# 2. The Bose Model and National Income

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In 1966, Sanjit Bose published the first optimal-growth version of the famous Fel'dman–Mahalanobis model. In a pioneering application of the maximum principle, Bose showed how the consumption- and investment-goods sectors of an economy should optimally be developed over time. Essentially, the optimal growth trajectory consists of two phases. In phase one, investment goods go to increase the capacity of whichever sector is 'under-balanced' relative to the other. In phase two, which begins as soon as balance has been achieved between the two sectors, balanced growth is thereafter maintained by allocating new investment goods to the two sectors in proportions that maintain the balance.

In this paper, I seek to redo an especially simple version of the 'Bose model' in order to emphasize the connection with a basic conceptual issue of national income accounting. For reasons that will become apparent, the Bose model is an ideal construct for examining the fundamental question 'what is income?'.

Let us first solve a one-sector growth model for a situation where homogeneous aggregate output is linearly proportional to aggregate capital (with output/capital coefficient $a$) and utility is a logarithmic function of consumption. Then we seek to pose and solve a natural two-sector generalization of the same problem, which is a special form of the model Bose solved in 1966.

We write the one-sector version here as being the optimal control problem to

$$\text{maximize} \int_0^\infty U(C(t))\, e^{-\rho t}\, dt \tag{2.1}$$

subject to

$$C(t) + I(t) = aK(t), \tag{2.2}$$

and

$$\dot{K}(t) = I(t), \tag{2.3}$$

and with the given initial condition

$$K(0) = K_0, \tag{2.4}$$

where the utility function is of the logarithmic from

$$U(C) = \log(C). \tag{2.5}$$

The 'net savings rate' at time $t$ for the above model is defined as

$$s(t) \equiv \frac{I(t)}{C(t) + I(t)}. \tag{2.6}$$

As is well known, we can completely characterize the solution to (2.1)–(2.6) as being to follow a policy of saving always at the constant rate

$$s^* = \frac{a - \rho}{a}, \tag{2.7}$$

which corresponds to having every part of the economy grow exponentially at the constant rate $g^* = a - \rho$.

The non-shiftable-capital model we shall analyse here consists of two sectors: the consumption-goods sector (Department 2), and the investment-goods sector (Department 1). We make the plausible assumption that at any given instant of time the productive capacity of each sector is quasi-fixed and non-shiftable, but that over time the proportions can be continuously altered by directing *new* investments to one or the other of the two sectors. Such a description accords well with the familiar putty-clay nature of real-world investment. The cement and steel of the investment-goods sector are pliable general-purpose construction materials that can be used to increase the capacity of either sector until they are hardened into concrete shells and bolted-down specific machinery dedicated to producing either more consumption (bread bakeries, urban housing, and so forth) or more investment (steel mills, cement factories, and so forth), at which point the two types of capital are considered to be as if frozen in place and are no longer shiftable.



To make the one- and two-sector versions comparable here, we assume that the utility of consumption for both models is the same logarithmic function (2.5), and that the output/capital coefficient is the same value $a$ in *both* sectors of the two-sector model (as well as for the single aggregated sector of the one-sector model). In his pioneering paper, Bose considered the more general case of an iso-elastic utility function and allowed capital to depreciate. The model we are considering here is thus a special case of the Bose model.

The simplest two-sector putty-clay analogue of the problem (2.1)–(2.6) is to

$$\text{maximize} \int_0^\infty U(C(t))\, e^{-\rho t}\, dt \tag{2.8}$$

subject to

$$C(t) = aK_2(t), \tag{2.9}$$

$$I(t) = aK_1(t), \tag{2.10}$$

and

$$\dot{K}_1(t) = I_1(t), \tag{2.11}$$

$$\dot{K}_2(t) = I_2(t), \tag{2.12}$$

and

$$I_1(t) + I_2(t) = I(t), \tag{2.13}$$

and

$$0 \le I_1(t) \le I(t), \tag{2.14}$$

$$0 \le I_2(t) \le I(t), \tag{2.15}$$

and with the given initial conditions

$$K_1(0) = K_0^1, \tag{2.16}$$

$$K_2(0) = K_0^2. \tag{2.17}$$

What is the relationship between the one-sector optimal growth model (2.1)–(2.6) and its two-sector putty-clay generalization (2.12)–(2.17)? To make both models tightly conformable, let us assume

$$K(0) = K_1(0) + K_2(0). \tag{2.18}$$

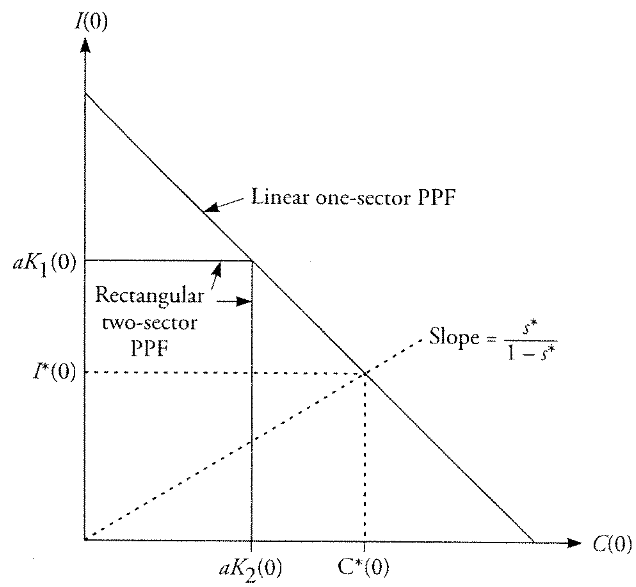


Figure 2.1: Shiftable versus Nonshiftable Capital (PPF: production possibility frontier)

In Figure 2.1 is depicted the relationship between the two models when (2.18) holds. At time zero (now), the one-sector version has a straight-line production possibilities frontier with a slope of $-1$, and the decision maker is free to choose *any* non-negative values of $C(0)$ and $I(0)$ satisfying

$$C(0) + I(0) = aK(0). \tag{2.19}$$

By contrast, the two-sector putty-clay version is 'stuck' at time zero with its historically inherited as-if-fixed-coefficient values of $C(0)$ and $I(0)$, which satisfy

$$C(0) = aK_2(0) \tag{2.20}$$

and

$$I(0) = aK_1(0). \tag{2.21}$$



Thus, the two-sector model here has a *rectangular*-shaped production possibilities frontier in Figure 2.1 [described by (2.20), (2.21)], while the one-sector aggregated version has a *line*-shaped production possibilities frontier in Figure 2.1 [described by (2.19)]. We know that the currently producible, historically given two-sector combination $(C(0), I(0)) = (aK_1(0), aK_2(0))$ is a point lying on the linear production possibilities frontier of the one-sector version. This *particular* point, however, need not represent an *optimal* combination of consumption and investment for the aggregate one-sector model. Imagine, though, that it *does*. Suppose, by pure coincidence, that the initial capital stocks of the two-sector model just so happen to satisfy the one-sector optimal savings condition, that is,

$$\frac{K_1(0)}{K_1(0) + K_2(0)} = s^*, \tag{2.22}$$

where $s^*$ is given by (2.7).

Notice that the one-sector aggregate version always has more production possibility options than the two-sector putty-clay version because the rectangular production possibilities set represented by (2.20), (2.21) is contained within the corresponding linear production possibilities set represented by (2.18). Notice too that, when (2.22) holds initially, the two-sector putty-clay model can choose to 'imitate' exactly the optimal one-sector constant-saving policy by selecting $I_1(t)$ at all times $t \geq 0$ so that

$$\frac{I_1(t)}{I(t)} = s^*. \tag{2.23}$$

Therefore, it follows, if (2.22) holds then both the one- and two-sector models have the identical optimal consumption trajectory and the same optimal value of the objective function. The relevant Hamiltonian for the two-sector optimal control problem is

$$H = U(aK_2) + p_1 I_1 + p_2 I_2. \tag{2.24}$$

Even without getting deeply involved in the details of the maximum principle here, it is now possible to make an 'educated guess' about the form of an optimal policy. If condition (2.22) holds initially, the optimal two-sector policy is to maintain these same ideal capital stock proportions forever thereafter by always obeying (2.23). If condition (2.22) does *not* hold initially, an 'educated guess' here might be that the optimal two-sector investment policy is a *most rapid approach* to the ideal savings ratio of capital stocks, which satisfies

$$\frac{K_1(t)}{K_1(t)+K_2(t)}=s^*, \tag{2.25}$$

and then, once condition (2.24) has been attained, the optimal policy remains forever in a state where (2.24) holds by following thereafter investment policy (2.23). A formal proof of the optimality of such a most rapid approach to the 'ideal proportions' (2.24) hinges on showing that the conditions

$$\frac{K_1(t)}{K_1(t)+K_2(t)}<s^*,\ p_1(t)>p_2(t),\ I_1^*(t)=aK_1(t),\ I_2^*(t)=0 \tag{2.26}$$

and

$$\frac{K_1(t)}{K_1(t)+K_2(t)}>s^*,\ p_1(t)<p_2(t),\ I_1^*(t)=0,\ I_2^*(t)=aK_1(t) \tag{2.27}$$

are mutually consistent with the corresponding price-differential equations of motion

$$\dot{p}_1(t) = \rho p_1(t) - a\max\{p_1(t), p_2(t)\}, \tag{2.28a}$$

$$\dot{p}_2(t) = \rho p_2(t) - U'(aK_2(t)), \tag{2.28b}$$

and also with the appropriate transversality conditions

$$\lim_{t\to\infty} p_1(t)K_1(t)e^{-\rho t} = \lim_{t\to\infty} p_2(t)K_2(t)e^{-\rho t} = 0. \tag{2.29}$$

(We leave the formal proof as an exercise with the additional hint that if $t = T$ is the *first time* when condition (2.24) holds, $p_1(t) = p_2(t)$ for all $t \geq T$.)

Having employed the relatively simple device of the two-sector non-shiftable-capital dynamic problem (2.12)–(2.17) to indicate how the maximum principle may be used to clarify the relationship between one- and two-sector versions of an optimal growth problem, we now utilize this same apparatus to explore in a very tentative way some aspects of the concept of national income. Let us observe first what emerges when we attempt to apply a standard well-known income concept here to both of our model economies.

For the sake of argument, suppose we are at time zero in a state of the two-sector putty-clay model where the two capital stocks $K_1(0)$ and are $K_2(0)$ are 'ideally balanced' in the sense that (2.22) holds. Then we know that with initial conditions satisfying (2.18) and (2.22), the optimal



trajectories of the one- and two-sector models are identical and both yield exactly the same dynamic welfare.

A well-known concept of income can perhaps be paraphrased as saying that income is 'the maximum amount that can presently be consumed without compromising future ability to consume at the same level'. I think this is a fair phrasing of a widespread notion of income that finds implicit expression throughout a broad range of contexts—from being a free translation of such populist public sources as the report on *Our Common Future* issued by the Bruntland Commission,[1] to being a transliteration of the financial/business concept of 'economic earnings',[2] to being a particular mutual strand of the conceptual apparatus used by three great economists who did fundamental theoretical work on the concept of income: Fisher, Lindahl, and Hicks.[3]

Irving Fisher (1930) was the first economist to note clearly that 'earnings' (what others would call 'income'—the headstrong Fisher had already appropriated the term 'income' for what others would call 'consumption') can fruitfully be conceptualized as a form of interest-like return paid on capital or wealth. Erik Lindahl (1933) argued cogently that the income of a period should be identified with the sum of consumption plus the net increase of capital value over the period. John Hicks (1946) introduced the modern idea, expressed in the definition of the previous paragraph, that income measures maximum present consumption subject to the sustainability-like condition of leaving intact future ability to consume at the same level. In the world of a Robinson Crusoe–like person whose single homogeneous capital good acts like a deposit in a bank account paying a constant interest rate, all three definitions (Fisher's 'income as interest on capital', Lindahl's 'income as consumption plus net capital increment', and Hicks's 'income as maximum sustainable consumption') coincide. But in almost any other world with even slightly more realistic complications, these three definitions generally differ, and it is not at all clear which one is better—or even, for what purposes it might be better. For the sake of

[1] World Commission on Environment and Development, 1987, p. 43, which defines 'sustainable development' as being 'development that meets the needs of the present without compromising the ability of future generations to meet their own needs'.

[2] *See*, Bodie, Kane, and Marcus, 2002, p. 611, who define 'economic earnings' as 'the sustainable cash flow that can be paid out to stockholders without impairing the productive capacity of the firm'.

[3] I hasten to add that several other concepts of income were also explored by them.

specificity, let us concentrate here on trying to apply the popular Hicksian concept ('the maximum amount that can presently be consumed without compromising future ability to consume at the same level') to this modified Bose model.

A glance at Figure 2.1 reveals that for the one-sector aggregate economy (with straight-line production possibilities), the Hicksian definition of income yields

$$aK(0), \tag{2.30}$$

while for the equivalent two-sector putty-clay version, the same definition of income gives

$$aK_2(0). \tag{2.31}$$

With conditions (2.22) and (2.25) holding, both economies are essentially equivalent, yet the *difference* in income (as defined above) between the one- and two-sector versions is

$$aK_1(0) = s^* aK(0). \tag{2.32}$$

Now, it should be apparent that something seems very puzzling about the fact that the above seemingly reasonable standard definition of income yields very different values for what are essentially identical economic situations. From Figure 2.1, the core problem here seems to be that such a definition of income effectively forces us to compare the hypothetical consumption-producing ability of economies in the region where net investment is zero (that is, where $s = 0$) even when we most emphatically prefer *not* to locate ourselves in such a region whenever $s^* > 0$.

'The maximum amount that can presently be consumed without compromising future ability to consume at the same level' might be a fine definition of income *if* we happen to *want* presently to be consuming at such a maximum sustainable rate in the sense that we *choose* $s^* = 0$. Otherwise, unfortunately, this definition of income depends artificially, and arbitrarily, on the elasticity of short-run substitution between the production of consumption and investment, which (at least in this example) is *not* related to the economy's ability to produce well-being over time. Such a definition, as Samuelson (1961) put it, essentially defines income as 'capacity to produce emergency consumption'—and this feature makes it quite idiosyncratically peculiar in any setting other than a situation where consumption and investment are infinitely substitutable. It seems natural enough to want income to be measuring some 'sustainable-like' property of 'present' and 'future' consumption possibilities; but this model is hinting very strongly that income should *not* be defined literally as the highest permanently maintainable level of consumption, and that a proper definition of income (if one exists) may have almost nothing to do with literal sustainability.



Without revealing all of the details, we merely state here what is the resolution of the seeming paradox, and which is the subject of a forthcoming book.

Let $V$ represent the maximized value of the objective function (2.8) subject to the constraints (2.9)–(2.17). It is $V$ that we are really interested in, because it is measuring welfare.

A natural definition of 'utility income' here is the Hamiltonian expression (2.24). The fundamental relationship between wealth or welfare and income is here

$$H = \rho V. \tag{2.33}$$

To make a long story short, it is (2.33) that gives the proper welfare underpinning for using here the natural definition of income $H$. The Hicksian parable is not literally true, but it is figuratively true. Although the constant utility income level $H$ is not literally attainable, because the production possibilities frontier is a rectangle instead of a straight line, it has the same allegorical meaning as Hicks intended. Income $H$ represents the *sustainable equivalent* or the *stationary equivalent* of the welfare that an optimal programme is actually able to deliver. That is to say, the present discounted utility of the optimal solution to the Bose model is exactly the same as the present discounted utility of the hypothetical constant utility level $H$. Because the relation (2.33) holds in a very broad class of economic models, this result is generic. To know $H$ is to know $V$. Hamiltonian income is the return on wealth (or welfare).

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