Journal of Economic Behavior & Organisation
Vol. 30 (1996) 25–43

JOURNAL OF
Economic Behavior
& Organization

Competition and the evolution of efficiency[1]

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Received 10 September 1995; revised 9 December 1995

Abstract

This paper presents an evolutionary model of the relationship between inter-firm competition and intra-firm organizational or X-efficiency. We model X-inefficiency within the firm as a prisoner's dilemma effort-monitoring problem, whose evolution is influenced by external competitive pressure from other firms. A closed form stochastic equilibrium displaying "survival of the fittest" dynastic cycles is derived and analyzed. The main result is that there exists a well defined sense in which competition is a surprisingly powerful force for efficiency.

JEL classification: C72; L10

Keywords: Competition; Evolution; Group selection; Prisoner's dilemma; X-inefficiency

1. Introduction

There are several possible reasons to favor competition. The argument most frequently encountered in "practical" policy debates, business discussions, and media reports is that exposure to competition compels firms to exert greater effort at improving their efficiency. Without competition, employees of a firm take their customers for granted, do "business as usual" or "work to rule", and generally lack the proper incentives to increase productivity, improve quality, develop new products, and so forth. Competition improves social welfare because it ensures that only the most efficient and innovative firms survive. According to this view, the Japanese invasion of the international automobile market improved global welfare not because automobiles were previously being produced from the wrong factor proportions, or because automobile prices were not close enough to marginal costs, or because the worldwide scale of automobile production

[1]We are grateful to two anonymous referees for helpful comments.

PII: S0167-2681(96)00840-2

had been insufficiently large to take advantage of increasing returns. Rather, the improvement came because the Japanese invaders forced home manufacturers everywhere to make a better product at lower cost or else face bankruptcy and extinction. Such a story could easily be repeated for a large number of important actual situations.

In this paper we study the balance between the forces of within-firm X-inefficiency and between-firm competitive replacement. We model X-inefficiency as a prisoner's dilemma problem caused by shirking among workers whose individual effort contributions cannot be perfectly monitored. The implied "free rider drift" toward a minimal level of X-efficiency is influenced by the between-firm force of competitive replacement, which itself is determined by the distribution of X-efficiency in the population of firms. We investigate the existence and stability of an equilibrium distribution of X-inefficiency among the firms. It is not our intention to present a particularly deep or innovative story about X-inefficiency or competitive takeovers per se. Rather, we simply assume that these forces are at work, and study the Schumpeterian cycles of creative destruction that result. The focus of the paper is on the statistical properties of equilibria, particularly the limiting distribution of efficiency. By assuming a continuum of firms, the resulting model becomes simple enough to allow us to calculate explicitly the equilibrium distribution of efficiency and to investigate how it depends on the parameters of the model.

Two very different strands of literature relate to these issues. First, there is the standard principal-agent theory with incomplete information. Hart (1983), Hermalin (1992), Horn et al. (1990), Scharfstein (1988, 1988). In this basically static approach, X-inefficiency is treated as maximizing behavior subject to incomplete contracts. This approach has provided some important insights. In particular, Scharfstein (1988a) and Hermalin (1992) show that more competition does not necessarily lead to more efficiency. However, our paper does not focus on the intricacies of the agency problem, and the mechanism leading to X-inefficiency is handled in much more of a bare bones manner than in the cited papers (basically, we simply assume the imitation of co-workers).

Our paper is much more closely related to the evolutionary strand. At a sufficiently high level of abstraction, there is an isomorphism between the themes of this paper and biological models of group selection (Wilson, 1983). In these models, a population is divided into locally isolated groups, corresponding to our firms. *Within* each group, there is the usual individual selection, but there is also selection *among* groups. A group with a low average fitness may become extinct, and its site occupied by the members of some more successful group. (Alternatively, groups may contribute genes to a common "mating pool", with successful groups contributing more genes to the pool.) The basic finding is that group selection may override the effects of individual selection.

Independently of our research, a model like this has been developed by Vega-Redondo (1993). In his model, workers play a coordination game with two Nash equilibria: an efficient "cooperative" equilibrium and an inefficient "uncooperative" equilibrium. Within each firm, all workers coordinate on some equilibrium, but firms where the workers coordinate on the inefficient equilibrium have an exogenously given lower chance of survival than firms where the workers play cooperatively. The long run outcome is that the whole population of firms end up at the same equilibrium, with the same level of efficiency.

The behavior of our model is totally different. We have a prisoner's dilemma within the firm and a stable (non-degenerate) *equilibrium distribution of efficiency* among firms, where each firm's survival probability is *endogenously* determined. In contrast to Vega-Redondo (1993), it is not an equilibrium for all firms to be maximally efficient. A firm's chances of survival are endogenously determined by its capacity to withstand challenges from other firms. If a firm is more efficient than the competition, it will not be invaded, but (because the prisoner's dilemma has a unique Nash equilibrium) it will eventually succumb to "free-rider drift". The more X-inefficient the firm gets, the more likely it will be replaced by a competitor. But only *relative* efficiency matters.

In "partial equilibrium", where one firm is subjected to an exogenously given distribution of challengers, there is a unique steady state distribution of efficiency for this firm. But in general equilibrium, where each firm lives in the same environment, there are many long run steady state distributions of efficiency. It turns out that they all correspond to different maximal efficiency levels. If there are no highly efficient firms present in the population, an inefficient firm is likely to survive for a long time, which supports a stable equilibrium with a low average level of efficiency. But if there are many efficient firms present, inefficient firms have a short life, which supports a stable "bootstrap" equilibrium with a high average level of efficiency. For a given maximal efficiency level, there exists a unique stable non-degenerate steady state distribution of efficiency. Thus, although the analysis is in the same spirit, the economic intuition behind our model is significantly different from Vega-Redondo (1993).

In a sense, it is not surprising that in a model like ours where more efficient firms replace less efficient firms there is some tendency to resist or slow down any reversions to inefficiency. However, once we consider the general equilibrium formulation it is not obvious that competitive challenges will provide anything but temporary relief from free-rider drift. Since each firm's chances of survival are endogenously determined by its capacity to withstand challenges from other firms, might it not be that all firms will lose efficiency together, so that the whole system will degenerate? But this does *not* happen. Thus, the novelty of this paper is to show rigorously that competitive challenges represent a surprisingly *strong* force for efficiency. Competitive challenges perform a "magic trick" by permanently maintaining, and even creating, efficiency in a system that otherwise would be running down over time.

That the long run survival, and even increase, of efficient firms is actually quite "paradoxical" can be seen by looking at general equilibrium in a model with only a finite number of firms. With a large finite number of firms, the system will have similar properties to those described above, with one significant difference. Eventually, a coincidental bad string of almost simultaneous downward mutations will lower the maximal efficiency level of the finite system, and indeed over time this will cause average efficiency to ratchet down toward zero. Since there are no exogenous upward mutations, competitive challenges provide only temporary relief from free-rider drift, and the whole population will eventually converge to zero. The finite system degenerates in the long run with probability one.

In contrast, in the continuum case the equilibrium distribution is non-degenerate, and indeed will approach maximum X-efficiency if competitive challenges are sufficiently frequent. Of course, the continuum assumption is only a mathematical abstraction, an idealization of the case where the number of firms is sufficiently large to make simultaneous downward mutations incredibly unlikely. But we think the important economic intuition, that a favorable distribution of efficiency can maintain itself by its own bootstraps, is best captured by our general equilibrium continuum model. Moreover, under reasonable assumptions there is no discontinuity in the limit as the number of firms grows large. Competitive challenges can maintain high efficiency in a large finite system if there is an arbitrarily small infusion of "new" high efficiency firms. We show in the appendix that, as long as there is an infinitesimal probability of exogenous "upward" mutations, a large finite world will have a unique steady state distribution that approximates the equilibrium for the continuum case.

2. X-inefficiency as a prisoner's dilemma problem

There is a large amount of evidence that internal-, organizational-, or X-efficiency is an empirically significant phenomenon (see Frantz, 1988). Following Leibenstein (1987), we understand X-inefficiency to be a prisoner's dilemma problem about the level of "effort" exerted by a firm's employees.

Consider a world populated by infinitely many identically-sized, symmetrical "island firms". Each island firm has the same fixed large number of employee–inhabitants. Let the variable x stand for an average employee's effort level. (Throughout the paper, all variables are normalized per employee.) If every employee is working at effort level x, this is also the firm's level of X-efficiency. A problem of observability and monitoring exists, since the firm does not know the contribution of any individual employee, although its aggregate level of X-efficiency is known to it.Let

$$R(x) \qquad R' > 0,'' < 0 \tag{1}$$

be the net revenue per employee when each employee's effort level is x. Let

$$C(x) \qquad C' > 0,'' < 0 \tag{2}$$

be the money-equivalent effort cost or disutility that each employee incurs when working at effort level x.

The "group optimal" level of effort for the firm and its employees is the value x^* that maximizes $R(x) - C(x)$, satisfying the first order condition:

$$R'(x^*) = C'(x^*). \tag{3}$$

The variable x could stand for many things. Essentially, x symbolizes "generalized effort." Depending on the context, x might stand for: working harder, working smarter, not working to rule, not doing business as usual, taking initiative, managerial tautness, having an enthusiastic attitude toward new methods or equipment or potential customers, being innovative, thinking up new and better ways of doing things, cooperating, being altruistic, having selfless genes, and so forth.

In a dynamic interpretation of the model, x could stand for the potential growth rate of the organization, which derives, say, from being more innovative or generally "more fit" for growth than competing organizations. With such an interpretation, a decreased x does not imply decreased efficiency in any absolute sense, but rather a failure to improve fast enough to keep up with the competition. (In this case $R(x)$ would be something like the present discounted net revenue value of growth at rate x, while $C(x)$ is the present discounted individual effort-cost of an employee being sufficiently innovative to attain level x).

Let U be the (fixed) reservation utility level for all island-firm employee–inhabitants, i.e. U is what an island inhabitant could get in the best alternative to working for the firm (for example, by being self-employed). A firm at X-efficiency level x pays its employees the smallest amount needed to retain them, namely

$$W(x) = \underline{U} + C(x) \tag{4}$$

Then long run profit (per employee) for the firm at X-efficiency level x is[2]

$$\Pi(x) \equiv R(x) - W(x) = R(x) - C(x) - \underline{U} \tag{5}$$

Π symbolizes "generalized performance" or, ultimately, relative fitness.

Under the assumptions made so far, the X-efficiency level that would yield highest profits or greatest fitness to the firm is the level of worker effort x^{**} that maximizes Eq. (5), satisfying the first order condition:

$$\Pi'(x^{**}) = 0 \tag{6}$$

Comparing Eq. (6) with Eq. (5) and Eq. (3), we see that $x^* = x^{**}$. Thus the profit-maximizing level of X-efficiency is the group-optimal level of individual effort. We will henceforth refer to x^* as "the" optimal level of x. If $x > x^*$, then a *decrease* in x increases efficiency. To avoid confusion on this point, in what follows we restrict attention to the interval $[0, x^*]$, where an increase in x indeed means higher efficiency.

For any isolated island-firm there is a tendency for x to drift down over time. The basic story behind this "free-rider drift" could be told as follows. Suppose the firm's employees are arrayed like discrete points around a circle. Every worker observes the two worker-neighbors on either side, but the firm cannot observe any workers directly. Let every employee except for one of *my* neighbor-workers be working at effort level x. Since there are a large number of workers, the X-efficiency of the firm is essentially x and each employee's pay is $W(x)$. Suppose my neighbor-worker is working at effort level y, observable to me. Then my behavior vis-a-vis my neighbor is essentially reactive. If my neighbor-worker has a lower utility level than mine, I act out of inertia and do not change my effort level. However, if my neighbor-worker is enjoying more utility than me, then I imitate the effort behavior that leads to such higher utility. My new effort level will be $x' = \min\{x, y\}$. In this environment any stochastic mutation causing a random change in individual employee effort will induce a downward ratcheting of X-efficiency over time. Unless the downward-ratcheting spiral of free rider drift is countered by some other force, the long run steady state equilibrium of any isolated island-firm is zero individual effort, which translates into zero X-efficiency. The group fitness of any isolated collection of

[2]Without significant loss of generality, in what follows we assume away the issue of break-even inequality conditions. (The analysis here of break-even corner solutions seems neither interesting nor insightful.)

individual units has a tendency over time to deteriorate from what are essentially selfish genes.

3. Free rider drift vs. competitive pressure

This section presents the partial equilibrium model. We are analyzing the characteristics of a single island-firm, called the "home" island-firm, embedded in a large archipelago of potentially invasive island-firms, called "challenger" island-firms, having exogenously given characteristics. The next section will extend the analysis to a general equilibrium framework within which the characteristics of the challenger island-firms are themselves endogenously determined.

Free rider drift within the home island-firm is caused by mutations, which are modeled as a homogenous Poisson process. With Poisson intensity μ, a mutation occurs in a single employee–inhabitant of the home island-firm. Formally, the probability of one mutation occurring in a given firm within any time interval $[t, t+\delta]$ is

$$\mu\delta + o(\delta) \tag{7}$$

and the probability of two or more mutations occurring is of the order $o(\delta)$. By definition,

$$\lim_{\delta\to 0} \frac{o(\delta)}{\delta} = 0. \tag{8}$$

The mutated employee works at effort level y, where y is a random variable drawn from an exogenously given cumulative probability distribution function $G(y)$ with support $[0, x^*]$. If the common effort level of the island-firm had been x, and an employee–inhabitant mutates to level y, then the new common effort level is (as discussed in Section 2)

$$x' = \min[x, y]. \tag{9}$$

Thus, we assume this adjustment is sufficiently rapid relative to the mutation rate so that, in effect, the firm goes instantaneously to its new effort level x'. A similar assumption is made by Vega-Redondo (1993), but (as pointed out by our referee) it is very strong. If this assumption is not made, there will be a distribution of effort levels not only among firms, but even within each firm. This would make the model richer. However, our assumption that imitation is fast is interesting because it creates the most unfavorable climate for efficiency. Any other assumption would lead to a higher average effort level.

More generally, we could consider I independent mutations at intensity rates

$$\mu_i = 1, 2, \ldots, I \tag{10}$$

and corresponding probability distribution functions

$$G_i(y_i) = 1, 2, \ldots, I. \tag{11}$$

It can readily be confirmed that the analysis goes through for this case 'as if' there were a single type of mutation at intensity rate

$$\mu \equiv \sum_{i=1}^{I} \mu_i \tag{12}$$

with a corresponding 'as if' probability distribution function

$$G(y) \equiv \frac{\sum_{i=1}^{I} \mu_i G_i(y)}{\sum_{i=1}^{I} \mu_i}. \tag{13}$$

The aggregate mutation frequency parameter μ might be influenced by a variety of internal cultural norms or sanctions within the home island-firm. But however small it might be, so long as μ is positive the mutation process described above, if unopposed, will induce free rider drift to a long run steady state level $x=0$.

Let the proportion of challenger island-firms having X-efficiency performance level no greater than z be $H(z)$. The exogenous specification of $H(\cdot)$ constitutes the partial equilibrium assumption. The support of $H(\cdot)$ is contained in the interval $[0, x^*]$.

Depending on the context, the home island-firm bumps into, interacts with, is exposed to, or is invaded by a randomly selected challenger island-firm. Again this is modeled as a Poisson process. The probability of one competitive challenge occurring in any time interval $[t, t+\delta]$ is

$$\lambda\delta + o(\delta) \tag{14}$$

and the probability of two or more challenges is of the order $o(\delta)$. Since challenges are uncorrelated with mutations the probability of both a challenge and a mutation occurring in the *same* time-interval $[t, t+\delta]$ is of the order $o(\delta)$.

The coefficient λ, called here the "challenge rate," parameterizes the degree of interaction, accessibility, competitiveness, openness, trade liberalization, ease of takeover, or the like. Conversely, $1/\lambda$ is a measure of the degree of isolation, remoteness, insularity, or protection of the home firm-island.

The following description would appear to represent the simplest possible reduced form model of the "takeover" process that occurs after a challenger island-firm invades the home island-firm. Suppose the X-efficiency of the home island-firm had been x before the invasion incident, with corresponding fitness level $\Pi(x)$. Let the X-efficiency of the challenger island-firm be z, with corresponding fitness level $\Pi(z)$. Then the challenger in effect "takes over" the home island-firm and induces its own performance level if and only if

$$\Pi(z) > \Pi(x) \tag{15}$$

i.e. if and only if

$$z > x. \tag{16}$$

The result of a competitive challenge to the home firm is "survival of the fittest" between invader and defender. The new efficiency level of the home firm (or whatever takes its place on the island) is x'', where

$$x'' \equiv \max[x, z]. \tag{17}$$

We believe Eq. (17) is the simplest way to model "survival of the fittest." It is perhaps easiest to think of the invading firm with a higher performance level displacing the existing firm by taking over its niche, in effect buying it out because the challenger is more profitable or more competitive or more fit, and therefore better able to outgrow the home firm on its original territory. Alternatively, a story could be told about the home firm rising to the challenge of a more efficient invader by somehow imposing a corporate culture that increases the common degree of effort to the challenger level. In any case the outcome is 'as if' the fittest or more profitable firm survives the competitive encounter.

Analogously to 10–13), the case of J independently challenging invader sub-populations can be aggregated 'as if' there were a single invader population with 'as if' challenge rate

$$\lambda \equiv \sum_{j=1}^{J} \lambda_j \tag{18}$$

and 'as if' cumulative population distribution function

$$H(z) \equiv \frac{\sum_{j=1}^{J} \lambda_j H_j(z)}{\sum_{j=1}^{J} \lambda_j}. \tag{19}$$

Theorem 1. The distribution of home island-firm efficiency converges globally to the unique steady state probability distribution function given by

$$F(x) \equiv \frac{\mu G(x)}{\mu G(x) + \lambda(1 - H(x))} \tag{20}$$

if $H(x) < 1$, and $F(x) = 1$ if $H(x) = 1$.

Proof. Let x_t be efficiency in the home island-firm at time t. Let $x_0 \epsilon [0, x^*]$ be a given initial level of efficiency. (More generally, we could consider a given initial distribution of efficiency.)

Fix $x \in [0, x^*]$. If $H(x) = 1$ then no challenger is ever more efficient than x. Thus once the home firm falls below x, it can never climb above x. Since on the other hand a mutation must eventually cause it to fall below x, the probability that the home firm is less efficient than x converges to 1.

Now suppose $H(x) < 1$. The relevant conditional probabilities are

$$Pr(x_{t+\delta} \leq x \mid x_t > x) = \mu G(x)\delta + o(\delta) \tag{21}$$

and

$$Pr(x_{t+\delta} \leq x \mid x_t \leq x) = 1 - \lambda(1 - H(x))\delta + o(\delta). \tag{22}$$

Let $p(t) \equiv Pr(x_t \leq x)$ be the unconditional probability that $x_t \leq x$. Using Eq. (21) and Eq. (22), we get

$$\begin{aligned} p(t+\delta) &= p(t)Pr(x_{t+\delta} \leq x \mid x_t \leq x) + (1 - p(t))Pr(x_{t+\delta} \leq x \mid x_t > x) \\ &= p(t)(1 - \lambda(1 - H(x))\delta) + (1 - p(t))\mu G(x)\delta + o(\delta). \end{aligned} \tag{23}$$

Rearranging and letting $\delta \rightarrow 0$, we obtain

$$\frac{\mathrm{d}p(t)}{\mathrm{d}t} = -a(x)(p(t) - F(x)) \tag{24}$$

where the constant $a(x)$ is defined as

$$a(x) \equiv \mu G(x) + \lambda(1 - H(x)). \tag{25}$$

The linear differential Eq. (24) has $p(t)$ converging exponentially to $F(x)$ at damping rate $a(x) > 0$. QED

As λ/μ is higher, the steady state distribution for the home firm is strictly improved in the sense of stochastic dominance. As $\lambda/\mu \to 0$, then $F(x) \to 1$ for any $x \in (0, x^*]$, so that all probability mass goes to $x = 0$. As $\lambda/\mu \to \infty$, then $F(x) \to 0$ for any $x \in [0, x^*)$, so that all probability mass becomes concentrated at $x = x^*$.

The global stability is intuitively obvious. The internal force of free rider drift pulls down efficiency over time, while the external force of competitive pressure acts to push up fitness. For x close to x^*, the home island-firm is extremely vulnerable to free rider drift, since any mutation will pull down x. But for x close to zero, the home island-firm is extremely vulnerable to challenge, since any invader can take over. These two opposing forces push the island-firm away from extremes and move it toward a middle ground of X-efficiency.

4. General equilibrium analysis

In the last section we treated the cumulative population distribution of the challenger island-firms as a given function $H(\cdot)$, proceeding from there to derive the limiting distribution of home-island efficiency $F(\cdot)$. We now endogenize $H(\cdot)$ in a general equilibrium setting with symmetric island-firms. In effect, the general equilibrium model is the same as the partial equilibrium model except for the added restriction that the home island-firm should share identical statistical properties with the other island-firms. Crudely speaking, the general equilibrium model of this section is the partial equilibrium model of last section with $H(x)$ chosen to satisfy the additional restriction $H(x) \equiv F(x)$, a substitution which essentially turns Eq. (20) into either $F(x) = \mu G(x)/\lambda$ or $F(x) = 1$. (The latter condition for $x = 0$ means that $F(0) = 1$ is a stationary distribution, since there are no upward mutations.)

There is an infinitely large number of perfectly symmetric island-firms. Let $\Phi(x;t)$ be the fraction of island firms having X-efficiency less than or equal to x at time t. There is no exogenous inflow of new island-firms into the system, nor are there any "upward" mutations in existing firms. Hence the maximal efficiency at any time $t > 0$ can be no greater than the maximal efficiency at time 0. Therefore, the support of the initial distribution of efficiency, $\Phi(\cdot;0)$, plays a significant role in determining the limiting distribution. Without loss of generality we suppose the greatest element in the support of $\Phi(\cdot;0)$ is x^*. If the maximal initial efficiency is $x' < x^*$, efficiency levels higher than x' are irrelevant, and we can replace x^* by x' throughout with identical results. We make no other assumptions about the initial distribution of firms $\Phi(\cdot;0)$.

As before, with exogenously given intensity μ a mutated employee appears who works at effort level y, where y is a random variable drawn from the given distribution $G(y)$. We suppose $G(x^*) > 0$ so that harmful mutations can occur. If an island-firm is at X-efficiency level x and an internal employee-mutation occurs to level y, the new

X-efficiency of the firm will be $x' = \min\{x, y\}$. Invasions by a challenger firm occur at intensity λ. The invader firm is drawn randomly from the population of island-firms. At time t the probability that a challenger has an efficiency below x is $\Phi(x;t)$. The reduced form outcome of the invasion of a home firm at efficiency level x by a challenger firm at efficiency level z is a "survival of the fittest" efficiency level $x'' = \max\{x, z\}$. Challenges occur independently, and are uncorrelated with mutations.

We now investigate what happens to $\Phi(x;t)$ as $t \to \infty$. Consider the population distribution function $\Phi(x)$ defined as follows:

Case 1: $\lambda > \mu$. Then

$$\Phi(x) \equiv \begin{cases} 1 & \text{if } x = x^* \\ \frac{\mu}{\lambda} G(x) & \text{if } x < x^* \end{cases}. \tag{26}$$

Case 2: $\lambda < \mu$. Then

$$\Phi(x) \equiv \begin{cases} 1 & \text{if } x \geq \bar{x} \\ \frac{\mu}{\lambda} G(x) & \text{if } x < \bar{x} \end{cases}. \tag{27}$$

where $\bar{x}$ is defined to be the (smallest) value of x satisfying

$$\frac{\mu}{\lambda} G(\bar{x}) = 1. \tag{28}$$

Case 3: $\lambda = \mu$. Then

$$\Phi(x) \equiv G(x). \tag{29}$$

Our basic result in the general equilibrium formulation is that, for all x and from any initial distribution, $\Phi(x;t)$ converges to $\Phi(x)$. The intuition behind this result is simple. Suppose only a few firms have efficiency greater than x, i.e. $\Phi(x;t)$ is close to (but not equal to) one. More precisely, suppose $\mu G(x)/\lambda < \Phi(x;t) < 1$. These more efficient firms essentially win every contest and will be very successful in replacing other firms. In fact such replacements will occur at a rate $\lambda(1 - \Phi(x;t))\Phi(x;t)$, which is the probability that a firm with efficiency greater than x challenges a firm with efficiency less than x. Firms with efficiency greater than x are mutated down below x at rate $\mu(1 - \Phi(x;t))G(x)$. The first effect dominates, implying a falling $\Phi(x;t)$, if and only if $\lambda(1 - \Phi(x;t))\Phi(x;t) > \mu G(x)(1 - \Phi(x;t))$. But this holds by the assumption $\mu G(x)/\lambda < \Phi(x;t) < 1$. Thus, the replacement of less efficient firms by more efficient firms dominates the downward drift of the more efficient firms. On the other hand, free rider drift will dominate, implying an increasing $\Phi(x;t)$, if $\Phi(x;t) < \mu G(x)/\lambda$. In a dynamic equilibrium, $\Phi(x;t) = \mu G(x)/\lambda$ whenever $\Phi(x;t) < 1$.We now provide a formal statement and proof of the theorem.

Theorem 2. For all $x \in [0, x^*]$, $\lim_{t\to\infty} \Phi(x;\ t) = \Phi(x)$.

Proof. Fix $x \in [0, x^*]$ and define $\Psi(t) \equiv 1 - \Phi(x;t)$. It is obvious that Ψ is continuous. If $x = x^*$, then $\Psi(t) = 0$ for all t, and the conclusion of the theorem holds for this x. So suppose $0 \leq x < x^*$. By assumption, the support of $\Phi(x;0)$ contains $x^* > x$, so $\Psi(0) = 1 - \Phi(x;0) > 0$. We need to show $\Psi(t) \to 1 - \Phi(x)$.

Consider a firm with efficiency less than (or equal to) x at time t. Challenges to this firm occur with Poisson intensity λ, and a challenge at time t will raise the challenged firm's efficiency level strictly above x if and only if the challenger's efficiency is strictly higher than x, which it is with probability $\Psi(t)$. As $\Psi(t)$ is approximately constant in a short interval $[t, t+\delta]$, the probability that a firm with efficiency less than x at time t has efficiency strictly greater than x at time $t+\delta$ is

$$\lambda\delta\Psi(t) + o(\delta). \tag{30}$$

As the proportion of firms with efficiency less than x at time t is $1 - \Psi(t)$, the measure of firms that rise above x in the interval $[t, t+\delta]$ is

$$\lambda\delta\Psi(t)(1 - \Psi(t)) + o(\delta). \tag{31}$$

The proportion of firms with efficiency strictly exceeding x at time t is $\Psi(t)$, and the probability that such a firm has efficiency no greater than x at time $t+\delta$ is

$$\mu G(x)\delta + o(\delta). \tag{32}$$

Thus we obtain

$$1 - \Psi(t+\delta) = (1 - \Psi(t)) - \lambda\delta\Psi(t)(1 - \Psi(t)) + \Psi(t)\mu G(x)\delta + o(\delta). \tag{33}$$

Rearranging, dividing by δ, and taking limits as $\delta \to 0$ yields the differential equation

$$\frac{\mathrm{d}\Psi(t)}{\mathrm{d}t} = \Psi(t)(\lambda - \mu G(x) - \lambda\Psi(t)). \tag{34}$$

There are two sub-cases to consider:

Sub-Case 1: $\lambda - \mu G(x) \leq 0$. This condition implies $x \geq \bar{x}$ ($x \geq \bar{x}$ was defined in Eq. (28)). Since the right hand side of Eq. (34) is strictly negative so long as $\Psi(t) > 0, \Psi(t) \to 0 = 1 - \Phi(x)$ for this sub-case.

Sub-Case 2: $\lambda - \mu G(x) > 0$. This condition implies $x < \bar{x}$ and hence $\Phi(x) < 1$. We rewrite Eq. (34) as a logistic differential equation

$$\frac{\mathrm{d}\Psi}{\mathrm{d}t} = r\Psi\left(1 - \frac{\Psi}{\mathrm{K}}\right) \tag{35}$$

where

$$r(x) \equiv \lambda - \mu G(x) > 0 \tag{36}$$

and

$$\mathrm{K}(x) \equiv 1 - \Phi(x) > 0. \tag{37}$$

It follows that $\Psi(t)$ increases whenever $0 < \Psi(t) < \mathrm{K}(x)$, and decreases whenever $\Psi(t) > K(x)$. Since $\Psi(0) = 1 - \Phi(x;0) > 0, \Psi(t) \to 1 - \Phi(x)$. QED

The proof of Theorem 2 shows that the behavior of $1 - \Phi(x;t)$ for $\Phi(x;t) > (\mu/\lambda)G(x)$ is *logistic growth*. Thus, if $\Phi(x;t)$ is close to one, so that almost all firms have efficiency below x, then the increase in the number of firms with efficiency greater than x is very fast, essentially growing exponentially at rate $\lambda - \mu G(x)$. The few existing firms with efficiency above x are tremendously successful at "invading" other firms before the invaders themselves succumb to free-rider drift. On the other hand, when

$\Phi(x;t)$ is close to zero, meaning almost all firms are more efficient than x, then the decline in the number of such efficient firms is basically exponential, at decay rate $\mu G(x)$. If $\Phi(x;t)$ is close to $\Phi(x)$, the change in $\Phi(x;t)$ is very slow.

The ratio λ/μ plays a critical role, as it does in the partial equilibrium case. An increase in λ/μ unambiguously improves efficiency, in the sense of stochastic dominance. When $\lambda > \mu$, challenges occur sufficiently frequently, relative to free-rider drift, that in the long run the fraction of island-firms massed together at the optimal efficiency level x^* is $1 - \mu/\lambda$, a positive number strictly increasing in λ/μ. The fraction of island-firms at suboptimal levels $x < x^*$ is approximately μ/λ, so that inefficiency disappears altogether as $(\lambda/\mu) \to \infty$. When $\lambda < \mu$, challenges occur so infrequently, or free rider drift is sufficiently strong, that all of the most efficient firms are eliminated in the long run. The maximal efficiency level that survives is $\bar{x} < x^*$, as defined by Eq. (28). Note that $\bar{x}$ declines strictly in λ/μ, approaching total X-inefficiency as $(\lambda/\mu) \to 0$.

We assumed above that x^* is in the support of $\Phi(\cdot;0)$. However, if instead $x' < x^*$ is the greatest efficiency level initially present, then Theorem 2 is still true if x^* is replaced by x' throughout and 26–29) are modified in the obvious way. In this case, if $x \leq \min\{x', \bar{x}\}$, then $\Phi(x;t) \to \mu G(x)/\lambda$. Otherwise, $\Phi(x;t) \to 1$. As a special case, if $x' = 0$ is the greatest efficiency level initially present, then the population of island-firms remains forever at the degenerate steady state with all probability mass concentrated on $x = 0$. This is because there is no mechanism by which more efficient firms are introduced into the system if they are not present initially. But if the initial distribution is *not* degenerate, we do *not* converge to the degenerate distribution. The reason is the bootstraps provided by high-efficiency firms "invading" low-efficiency firms.

As was noted in the introduction, the assumption of an infinite number of island-firms is important because it guarantees an adequate supply of efficient firms. In the finite case, a coincidental bad string of almost simultaneous downward mutations may lower the maximal efficiency level, and indeed over time this will cause average efficiency to ratchet down toward zero. In the long run the finite system degenerates with probability one.[3]

Fortunately there are various ways of obtaining limit results for finite systems. It turns out that under reasonable assumptions there is no discontinuity in the limit. In the appendix we show that a large finite world will have a unique steady state distribution that approximates $\Phi(\cdot)$ if there is but an infinitesimally small probability of exogenous "upward" mutations. Suppose upward jumps in efficiency occur at a Poisson rate $\epsilon > 0$, and suppose the number of island-firms is $n < \infty$. In this case there exists a unique stationary distribution of efficiency. Theorem 3 in the appendix establishes that this distribution approximates $\Phi(\cdot)$ for small ϵ and large n.

5. Conclusion

No economist will be surprised by the idea that competition is a force for efficiency. Furthermore, it is hardly surprising that in a model where more efficient firms replace less

[3]In a loose way the finite case exhibits characteristics roughly analogous to genetic drift in biology, whereas the infinite case has some rough analogies to Hardy–Weinberg equilibrium.

efficient firms there is some tendency to resist or slow down any reversions to inefficiency. However, in our model competitive challenges represent a surprisingly *strong* force for efficiency. Competitive challenges perform a "magic trick" by maintaining, and even creating, efficiency in a system that otherwise would be running down over time. These results are derived for the case of an infinite number of firms. Competitive challenges can also maintain high efficiency in a large finite system if there is an arbitrarily small infusion of "new" high efficiency firms.

Appendix A

In this appendix we consider the case of a finite number of island-firms. We show that if there is a small probability of "upward" mutation and a large but finite number of firms, the unique stationary distribution approximates arbitrarily closely $\Phi(x)$ (as defined by Eqs. (26–29)).

Let $n < \infty$ be the number of firms. Fix $x < x^*$ and let $i \leq n$ be the number of firms with efficiency strictly greater than x at time t. Under the assumptions made previously, the probability that one of the i firms with efficiency strictly above x is brought down below x during the interval $[t, t+\delta]$ is

$$p_{i,i-1}(\delta) = i\mu\delta G(x) + o(\delta) = \mu_i\delta + o(\delta) \tag{38}$$

where

$$\mu_i \equiv i\mu G(x). \tag{39}$$

The probability that more than one firm is brought down is $o(\delta)$.

Consider now one of the $n-i$ firms with efficiency below x at time t. Referring to the arguments developed earlier, the probability that a challenge raises efficiency in this firm strictly above x during $[t, t+\delta]$ is

$$\lambda\delta\frac{i}{n} + o(\delta) \tag{40}$$

since i/n is the proportion of the firms with efficiency strictly greater than x at time t. As before, the probability of multiple challenges is of the order $o(\delta)$.

Now we make an important new assumption. With an exogenously given Poisson intensity $\varepsilon(x)$, an "upward" mutation raises a firm's efficiency above x, even in the absence of any challenge. We suppose $\varepsilon(x) > 0$ for $x < x^*$ and $\varepsilon(x) = 0$ for $x \geq x^*$. (In contrast to the earlier analysis, we do not need to assume that firms with efficiency x^* are present initially.) Under this assumption, the probability that one of the $n-i$ firms with efficiency below $x \in (0, x^*)$ is brought up above x is

$$p_{i,i+1}(\delta) = (n-i)\left(\lambda\frac{i}{n} + \varepsilon(x)\right)\delta + o(\delta) = \lambda_i\delta + o(\delta) \tag{41}$$

where

$$\lambda_i \equiv (n-i)\left(\lambda\frac{i}{n} + \varepsilon(x)\right). \tag{42}$$

The probability that more than one firm is brought up is $o(\delta)$.

In what follows, we write $\varepsilon(x) = \varepsilon$ for convenience (x will be fixed throughout). Define

$$p_{i,i}(\delta) = 1 - p_{i,i-1}(\delta) - p_{i,i+1}(\delta). \tag{43}$$

Let $p_i(t)$ be the unconditional probability that exactly i firms have efficiency strictly greater than x. Using Eq. (38) and Eq. (41),

$$\begin{aligned} p_i(t+\delta) &= p_{i-1}(t)p_{i-1,i}(\delta) + p_{i+1}(t)p_{i+1,i}(\delta) + p_i(t)p_{i,i}(\delta) + o(\delta) \\ &= p_{i-1}(t)\lambda_{i-1}\delta + p_{i+1}(t)\mu_{i+1}\delta + p_i(t)(1 - \lambda_i\delta - \mu_i\delta) + o(\delta). \end{aligned} \tag{44}$$

Rearranging and taking limits as $\delta \to 0$, we get

$$\frac{\mathrm{d}p_i(t)}{\mathrm{d}t} = p_{i-1}(t)\lambda_{i-1} + p_{i+1}(t)\mu_{i+1} - p_i(t)(\lambda_i + \mu_i). \tag{45}$$

In steady state Eq. (45) has to equal zero for $i = 0, 1, \ldots, n$. This leads to the system of equations

$$p_{i-1}\lambda_{i-1} + p_{i+1}\mu_{i+1} - p_i(\lambda_i + \mu_i) = 0 \tag{46}$$

where p_i is the steady-state probability that precisely i firms have efficiency strictly greater than x. This system can be solved recursively. For $i = 0$ we get (using the fact $p_{-1}(t) = 0$ for all t)

$$p_1\mu_1 - p_0(\lambda_0 + \mu_0) = 0 \tag{47}$$

so that

$$p_1 = \frac{\lambda_0 + \mu_0}{\mu_1}p_0 = \frac{\lambda_0}{\mu_1}p_0 \tag{48}$$

Continuing recursively, we obtain steady state probabilities

$$p_i = \pi_i p_0 \tag{49}$$

for $i = 0, 1, \ldots, n$, where

$$\pi_i = \frac{\lambda_0 \cdot \lambda_1 \cdot \ldots \cdot \lambda_{i-1}}{\mu_1 \cdot \mu_2 \cdot \ldots \cdot \mu_i} \tag{50}$$

for $i = 1, 2, \ldots, n$ and

$$\pi_0 = 1 \tag{51}$$

Since the probabilities must add to one,

$$\sum_{i=0}^{n} p_i = p_0 \sum_{i=0}^{n} \pi_i = 1. \tag{52}$$

Combining Eq. (49) and Eq. (52), we see that

$$p_i = \frac{\pi_i}{\sum_{j=0}^{n} \pi_j}. \tag{53}$$

To indicate that most variables defined so far depend on n and ε, write $p_i = p_i(n, \varepsilon)$ for the ith steady state probability, $\pi_i = \pi_i(n, \varepsilon)$ for the ith coefficient defined by 50 and 51), etc. For $j > 0$, Eq. (50) can be rewritten as

$$\pi_j(n,\varepsilon) = \prod_{i=0}^{j-1} \frac{\lambda_i(n,\varepsilon)}{\mu_{i+1}(n,\varepsilon)} = \prod_{i=0}^{j-1} \frac{(n-i)\left(\lambda\frac{i}{n}+\varepsilon\right)}{(i+1)\mu G(x)} = \binom{n}{j} \prod_{i=0}^{j-1} \left(\frac{\lambda}{\mu G(x)}\frac{i}{n} + \frac{\varepsilon}{\mu G(x)}\right). \quad (54)$$

We want to prove that the finite world approximates the infinite world if $\varepsilon(x)$ is small for each x and n is large. Let $\Phi(x)$ be defined by 26–29). The number of firms with fitness greater than x is a birth-death process with steady-state probabilities given by Eq. (54). Theorem 3 shows that as the world grows, in this unique steady-state all probability mass is concentrated in a neighborhood of $1-\Phi(x)$. This neighborhood can be made arbitrarily small by choosing $\varepsilon(x)$ sufficiently small.

Theorem 3. For any $\delta > 0$, there exists $\varepsilon^* > 0$ such that if $\varepsilon < \varepsilon^*$, then as $n \to \infty$ we have

$$\sum p_i(n,\varepsilon) \to 1 \quad (55)$$

where the sum is taken over the set $\{i : 1-\Phi(x)-\delta < i/n < 1-\Phi(x)+\delta\}$.

Proof. Fix x and define $q^* = 1-\Phi(x)$, where $\Phi(x)$ is given by 26–29). If $q^* = 1$ then $G(x) = 0$ so that there is no possibility of harmful mutations: in this case the proof is trivial. The case $q^* = 0$ needs a few straightforward modifications of our argument, which we leave to the reader. Hence, suppose $0 < q^* < 1$.

Throughout the proof, q will represent the proportion of islands with efficiency greater than x, so that the formulas will make sense only under the implicit assumption that qn is an integer between zero and n.

By Stirling's approximation (Courant, 1988, p.361),

$$\frac{1}{\left(1+\frac{1}{4(1-q)n}\right)\left(1+\frac{1}{4qn}\right)} < \frac{\binom{n}{qn}}{q^{-qn}(1-q)^{-(1-q)n}(2q(1-q)\pi n)^{-1/2}} < 1+\frac{1}{4n} \quad (56)$$

for all n and $q \in (0,1)$.

By a straightforward computation,

$$\begin{aligned} \log\left(\prod_{i=0}^{qn-1}\left(\frac{\lambda}{\mu G(x)}\frac{i}{n}+\frac{\varepsilon}{\mu G(x)}\right)\right) &= \sum_{i=0}^{qn-1}\log\left(\frac{\lambda}{\mu G(x)}\frac{i}{n}+\frac{\varepsilon}{\mu G(x)}\right) \\ &\le \sum_{i=0}^{qn-1}\int_i^{i+1}\log\left(\frac{\lambda}{\mu G(x)}\frac{s}{n}+\frac{\varepsilon}{\mu G(x)}\right)\mathrm{d}s \\ &= \int_0^{qn}\log\left(\frac{\lambda}{\mu G(x)}\frac{s}{n}+\frac{\varepsilon}{\mu G(x)}\right)\mathrm{d}s \\ &= \log\left(\left(\left(\frac{\lambda q+\varepsilon}{\mu G(x)}\right)^q\left(1+\frac{\lambda q}{\varepsilon}\right)^{\varepsilon/\lambda}\exp(-q)\right)^n\right). \end{aligned} \quad (57)$$

Similarly,

$$\begin{aligned}
\log\left(\prod_{i=0}^{qn-1}\left(\frac{\lambda}{\mu G(x)}\frac{i}{n}+\frac{\varepsilon}{\mu G(x)}\right)\right) &= \sum_{i=0}^{qn-1}\log\left(\frac{\lambda}{\mu G(x)}\frac{i}{n}+\frac{\varepsilon}{\mu G(x)}\right)\\
&\geq \log\left(\frac{\varepsilon}{\mu G(x)}\right)-\log\left(\frac{\lambda q}{\mu G(x)}+\frac{\varepsilon}{\mu G(x)}\right)\\
&\quad+\sum_{i=1}^{qn}\int_{i-1}^{i}\log\left(\frac{\lambda}{\mu G(x)}\frac{s}{n}+\frac{\varepsilon}{\mu G(x)}\right)\mathrm{d}s\\
&= \log\left(\frac{\varepsilon}{\lambda q+\varepsilon}\right)+\int_{0}^{qn}\log\left(\frac{\lambda}{\mu G(x)}\frac{s}{n}+\frac{\varepsilon}{\mu G(x)}\right)\mathrm{d}s\\
&= \log\left(\frac{\varepsilon}{\lambda q+\varepsilon}\right)\\
&\quad+\log\left(\left(\left(\frac{\lambda q+\varepsilon}{\mu G(x)}\right)^{q}\left(1+\frac{\lambda q}{\varepsilon}\right)^{\varepsilon/\lambda}\exp(-q)\right)^{n}\right)
\end{aligned} \tag{58}$$

Using Eq. (56) and Eq. (57) in Eq. (54), we have an upper bound for $\pi_{qn}(n,\varepsilon)$ for $0<q<1$:

$$\begin{aligned}
\pi_{qn}(n,\varepsilon) &< \left(1+\frac{1}{4n}\right)q^{-qn}(1-q)^{-(1-q)n}(2q(1-q)\pi n)^{-1/2}\\
&\quad\times\left(\left(\frac{\lambda q+\varepsilon}{\mu G(x)}\right)^{q}\left(1+\frac{\lambda q}{\varepsilon}\right)^{\varepsilon/\lambda}\exp(-q)\right)^{n}\\
&= \left(1+\frac{1}{4n}\right)(2q(1-q)\pi n)^{-1/2}(\Omega(q,\varepsilon))^{n},
\end{aligned} \tag{59}$$

where

$$\Omega(q,\varepsilon) \equiv q^{-q}(1-q)^{-(1-q)}\left(\frac{\lambda q+\varepsilon}{\mu G(x)}\right)^{q}\left(1+\frac{\lambda q}{\varepsilon}\right)^{\varepsilon/\lambda}\exp(-q). \tag{60}$$

Using Eq. (56) and Eq. (58) in Eq. (54), we get (for $0<i<n$) the lower bound

$$\pi_i(n,\varepsilon) \geq \frac{(\Omega(i/n,\varepsilon))^{n}}{(2\pi n)^{1/2}\mathrm{K}(i,n,\varepsilon)}, \tag{61}$$

where

$$\mathrm{K}(i,n,\varepsilon) \equiv \left(1+\frac{1}{4(n-i)}\right)\left(1+\frac{1}{4i}\right)\left(1+\frac{\lambda i}{n\varepsilon}\right)\left(\frac{i}{n}\left(1-\frac{i}{n}\right)\right)^{1/2}. \tag{62}$$

From Eq. (57) and Eq. (58) it follows that

$$\left(\frac{\varepsilon}{\lambda+\varepsilon}\right)(\Omega(1,\varepsilon))^n \leq \pi_n(n,\varepsilon) = \prod_{i=0}^{n-1}\left(\frac{\lambda}{\mu G(x)}\frac{i}{n}+\frac{\varepsilon}{\mu G(x)}\right) \leq (\Omega(1,\varepsilon))^n, \tag{63}$$

where by definition

$$\Omega(1,\varepsilon) \equiv \left(\frac{\lambda+\varepsilon}{\mu G(x)}\right)\left(1+\frac{\lambda}{\varepsilon}\right)^{\varepsilon/\lambda}\exp(-1). \tag{64}$$

We also define $\Omega(0,\varepsilon) \equiv 1$ and note that

$$\pi_0(n,\varepsilon) = 1 = 1^n = (\Omega(0,\varepsilon))^n. \tag{65}$$

With these definitions $\Omega(\cdot,\varepsilon)$ is continuous on [0, 1]. We will find q that maximizes $\Omega(q,\varepsilon)$ for given $\varepsilon > 0$. For $q \in (0,1)$ we have

$$\log\Omega(q,\varepsilon) \equiv -q\log q - (1-q)\log(1-q) + q\log\left(\frac{\lambda q+\varepsilon}{\mu G(x)}\right) + \frac{\varepsilon}{\lambda}\log\left(\frac{\varepsilon+\lambda q}{\varepsilon}\right) - q \tag{66}$$

and

$$\begin{aligned}\frac{\partial\log\Omega(q,\varepsilon)}{\partial q} &= -\log q - 1 + \log(1-q) + 1 + \log\left(\frac{\lambda q+\varepsilon}{\mu G(x)}\right) + \frac{\lambda q}{\lambda q+\varepsilon} + \frac{\varepsilon}{\varepsilon+\lambda q} - 1\\ &= \log\left(\frac{1-q}{q}\frac{\lambda q+\varepsilon}{\mu G(x)}\right).\end{aligned} \tag{67}$$

Moreover, for $q \in (0,1)$,

$$\frac{\partial^2\log\Omega(q,\varepsilon)}{\partial q^2} = -\frac{\varepsilon+\lambda q^2}{(1-q)q(\lambda q+\varepsilon)} < 0. \tag{68}$$

Setting Eq. (67) equal to zero yields

$$\frac{1-q}{q}\frac{\lambda q+\varepsilon}{\mu G(x)} = 1. \tag{69}$$

Eq. (69) has a unique positive root $q = q(\varepsilon) \in (0,1)$ which by Eq. (68) is the unique maximizer of $\Omega(q,\varepsilon)$ in the interval [0,1]. It is easy to check that $q(\varepsilon) \to q^*$ as $\varepsilon \to 0$. It can also be verified that

$$\lim_{\varepsilon\to 0,\, q\to q^*}\Omega(q,\varepsilon) = \exp(-q^*)(1-q^*)^{-1} > 1. \tag{70}$$

Fix $\delta > 0$. By Eq. (68), Ω is uniformly strictly concave in q in some small neighborhood of $(\varepsilon, q) = (0, q^*)$, i.e.

$$\frac{\partial^2\log\Omega(q,\varepsilon)}{\partial q^2} < -\frac{1}{2(1-q^*)} \tag{71}$$

in some such neighborhood. By Eq. (71) and the fact that $q(\varepsilon) \to q^*$ we can find $\varepsilon' > 0$ and $A > 0$ such that if $\varepsilon < \varepsilon'$ and $| q - q^* | > \delta$, then

$$\Omega(q,\varepsilon) \leq \Omega(q^*,\varepsilon) - 2A. \tag{72}$$

But Eq. (70) implies, by definition of double limit, that there exists $v > 0$ and $\varepsilon'' > 0$ such that if $| q' - q^* | < v$ and $\varepsilon < \varepsilon''$, then

$$\Omega(q', \varepsilon) > \Omega(q^*, \varepsilon) - A. \tag{73}$$

Let $\varepsilon^* = \min\{\varepsilon', \varepsilon''\}$. Combining Eq. (72) and Eq. (73) yields

$$\Omega(q', \varepsilon) > \Omega(q, \varepsilon) + A \tag{74}$$

whenever $| q' - q^* | < v$, $| q - q^* | \geq \delta$, and $\varepsilon < \varepsilon^*$.

For each $n > 1/v$ we can select a positive integer $i(n)$ such that $| i(n)/n - q^* | < v$. From Eq. (74), if $| i/n - q^* | > \delta$ and $\varepsilon < \varepsilon^*$, then

$$\Omega\left(\frac{i}{n}, \varepsilon\right) < \Omega\left(\frac{i(n)}{n}, \varepsilon\right) - A. \tag{75}$$

Since $\Omega(i(n)/n)$ is bounded as $n \to \infty$, Eq. (75) implies that there exists $B < 1$ such that

$$\frac{\Omega(\frac{i}{n}, \varepsilon)}{\Omega\left(\frac{i(n)}{n}, \varepsilon\right)} \leq B \tag{76}$$

whenever $| i/n - q^* | > \delta$. Note that this includes the cases $i = 0$ and $i = n$.

The final step is to compute, for $\varepsilon < \varepsilon^*$ and $n > 1/v$,

$$\begin{aligned}
&\sum_{\{i:|(i/n)-q^*|>\delta\}} p_i(n, \varepsilon) \\
&= \frac{\sum_{\{i:|(i/n)-q^*|>\delta\}} \pi_i(n, \varepsilon)}{\sum_{j=0}^{n} \pi_j(n, \varepsilon)} < \frac{\sum_{\{i:|(i/n)-q^*|>\delta\}} \pi_i(n, \varepsilon)}{\pi_{i(n)}(n, \varepsilon)} \\
&\leq \mathrm{K}(i(n), n, \varepsilon) \frac{\pi_0(n, \varepsilon) + \pi_n(n, \varepsilon)}{(2\pi n)^{-1/2}\left(\Omega\left(\frac{i(n)}{n}, \varepsilon\right)\right)^n} \\
&\quad + \mathrm{K}(i(n), n, \varepsilon) \frac{\sum_{\{i:0<i<n, |(i/n)-q^*|>\delta\}} \left(1 + \frac{1}{4n}\right)\left(\frac{i}{n}\left(1 - \frac{i}{n}\right)2\pi n\right)^{-1/2} \left(\Omega(\frac{i}{n}, \varepsilon)\right)^n}{(2\pi n)^{-1/2}\left(\Omega\left(\frac{i(n)}{n}, \varepsilon\right)\right)^n} \\
&\leq 2\mathrm{K}(i(n), n, \varepsilon)\left((2\pi n)^{1/2} + n^2\right)B^n
\end{aligned} \tag{77}$$

In Eq. (77), the first equality uses Eq. (53), the first inequality is obvious, the second inequality uses Eq. (59) and Eq. (61), the third inequality uses 63,65 and 76). But $K(i(n), n, \varepsilon)$ is bounded as $n \to \infty$, and $B < 1$. Therefore, the final expression in Eq. (77) converges to zero. Therefore, Eq. (55) holds. QED

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