_{2h-1}(0)])$$

So

$$\begin{aligned} E(D_k D_h) &= \frac{1}{4} E([Y_{2k-1}(1) - Y_{2k}(0)][Y_{2h-1}(1) - Y_{2h}(0)]) \\ &+ \frac{1}{4} E([Y_{2k-1}(1) - Y_{2k}(0)][Y_{2h}(1) - Y_{2h-1}(0)]) \\ &+ \frac{1}{4} E([Y_{2k}(1) - Y_{2k-1}(0)][Y_{2h-1}(1) - Y_{2h}(0)]) \\ &+ \frac{1}{4} E([Y_{2k}(1) - Y_{2k-1}(0)][Y_{2h}(1) - Y_{2h-1}(0)]) \end{aligned}$$

$$\begin{aligned} &E([Y_{2k-1}(1) - Y_{2k}(0)][Y_{2h-1}(1) - Y_h(0)]) \\ &\quad = E(\theta_{2k-1}\theta_{2h-1}) + E(\theta_{2k-1} r(X_{2h-1})) + E(\theta_{2k-1}\epsilon_{2h-1}) + E(r(X_{2k-1})\theta_{2h-1}) \\ &\qquad + E(r(X_{2k-1}) r(X_{2h-1})) + E(\epsilon_{2k-1}\theta_{2h-1}) - E(\theta_{2k-1} r(X_{2h})) - E(\theta_{2k-1}\epsilon_{2h}) \\ &\qquad - E(r(X_{2k-1}) r(X_{2h})) - E(r(X_{2k})\theta_{2h-1}) - E(r(X_{2k}) r(X_{2h-1})) - E(\epsilon_{2k}\theta_{2h-1}) \\ &\qquad + E(r(X_{2k}) r(X_{2h})) \end{aligned}$$

$$\begin{aligned} &E([Y_{2k-1}(1) - Y_{2k}(0)][Y_{2h}(1) - Y_{2h-1}(0)]) \\ &\quad = E(\theta_{2k-1}\theta_{2h}) + E(\theta_{2k-1} r(X_{2h})) + E(\theta_{2k-1}\epsilon_{2h}) \\ &\qquad + E(r(X_{2k-1})\theta_{2h}) + E(r(X_{2k-1}) r(X_{2h})) + E(\epsilon_{2k-1}\theta_{2h}) \\ &\qquad - E(\theta_{2k-1} r(X_{2h-1})) - E(\theta_{2k-1}\epsilon_{2h-1}) - E(r(X_{2k-1}) r(X_{2h-1})) \\ &\qquad - E(r(X_{2k})\theta_{2h}) - E(r(X_{2k}) r(X_{2h})) - E(\epsilon_{2k}\theta_{2h}) \\ &\qquad + E(r(X_{2k}) r(X_{2k-1})) \end{aligned}$$

$$\begin{aligned} &E([Y_{2k}(1) - Y_{2k-1}(0)][Y_{2h-1}(1) - Y_{2h}(0)]) \\ &\quad = E(\theta_{2k}\theta_{2h-1}) + E(r(X_{2k})\theta_{2h-1}) + E(\epsilon_{2k}\theta_{2h-1}) \\ &\qquad + E(\theta_{2k} r(X_{2h-1})) + E(r(X_{2k}) r(X_{2h-1})) + E(\theta_{2k}\epsilon_{2h-1}) \\ &\qquad - E(r(X_{2k-1})\theta_{2h-1}) - E(r(X_{2k-1}) r(X_{2h-1})) - E(\epsilon_{2k-1}\theta_{2h-1}) \\ &\qquad - E(\theta_{2k} r(X_{2h})) - E(\theta_{2k}\epsilon_{2h}) - E(r(X_{2k}) r(X_{2h})) \\ &\qquad + E(r(X_{2k-1}) r(X_{2h})) \end{aligned}$$

$$
\begin{aligned}
&E([Y_{2k}(1) - Y_{2k-1}(0)][Y_{2h}(1) - Y_{2h-1}(0)]) \\
&\quad = E(\theta_{2k}\theta_{2h}) + E(\theta_{2k} r(X_{2h})) + E(\theta_{2k}\epsilon_{2h}) \\
&\qquad + E(r(X_{2k})\theta_{2h}) + E(r(X_{2k}) r(X_{2h})) + E(\epsilon_{2k}\theta_{2h}) \\
&\qquad - E(\theta_{2k} r(X_{2h-1})) - E(\theta_{2k}\epsilon_{2h-1}) - E(r(X_{2k}) r(X_{2h-1})) \\
&\qquad - E(r(X_{2k-1})\theta_{2h}) - E(\epsilon_{2k-1}\theta_{2h}) - E(r(X_{2k-1}) r(X_{2h})) \\
&\qquad + E(r(X_{2k-1}) r(X_{2h-1}))
\end{aligned}
$$

So,

$$
\begin{aligned}
&4[cov(D_k, D_h) + \theta^2] \\
&\quad = E(\theta_{2k-1}\theta_{2h-1}) + E(\theta_{2k-1} r(X_{2h-1})) + E(\theta_{2k-1}\epsilon_{2h-1}) + E(r(X_{2k-1})\theta_{2h-1}) \\
&\qquad + E(r(X_{2k-1}) r(X_{2h-1})) + E(\epsilon_{2k-1}\theta_{2h-1}) - E(\theta_{2k-1} r(X_{2h})) - E(\theta_{2k-1}\epsilon_{2h}) \\
&\qquad - E(r(X_{2k-1}) r(X_{2h})) - E(r(X_{2k})\theta_{2h-1}) - E(r(X_{2k}) r(X_{2h-1})) - E(\epsilon_{2k}\theta_{2h-1}) \\
&\qquad + E(r(X_{2k}) r(X_{2h})) \\
&\qquad + E(\theta_{2k-1}\theta_{2h}) + E(\theta_{2k-1} r(X_{2h})) + E(\theta_{2k-1}\epsilon_{2h}) \\
&\qquad + E(r(X_{2k-1})\theta_{2h}) + E(r(X_{2k-1}) r(X_{2h})) + E(\epsilon_{2k-1}\theta_{2h}) \\
&\qquad - E(\theta_{2k-1} r(X_{2h-1})) - E(\theta_{2k-1}\epsilon_{2h-1}) - E(r(X_{2k-1}) r(X_{2h-1})) \\
&\qquad - E(r(X_{2k})\theta_{2h}) - E(r(X_{2k}) r(X_{2h})) - E(\epsilon_{2k}\theta_{2h}) \\
&\qquad + E(r(X_{2k}) r(X_{2k-1})) \\
&\qquad + E(\theta_{2k}\theta_{2h-1}) + E(r(X_{2k})\theta_{2h-1}) + E(\epsilon_{2k}\theta_{2h-1}) \\
&\qquad + E(\theta_{2k} r(X_{2h-1})) + E(r(X_{2k}) r(X_{2h-1})) + E(\theta_{2k}\epsilon_{2h-1}) \\
&\qquad - E(r(X_{2k-1})\theta_{2h-1}) - E(r(X_{2k-1}) r(X_{2h-1})) - E(\epsilon_{2k-1}\theta_{2h-1}) \\
&\qquad - E(\theta_{2k} r(X_{2h})) - E(\theta_{2k}\epsilon_{2h}) - E(r(X_{2k}) r(X_{2h})) \\
&\qquad + E(r(X_{2k-1}) r(X_{2h})) \\
&\qquad + E(\theta_{2k}\theta_{2h}) + E(\theta_{2k} r(X_{2h})) + E(\theta_{2k}\epsilon_{2h}) \\
&\qquad + E(r(X_{2k})\theta_{2h}) + E(r(X_{2k}) r(X_{2h})) + E(\epsilon_{2k}\theta_{2h}) \\
&\qquad - E(\theta_{2k} r(X_{2h-1})) - E(\theta_{2k}\epsilon_{2h-1}) - E(r(X_{2k}) r(X_{2h-1})) \\
&\qquad - E(r(X_{2k-1})\theta_{2h}) - E(\epsilon_{2k-1}\theta_{2h}) - E(r(X_{2k-1}) r(X_{2h})) \\
&\qquad + E(r(X_{2k-1}) r(X_{2h-1})) \\
&\quad = E(\theta_{2k-1}\theta_{2h-1}) + E(\theta_{2k-1}\theta_{2h}) + E(\theta_{2k}\theta_{2h-1}) + E(\theta_{2k}\theta_{2h}) \quad \square
\end{aligned}
$$