A DIFFERENTIAL APPROACH TO ANCHORING AND ADJUSTMENT FOR BINARY LOTTERIES

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Abstract. This letter provides a novel approach for modeling observed behaviors by employing differential equations to describe heuristic anchoring and adjustment. To demonstrate the value of such an approach, I use it to provide a unified framework for the various heuristics employed in the valuation of binary lotteries. The theory is then validated on the individual level using the Gonzalez and Wu (1999) dataset. My approach also allows for subject heterogeneity in larger datasets to modeled. This is illustrated and validated out of sample using a number of lottery valuation datasets. Stylized facts are also presented concerning the distribution of subjects' choice of heuristic, their certainty in the effectiveness of their heuristic and the noise in subjects' answers. Despite being required to pass comprehension checks a significant percentage of subjects' responses are a result of guessing and noise.

1. Introduction

Models of human behavior in Behavioral Economics and Psychology have generally been built through either set-theoretical assumptions, the optimization of an objective function or others concerning particular functional forms (e.g. Weber and Stevens' laws). Even Kahneman and Tversky's notion of subjects anchoring and adjusting has been reduced to mean convex combinations of functions from other established methods. This letter proposes understanding anchoring and adjustment in a differential light, in which the anchors are boundary conditions and the adjustment is governed in a non-linear manner by a differential equation. To illustrate this approach, I consider the open problem in Behavioral Economics and Psychology in discovering how subjects make choices between risky options.

Since the St. Petersburg paradox was posed in the 18^{th} century, Economics and Psychology have been interested in formally modeling people's choices between risky and safe options. Such a theory would have broad impact with implications in industry (e.g. insurance pricing), criminal justice (e.g. plea bargaining) and international relations. As a starting point, the simplest choice to model concerns one between a binary lottery with one non-zero payoff and another fixed sum. Formally, we're interested in the amount of a money, x, such that a person would be indifferent between receiving x or the lottery, L.

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$$L = \begin{cases} l, & p \\ 0 & (1-p) \end{cases} \qquad p \in (0,1), \quad l \geq 0 \tag{1}$$

We denote this amount, x, as the *certainty equivalent* of the lottery, L. In the case of binary lotteries, it is generally more instructive to discuss the quantity x/l which we denote as the *normalized certainty equivalent.* In this simple case nearly all proposed models (e.g. Savage's Subjective Expected Utility, Kahneman and Tversky's Cumulative Prospect Theory, etc.) coincide to predict that the amount x will satisfy:

$$v(x) = \pi(p)v(l) \tag{2}$$

in which $v \in C^2$ is a cardinal utility function and $\pi \in \{C^1 | \pi' > 0, \pi(0) = 0, \pi(1) = 1\}$ measures subjective probability. Various functions have been proposed for v and π using different frameworks. The most popular choices, though, appear to be a power utility function, given in eq. (3), and the Goldstein-Einhorn weighting function[1], given in eq. (4), which were popularized and tested experimentally by Gonzalez and Wu on ten subjects, [4].

$$v(x) = \theta x^{\alpha} \tag{3}$$

$$\pi(p) = \frac{\delta p^{\gamma}}{\delta p^{\gamma} + (1-p)^{\gamma}} \tag{4}$$

While other probability weighting functions have been proposed, we take the form above as representative since the various functional forms are nearly impossible to differentiate given suitable choices of parameters. While this framework is an improvement over assuming people calculate the expected value it still suffers from both theoretical and experimental flaws. Firstly, it assumes that the normalized certainty equivalent of a binary lottery (1) is fixed for a given probability of winning. In other words, the normalized certainty equivalent of L is given by:

$$x/l = \pi^{1/\alpha}(p) \tag{5}$$

so that the normalized certainty equivalent isn't a function of the amount at stake. However, this claim is false for a significant portion of experimental subjects including at least four[2] of Gonzalez and Wu's ten subjects.

A second issue concerns behavior at low probabilities in which subjects normalized certainty equivalents don't appear to converge to zero as the probabilities decrease (Cf. App. Fig. 1). Given the requirement that $\pi(0) = 0$ and that π be continuous the appearance of a non-zero "intercept" leads to unrealistic assumptions of sensitivity near the boundary.

[1]This functional form for the subjective probability is derived through assuming that a person subjectively understands probabilities as the logarithm of the corresponding betting odds and through assuming that a person's internal biases may be approximated by an affine transformation in this log-odds space, [4].

[2]Plots of their normalized certainty equivalents are shown in Appendix Fig. 1

For instance, the Gonzalez-Wu estimates for one subject[3] lead one to the conclusion that the difference in value between two binary lotteries with $p = 0.993$ and $p = 0.998$ is greater than that between lotteries with $p = 0.55$ and $p = 0.45$. Without the framing effect of having another digit (e.g. $0.99 \to 0.999$) such behavior is implausible.

One potential solution is to propose that subjects employ a logarithmic utility function. Recent work by Khaw, Li, and Woodford argues for a log-log relationship between certainty equivalents and lottery amounts (with the probabilities determining the slope and intercept) as an optimal strategy given logarithmic perception and cognitive noise [6]. Formally, in the event that cognitive precision is to be estimated from the data, they claim that the normalized certainty equivalent satisfies:

$$\log|x/pl| = \alpha(p) + \beta(p)\log|l|, \quad 0 \leq \beta(p) \leq 1 \tag{6}$$

which nests the popular approach of Gonzalez and Wu in the case $\beta(p) = 0$ and that of all prior approaches in the case of a power or logarithmic value function. However, I have two principal concerns with their approach. Firstly, given their modeling assumptions, in the event that subjects perceive the lottery amounts and probabilities nearly perfectly[4] their predicted mean response is proportional to the expected value. This is certainly inconsistent with the behavior found in lab experiments. Secondly, by not specifying the functions $\alpha(p), \beta(p)$ in their general model the authors allow far too many degrees of freedom for their model to have applicability in general tests of behavior under risk and certainly miss many of the regularities found in the data. It is only with further assumptions that the authors yield conditions for α, β and even then the results aren't readily available in closed form.

Thirdly, recent work by Oprea [7], provides evidence that behavior that appears similar to "probability weighting" can be elicited even in situations without stochasticity. In Oprea's experiment, subjects displayed probability weighting style behavior even in response to multiple price lists eliciting the value of a percentage of a fixed sum e.g. 35% of $20. Furthermore, the magnitude of these deviations were found to be correlated with the degree of probability weighting elicited when those same subjects completed multiple price lists for lotteries. Accordingly, our models for describing subject behavior in these experiments need to account for the fact that heuristic behavior can also generate similar behavior and may perhaps even be responsible for the behavior we observe in most experiments.

To resolve these issues I propose a new approach that jointly derives pairs of value functions and weighting functions on the basis of simple heuristics. The existence of subjects both displaying choices consistent with logarithmic valuation and otherwise is justified on the basis of heterogeneity among subjects in their choice of heuristic. Subjects are assumed to employ one of four heuristics to value lotteries leading to a mixture distribution of responses. Unlike existing models, the heuristic approach allows sufficient flexibility to

[3]Subject 8

[4]If subjects can type the correct lottery amount and probability after submitting their certainty equivalent then this would likely be the case and when the correct values are no longer directly in front of them.

fit the following empirical regularities that can not be explained by current models: (a) that there exists a minority of subjects whose implied probability weights are functions of the lottery amount; (b) that probability weighting can be observed in situations without stochasticity. This new model is calibrated using experimental datasets and validated out of sample. The usage of different heuristics is validated using subjects' self-reported uncertainty. The distribution of error parameters across sub-populations is highly correlated with their subjects' degree of inconsistency in responding to repeated questions. And lastly, a significant proportion of subjects appear to be guessing instead of employing a consistent methodology despite having passed comprehension checks.

2. The Individual Model

I propose that four heuristics are employed by subjects to value binary lotteries: the calculation of the expected value, two heuristics involving anchoring and adjustment and that of arbitrarily guessing an answer (i.e. noise). In the following subsection I describe the adjustment heuristics subjects employ and derive their functional forms.

2.1. Adjustment Heuristics.

Assume that numbers are perceived as abstract scalars or seen as amounts/distances, that is, values with units. When they're perceived as distances I assume that subjects follow Weber's law[5] such that the perceived value of the number is logarithmic. When they are perceived as scalars, I assume that they are perceived arithmetically for one of two reasons. If a subject is employing higher order thinking then they treat scalars according to standard arithmetic. Otherwise, a scalar is treated as an unknown quantity and by an indifference principle one arrives at the notion that the differences between ascending probabilities ought to be treated without relation to their respective underlying probabilities.

2.1.1. *The Log-Heuristic.*

Consider a person that has a poor[6] understanding of probability such that they only know that increases in the lottery probability are preferred *ceteris paribus*. Due to their uncertainty as to the cardinal value of probabilities we assume that they perceive probabilities as scalars. Consequently, we can represent the perceived probability as a positive affine transformation of the true probability. The lottery amount and certainty equivalent, though, are governed by Weber's law since it's an intuitively understandable amount. We then arrive at:

$$s_l = \log |l|, s_c = \log |c|, \qquad s_p = a_1 + a_2 p$$

in which s_l, s_c, s_p are the subjective values of l, c, p. We're interested in how such a person treats the marginal value of an increase in the lottery amount, l. The more unease the person feels about their chances of winning the lottery, the less valuable they find such an

[5]While this assumption is particularly strong given the evidence suggesting that Steven's law may be more appropriate for large numbers, our assumption of logarithmic perception will suffice for the current exercise [1].

[6]This includes people that are answering heuristically without employing their knowledge of probability thereby exhibiting a temporary poor understanding of probability.

increase. Since the relationship between lottery probabilities and the sensations of unease they engender is monotonically decreasing we may express our claim as:

$$\frac{\partial^2 s_c}{\partial u \partial s_l} = f(u) < 0 \quad \longrightarrow \quad \frac{\partial^2 s_c}{\partial s_p \partial s_l} \propto \frac{du}{ds_p} f(u) > 0 \tag{7}$$

in which s_c is the perceived certainty equivalent and u denotes one's sensation of unease so that du/ds_p is negative. A first order approximation of how $\partial s_c/\partial s_l$ behaves with respect to s_p is then

$$\frac{\partial s_c}{\partial s_l} = b_1 + b_2 s_p = a + bp, \quad a, b > 0 \tag{8}$$

The simplest[7] solution to this equation provides that the certainty equivalent for the interior probabilities is given by:

$$c = l^{a+bp} \tag{9}$$

Imposing the condition that $s_c = s_l$ when there isn't any risk yields the constraint

$$a + b = 1 \tag{10}$$

It is worth noting that this approach is consistent with Woodford's model [6]; however, whereas he derives his model on the basis of an optimization our approach allows this behavior to be derived from simple intuitions.

2.1.2. *The Beta Heuristic.* The next heuristic concerns subject behavior that overweights changes from certainty to uncertainty in which subjects essentially overreact to the introduction of minor uncertainty into what was previously a certain outcome. While this phenomena was formally described by many others, notably Tversky and Wakker, our goal is to demonstrate how such an intuition and Weber's law directly lead to the derivation of a functional form [9].

Consider an individual with three variables on their mind, l, the lottery amount, p, the probability of winning the lottery and, q, that of losing it. Even though p and q are trivially related, they are treated independently by the heuristic because they reflect separate notions. The value, q being the distance from certainty and the value, p, being the distance from impossibility. We define the marginal value of the lottery given an increase in the lottery probability by $\xi \coloneqq \partial c/\partial p$ as a subject's sensitivity to the lottery probability. Since l, p, q, ξ are either in units of currency or perceived as distances, they are all perceived logarithmically.

[7]The general solution, given an uncertainty function, $u(p)$ is,

$$c = \pi(p) l^{a+bp}$$

for some function $\pi(p)$. Enforcing the boundary conditions $c(p,l)\big|_{p=0} = 0, \quad c(p,l)\big|_{p=1} = l$ leads to the conditions: $\pi(0) = 0, \quad \pi(1) = 1, \quad b = 1 - a$.

Firstly, we note that subjects are more sensitive to the lottery probability when there is more at stake. This notion doesn't depend on the lottery amount or probability.

$$\frac{\partial s_\xi}{\partial s_l} \propto 1 \tag{11}$$

Next, we note that a person intuitively understands the categorical difference between a state in which it is possible to gain and one in which no gain is possible. Accordingly, the closer the probability is to 0 the more sensitive they are to changes in the lottery probability. This intuition, to first order, is captured by:

$$\frac{\partial s_\xi}{\partial s_p} \propto -1 \tag{12}$$

Finally, we note that the same effect may be observed near certainty since such a person intuitively understands the difference between a state of certain gain and state in which it is possible not to. Similarly, the closer the probability of losing is to 0 the more sensitive they are to changes in the lottery probability. This intuition is captured, to first order by:

$$\frac{\partial s_\xi}{\partial s_q} \propto -1 \tag{13}$$

Solving eqs. 11, 12 and 13 yields:

$$s_\xi = k + \gamma s_{l_1} + \alpha s_p + \beta s_q \tag{14}$$

$$\xi = K p^\alpha q^\beta l_1^\gamma \tag{15}$$

with $\alpha, \beta \leq 0, \gamma > 0$.

We now turn our attention to the constraints. Firstly, the probability constraint of:

$$q = 1 - p \tag{16}$$

which we temporarily relaxed to describe the heuristics. Secondly, the boundary conditions:

$$c(l, 1) = l_1 \qquad c(l, 0) = 0$$

imply that our solution must satisfy

$$\int_0^1 \xi dp = c(l, 1) - c(l, 0) = l \tag{17}$$

These two constraints are satisfied if

$$K = 1/B(\alpha + 1, \beta + 1), \qquad \gamma = 1 \tag{18}$$

where $B(x, y)$ is the Beta function. If we substitute in the values from eq. 18 into eq. 15 and change variables under $(\alpha + 1, \beta + 1) \mapsto (\alpha, \beta)$ then we may write the solution as:

$$c = l \int_0^p \frac{p^{\alpha-1}(1-p)^{\beta-1}}{B(\alpha, \beta)} dp = l I_p(\alpha, \beta), \qquad \alpha, \beta \leq 1 \tag{19}$$

in which $I_x(a,b)$ is the regularized incomplete Beta function (the Beta CDF). In the case that a subject is equally sensitive to changes in probability near 0 and 1 we satisfy the restriction that

$$\alpha = \beta \tag{20}$$

which ensures that the Beta function is symmetric about $p = 0.5$.

It should be noted that both solutions we developed fit within the broad framework proposed by Subjective Expected Utility Theory. However, we derive our functional forms from first principles and may therefore justify these different functional forms on the basis of known heuristics in lieu of the axioms of other theories. Importantly, our approach implies that the value function and probability weighting function can, on a conceptual level, be dependent on one another. If this is broadly the case, attempts to decouple the two through experimentation are likely to fail out of sample. Furthermore, it explains and likely requires that there be heterogeneity in the heuristic employed unlike other models [5], [6].

2.2. **The Distribution of the Noise.** It is important to take a moment and consider what we expect the distribution of the noise. In the case of the Log-heuristic and Beta heuristic, we expect the noise term to have a standard deviation roughly proportional the amount at play, l, which reflects the logarithmic perception inherent to those heuristics. Subjects calculating the expected value likely also have errors that are proportional to the amount at stake due to the multiplicative nature of the expected value. Accordingly, I estimate the distribution of the normalized certainty equivalent, which ought to have approximately constant variance for each sub-population, in lieu the certainty equivalent itself.

3. A Population Model

To account for heterogeneity in the general population I model the general population using a mixture model that incorporates the various heuristics. Based on the work of Bruhin, Fehr-Duda and Epper (2010) I assume that there are four sub-populations present in general lottery experiments: expected value maximizers, a Log-heuristic population, a Beta-heuristic population and a noise population that arbitrarily chooses responses, [2]. I assume that each sub-population has default parameters. Individual subjects' parameters are then sampled from a distribution centered on the default parameters. For instance, each person employing the Beta-heuristic has Beta parameters satisfying

$$\alpha = \alpha_0 + \varepsilon, \qquad \beta = \beta_0 + \varepsilon$$

in which α_0, β_0 are the sub-population's default parameters and ε is approximately normal mean zero noise. Monte Carlo simulations illustrate that this noise, when propagated through the various functional forms, leads to a unimodal distribution of errors about the normalized certainty equivalent implied by the default parameters. In addition to the error generated by heterogeneity in the parameters, each sub-population also has an error from generic errors in thinking and myriad other small idiosyncratic behaviors. I approximate the combination of these two sources of error using Gaussians due to both the CLT and its tractability. Accordingly, if the i^{th} sub-population's default normalized certainty equivalent is given by $\mu_i(p, l_1)$, then I approximate the distribution of its normalized certainty equivalents by

$$nce \sim \mathcal{N}(\mu_i(p, l_1), \sigma_i^2) \tag{21}$$

The presence of the noise-population is meant to capture those subjects who mostly guess in their responses. I assume that they employ the least amount of thinking and also model their choices using the Log-heuristic. Since their answers are mostly noise, I expect them to have a larger error parameter

$$\sigma^2_{\text{Noise}} > \sigma^2_{\log} \tag{22}$$

If the chance of a randomly selected individual being in the i^{th} sub-population is denoted by π_i then we may describe the CDF of normalized certainty equivalents for a given simple lottery by

$$F_{nce}(nce) = \sum_{i=1}^{4} \pi_i \Phi\left(\frac{nce - \mu_i(p, l_1)}{\sigma_i}\right) \tag{23}$$

with a natural generalization when lottery questions are asked.

Bruhin, Fehr-Duda, and Epper convincingly demonstrate that female responses differed significantly from male responses [2]. Accordingly, we estimate eq. (23) separately for both men and women.

4. Estimating the Individual Model

As a first test of our ideas, it is necessary to demonstrate that the inclusion of heterogeneity provides a better explanation of the data than that of other models such as Cumulative Prospect Theory or that of Khaw, Li and Woodforf [6]. My model for individuals effectively has 7 parameters[8] in contrast to Gonzalez and Wu's CPT model which has 4. Accordingly, we can decide between the models on the basis of their information criteria. As is standard, I use the AIC criterion.

To test the individual model we require a large number of responses for each subject. As such, I chose to employ Gonzalez and Wu's dataset [4] which includes ten subjects' certainty equivalents for 92 simple lotteries. We find a $\Delta AIC > 50$ in favor of my model providing strong evidence for heterogeneity in subjects' choice of heuristic. As to the breakdown of subject behavior, 4 subjects are best described by the Beta heuristic and 6 subjects are best described by the Log heuristic.

[8]Since the EV maximizer is nested in the Beta model and Noise is nested in the Log model, we only have to estimate parameters, $\alpha, \beta, a, b, \sigma_{\text{beta}}, \sigma_{\log}, \pi$. The last parameter, π, is the probability that the person is better described by the Log model than the Beta model.

5. Estimating and Validating the Population Model

To test the population mixture model, I employed the EM algorithm to classify subjects by heuristic used and fit the heuristic model parameters. Since this algorithm is sensitive to the starting parameters (i.e. initial probabilities of a subject belonging to a given sub-population), I assigned them on the basis using an *a priori* analysis[9] of the data that introduced a preliminary classification of subjects.

While developing the model, I analyzed the 2003 and 2006 Swiss datasets collected by Bruhin, Fehr-Duda and Epper as well as the 2022 Prolific dataset collected by Enke and Graeber consisting of elicited certainty equivalents for binary lotteries [2], [3]. They essentially served as the training set for the model. It is worth noting that the experiments that generated these datasets differ considerably in the manner in which they were collected. Whereas Enke and Graeber collected data online from US residents through the direct elicitation of subjects' certainty equivalents, Fehr-Duda *et al* asked Swiss students in person to fill out multiple price lists. This provides a further check on the robustness of my findings. After training my model I froze the analysis code.

To validate my conclusions out of sample, I employed the certainty equivalent data collected by Enke and Graeber in 2019 as a validation set and ran the existing code on it. Unfortunately, the 2019 dataset suffers from two major weakness. Firstly, the dataset only has a maximum of 3 points per subject. While this is still easily sufficient to separate out those following an EV or Noise heuristic from the others it is more difficult to classify the Log and Beta populations due to the fact that the difference in their behavior need only occur near the boundaries. Secondly, the subjects in Enke & Graeber's 2019 Prolific dataset were also asked subjects questions that are irrelevant to our inquiries (e.g. compound lotteries, ambiguous lotteries, etc.) that potentially could have biased their responses to the questions we are concerned with.

The results of this analysis are found in the tables below. The categorization of subjects is found in Table 1; the heuristic parameters are found in Table 2 and the values of σ for each sub-population are found in Table 5.

[9]Initial probabilities of belonging in the various sub-populations are generated using behaviors predicted by the model. Subjects whose normalized certainty equivalents are close to linearly related (slopes within 7.5% of 1) with the lottery probability are assigned a high initial probability of being an EV maximizer. Those with normalized certainty equivalents on average more than 0.35 from the lottery probability are given a high initial probability of being noise. The remaining population is split into Log and Beta based on their behavior near $p = 0$ in which the Log population is expected to have a non-zero intercept.

Table 1. Categorization of Subjects

Dataset	Expected Value	Log	Beta	Noise
Swiss 2003 (M)	0.06	0.19	0.55	0.20
Swiss 2006 (M)	0.19	0.24	0.48	0.09
US 2022 (M)	0.08	0.43	0.26	0.23
Swiss 2003 (F)	0.06	0.22	0.54	0.17
Swiss 2006 (F)	0.00	0.26	0.53	0.20
US 2022 (F)	0.21	0.31	0.38	0.1
Validation (M)	0.17	0.46	0.16	0.22
Validation (F)	0.10	0.18	0.67	0.06

5.1. **The Categorization of Subjects.** As shown above, the proportion of subjects categorized into each sub-population is generally consistent across datasets. Crucially, my analysis suggests that a fifth of subjects consistently act as if they are noisily answering questions despite passing comprehension checks. If these subjects are not accounted for separately then probability weighting estimates will be biased towards an inverse-S shape. Likewise, if the expected value maximizers are not removed then probability weighting estimates are biased in the direction of risk neutrality. Accordingly, all estimates of behavioral theories can be severely biased if they do not account for this persistent form of noise. Secondly, it is worth noting that in the Validation (M) set the relative proportions between the Log and Beta sub-population proportions seem to be opposite those in the other datasets. This potentially reflects a misclassification of the Log and Beta sub-populations by the algorithm which, as mentioned before, is possible in the Validation set due to the paucity of data.

Table 2. Heuristic Parameters for Sub-populations

	Log		Beta		Noise	
Dataset	a	b	α	β	a	b
Swiss 2003 (M)	0.64	0.30	0.51	0.43	0.58	0.28
Swiss 2006 (M)	0.65	0.26	0.60	0.45	0.45	0.34
US 2022 (M)	0.52	0.46	0.82	0.81	0.81	0.04
Swiss 2003 (F)	0.59	0.37	0.37	0.21	0.74	0.14
Swiss 2006 (F)	0.52	0.28	0.53	0.39	0.58	0.32
US 2022 (F)	0.77	0.20	0.29	0.22	0.95	−0.18
Validation (M)	0.43	0.55	0.13	0.02	0.76	0.07
Validation (F)	0.18	0.31	0.34	0.29	0.86	−0.04

5.2. **The Estimated Parameters and the Relations Between Them.** As may be seen in Table 2 the parameter estimates are generally stable across datasets with one another with the exception of the female Validation set.

I test the restriction imposed by eq. 9, that $b = 1 - a$ on the training data using a t-test and find that we can not reject their equality (t = 1.35). As a second method of investigating their relationship I regress b on a with the result displayed in Table 3. Though the result resembles the postulated relationship the slope is not statistically significant. While this may be a result of having too little data there are other reasons to be skeptical. In the validation set, this relationship is present in the male dataset but not in the female dataset.

TABLE 3. The relationship between a and b in the training datasets

	Dependent variable:
	b
a	$-0.741^{\cdot}$
	(0.34)
Constant	0.85^{**}
	(0.11)
Observations	6
Adjusted R^2	0.55
Residual Std. Error	0.07
F Statistic	4.92^{*}
Note:	$^{*}p<0.1$; $^{**}p<0.05$; $^{***}p<0.01$

In light of these results I conclude that while there may be suggestive evidence in favor of this relationship it is not conclusive. Interestingly, the lack of evidence in favor of this result indicates that our conjecture requiring subjects' certainty equivalents to be continuous with respect to probability while satisfying the boundary conditions for certain events may be false. It may be that subjects treat the interior probabilities as categorically different quantities than those on the boundary.

I also test my conjecture from eq. (20) that $\alpha = \beta$ using a t-test and find that we can not reject their equality (t = 0.87). However, to better investigate the relationship between these two parameters I regress the values of α on β from the training set in Table 4 and find a significant positive relationship between. Furthermore, this relationship is consistent with the Validation sets since its predicted values of α lie within 2σ of their estimated values.

TABLE 4. The relationship between α and β in the training datasets

	Dependent variable:
	α
β	0.831***
	(0.107)
Constant	0.171**
	(0.049)
Observations	6
R^2	0.938
Adjusted R^2	0.922
Residual Std. Error	0.051
F Statistic	60.259***
Note:	*p<0.1; **p<0.05; ***p<0.01

The existence of a positive constant term is interesting since it indicates a bias in favor of the α term over the β term. Using the logic of the heuristic, this suggests that subjects find the distinction between possibility and impossibility to be greater than possibility and certainty. As expected, though, the coefficient is also significant indicating that overall sensitivity near the boundary probabilities is what drives the heuristic. Lastly, the strength of the relationship shown by the high Adjusted R^2 indicates that the Beta heuristic likely only requires one parameter.

TABLE 5. Error Parameters (σ) for Sub-populations $l_2 = 0$

Dataset	Expected Value	Log	Beta	Noise
Swiss 2003 (M)	0.04	0.12	0.12	0.27
Swiss 2006 (M)	0.05	0.12	0.08	0.30
US 2022 (M)	0.00	0.14	0.05	0.38
Swiss 2003 (F)	0.12	0.12	0.14	0.28
Swiss 2006 (F)	N/A	0.08	0.09	0.25
US 2022 (F)	0.07	0.15	0.18	0.43
Validation (M)	0.03	0.11	0.11	0.34
Validation (F)	0.05	0.11	0.18	0.58

5.3. **The Estimated Error Parameters.** As expected, it is clear that the EV population has the smallest value of σ in each dataset. Similarly, our supposition in eq. (22) is confirmed since the Noise population consistently has the largest value of sigma. Moreover, given that the range of possible values for the normalized certainty equivalent is bounded between 0 and 1, the fact that values of σ for the Noise sub-population are generally greater than 0.25 indicates that nearly the entire range of possible values is contained within 2σ of the mean response. This supports our identification of the Noise sub-population.

5.4. **An Analysis of the Variance.** Recall that in the population model there are two major sources of noise: subject level error that manifests itself in the form of inconsistency when faced with the same trial at a later time and heterogeneity in parameters within a given sub-population. Since subjects faced two repeated lotteries valuation questions in the US 2022 dataset we can directly estimate the variance due to inconsistency.

If members of a given sub-population answer questions with an error of variance σ^2_{Incon} then the sum of the differences between the two sets of responses will have variance $4\sigma^2_{\text{Incon}}$. Accordingly, we can directly estimate σ_{Incon} for each sub-population. We then employ NLS to estimate σ for each sub-population. To avoid trivial correlations between these two measures, the dataset the NLS routine is run on excludes the data employed in estimating σ_{Incon} through inconsistencies. The positive relationship between the two is tested using a linear regression and the results are in the table below.

TABLE 6. The relationship between various estimates of the variance

	Dependent variable:
	$\hat{\sigma}^2_{\text{NLS}}$
$\hat{\sigma}^2_{\text{Incon}}$	2.621^{***}
	(0.456)
Constant	-0.002
	(0.010)
Observations	8
R^2	0.846
Adjusted R^2	0.821
Residual Std. Error	0.019
F Statistic	33.031^{***}
Note:	*p<0.1; **p<0.05; ***p<0.01

As expected, there exists a significant positive relationship between the two estimates of the variance. Using the estimates in Table 6 we see that that inconsistency accounts for approximately 40% of the variance in a given sub-population's answers. Lastly, it would appear that the variance due to heterogeneity is correlated with the variance due to inconsistency since the constant term is negligible. This is unsurprising given that it would make sense for individuals within the noise sub-population to have greater variance both within and between subjects' responses and the reverse for the EV sub-population.

6. On the Nature of Cognitive Uncertainty

The complexity of heuristics follows from their corresponding intuitions and is given by:

$$Noise < Log \sim Beta < EV \tag{24}$$

Assuming that a subject's use of less complex heuristics is correlated with their degree of uncertainty, we can provide evidence for our hierarchy of heuristic complexity using Enke and Graeber's (2019, 2022) measure of cognitive uncertainty. In their experiment, they asked subjects to rate their uncertainty on a scale of one to twenty. Since the EM algorithm provides us with posterior probabilities of each subject belonging to a given sub-population we can regress subjects' cognitive uncertainty on these posterior probabilities to verify that our proposed hierarchy holds. It is also necessary to add linear and quadratic terms of the lottery probability as regressors to control for the fact that it is easier to provide certainty equivalents near the boundaries than in the interior. The results are seen in the Table 7.

Not only do these group membership probabilities explain the variation in cognitive uncertainty, a fact attested to by the large Adjusted R^2 value, but, the coefficients are all significant and decrease in the order of the associated heuristics' complexity in line with our expectations.

Furthermore, the coefficients of the lottery probability and its square are consistent with the claim that cognitive uncertainty decreases as one approaches a boundary in line with our intuition for the Beta heuristic. To see this, note that the maximum of a quadratic: $ax^2 + bx + x$ is at $x = -b/2a$ if $a < 0$. As may be seen in Tab. 7 $a < 0$ for each dataset and the maxima for the datasets are found at the interior probabilities of: 0.53, 0.47, 0.31, 0.23 respectively. The case $\alpha = \beta$ therefore corresponds to having an interior maximum at $p = 0.5$ whereas $\alpha > \beta$ corresponds to an interior max with $p < 0.5$.

7. Concluding Remarks

In this paper, I demonstrated how an understanding of the heuristics experimental subjects employ in tandem with Weber's Law allows one to derive functional forms for subject behavior in experimental settings. Heterogeneity in the choice of heuristic is explicitly modeled in such a setting and is necessary to avoid biasing the estimates for parameters of interest. I model the variance of subject responses and show that obey predictable patterns between sub-populations. My theory is tested and validated both on the individual and population level using a variety of datasets. Furthermore, I validate my understanding of the intuition of the heuristics using data on subjects' reported uncertainty when providing certainty equivalents.

However, in developing this approach I employed simple binary lotteries with payoffs $(l, 0)$ for simplicity and small stakes in which there isn't a serious presence of diminishing sensitivity. Further efforts are necessary to extend the model these settings. Diminishing

Table 7. Cognitive Uncertainty by Sub-Population

	Dependent variable:			
	Cognitive Uncertainty			
	2019 (M)	2019 (F)	2022 (M)	2022 (F)
P_{EV}	0.083***	0.116**	0.116***	0.303***
	(0.028)	(0.046)	(0.023)	(0.018)
P_{Beta}	0.196***	0.130***	0.199***	0.361***
	(0.031)	(0.028)	(0.016)	(0.016)
P_{Log}	0.118***	0.203***	0.253***	0.351***
	(0.025)	(0.036)	(0.015)	(0.016)
P_{Noise}	0.210***	0.308***	0.343***	0.448***
	(0.030)	(0.056)	(0.016)	(0.023)
Lottery Probability	0.276**	0.484***	0.232***	0.161*
	(0.119)	(0.144)	(0.065)	(0.068)
(Lottery Probability)^2	−0.294**	−0.515***	−0.379***	−0.353***
	(0.115)	(0.142)	(0.062)	(0.067)
Observations	822	726	1,494	1,494
Adjusted R^2	0.365	0.368	0.567	0.661
Residual Std. Error	0.240	0.280	0.209	0.224
Note:		*p<0.1; **p<0.05; ***p<0.01		

sensitivity in the lottery amount is potentially accounted for through the notion of log-log perception as advocated for Prelec, [8]; but, lotteries with multiple outcomes likely require a holistic approach to understand how subjects summarize visual data as an intermediate step before constructing their certainty equivalents.

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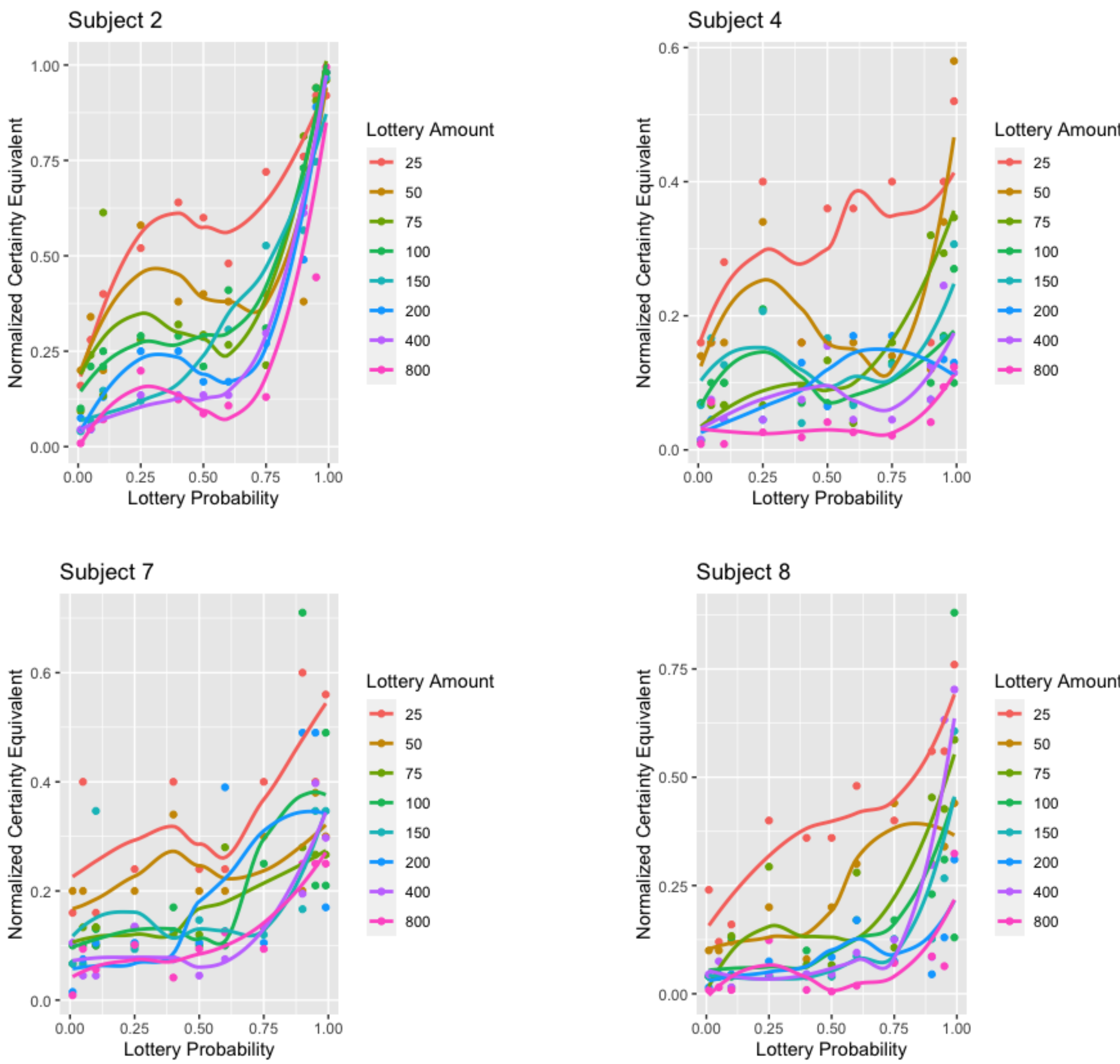

FIGURE 1. Plots of normalized certainty equivalents and LOESS fits as a function of lottery probability for binary lotteries for which $l_2 = 0$. Data taken from Gonzalez & Wu (1999)

Appendix A. Robustness

To test how robust the original estimates were for the 2022 (USA) dataset I reran the analysis code on each of the three main sessions that generated the 2022 US dataset separately and used these categorizations to fit parameter estimates. These three sessions account for 75% of the data in the 2022 dataset. I denote these estimates by "Bootstrap." The reason I avoided regular bootstrapping is because Prolific uses hidden demographic variables to balance the subject pool in each session. The differences between the "Bootstrap" estimates and the original categorization stem from the fact that in the original categorization, the EV group only consisted of those who were risk neutral and answered with decimal level precision whereas in the "Bootstrap" categorization, these individuals are binned with those who answer the EV but round their answer.

The corresponding tables are found below.

Table 8. Categorization of Subjects for $l_2 = 0$

Dataset		Expected Value	Log	Beta	Noise
Swiss 2003	(M)	0.06	0.19	0.55	0.20
Swiss 2006	(M)	0.19	0.24	0.48	0.09
"Bootstrap"	(M)	0.1	0.22	0.42	0.26
Swiss 2003	(F)	0.06	0.22	0.54	0.17
Swiss 2006	(F)	0.00	0.26	0.53	0.20
"Bootstrap"	(F)	0.13	0.38	0.34	0.15
Validation	(M)	0.17	0.46	0.16	0.22
Validation	(F)	0.10	0.18	0.67	0.06

Table 9. Heuristic Parameters for Sub-populations $l_2 = 0$

		Log		Beta		Noise	
Dataset		a	b	α	β	a	b
Swiss 2003	(M)	0.64	0.30	0.51	0.43	0.58	0.28
Swiss 2006	(M)	0.65	0.26	0.60	0.45	0.45	0.34
"Bootstrap"	(M)	0.47	0.52	0.55	0.61	0.80	0.04
Swiss 2003	(F)	0.59	0.37	0.37	0.21	0.74	0.14
Swiss 2006	(F)	0.52	0.28	0.53	0.39	0.58	0.32
"Bootstrap"	(F)	0.72	0.21	0.29	0.33	0.87	−0.08
Validation	(M)	0.43	0.55	0.13	0.02	0.76	0.07
Validation	(F)	0.18	0.31	0.34	0.29	0.86	−0.04

TABLE 10. Error Parameters (σ) for Sub-populations $l_2 = 0$

Dataset		Expected Value	Log	Beta	Noise
Swiss 2003	(M)	0.04	0.12	0.12	0.27
Swiss 2006	(M)	0.05	0.12	0.08	0.30
"Bootstrap"	(M)	0.03	0.12	0.12	0.29
Swiss 2003	(F)	0.12	0.12	0.14	0.28
Swiss 2006	(F)	N/A	0.08	0.09	0.25
"Bootstrap"	(F)	0.03	0.20	0.19	0.36
Mean	(Training Set)	0.05	0.13	0.12	0.28
Std. Dev.	(Training Set)	0.04	0.04	0.04	0.02
Validation	(M)	0.03	0.11	0.11	0.34
Validation	(F)	0.05	0.11	0.18	0.58

TABLE 11. The relationship between various estimates of the variance

	Dependent variable:
	$\hat{\sigma}^2_{\text{NLS}}$
$\hat{\sigma}^2_{\text{Incon}}$	2.509***
	(0.584)
Constant	0.008
	(0.012)
Observations	8
Adjusted R^2	0.714
Residual Std. Error	0.026
F Statistic	18.439***
Note:	*p<0.1; **p<0.05; ***p<0.01

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TABLE 12. Cognitive Uncertainty by Sub-Population

	Dependent variable:			
	Cognitive Uncertainty			
	2019 (M)	2019 (F)	"Bootstrap" (M)	"Bootstrap" (F)
P_{EV}	0.083***	0.116**	0.134***	0.329***
	(0.028)	(0.046)	(0.024)	(0.024)
P_{Beta}	0.196***	0.130***	0.233***	0.335***
	(0.031)	(0.028)	(0.018)	(0.019)
P_{Log}	0.118***	0.203***	0.255***	0.392***
	(0.025)	(0.036)	(0.020)	(0.019)
P_{Noise}	0.210***	0.308***	0.351***	0.446***
	(0.030)	(0.056)	(0.019)	(0.023)
Lottery Probability	0.276**	0.484***	0.252***	0.139*
	(0.119)	(0.144)	(0.079)	(0.081)
(Lottery Probability)^2	−0.294**	−0.515***	−0.405***	−0.355***
	(0.115)	(0.142)	(0.076)	(0.078)
Observations	822	726	1,062	1,056
Adjusted R^2	0.365	0.368	0.580	0.674
Residual Std. Error	0.240	0.280	0.213	0.223
Note:				*p<0.1; **p<0.05; ***p<0.01