

# CONSUMER'S SURPLUS AS AN EXACT APPROXIMATION WHEN PRICES ARE APPROPRIATELY DEFLATED\*

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A canonical price-normalized form is proposed as a generalization of the ordinary consumer's surplus expression commonly used to evaluate changes in economic welfare. This familiar-looking formula, it is proved, can be rigorously interpreted as representing the first- and second-order terms of a Taylor-series expansion for the equivalent-variation or willingness-to-pay function of a single consumer. In principle, the lowly consumer's surplus triangle-and-rectangle methodology can be rigorously defended as an exact approximation to a theoretically meaningful measure as long as prices are appropriately deflated. The appropriate price deflator is derived, and some implications are discussed.

## INTRODUCTION

Consider heuristically how to evaluate the welfare change between two different situations. The first-order effect should presumably be the change in real income. An index number problem is present, but some term of the form,

$$(1) \quad P^0 \Delta Q,$$

can probably serve as a reasonable approximation. In addition, it is often argued, there ought also to be tacked on some sort of triangle-like term of the form.

$$(2) \quad \frac{1}{2} \Delta P \Delta Q,$$

which quantifies a second-order welfare correction standing for the value of substitution possibilities. This second-order correction presumably depends on appropriately normalized price changes, an issue seldom discussed explicitly in the literature but a central theme of the approach taken here.

An important question for a long time has been what, exactly, do such expressions as the sum of (1) plus (2) mean. They are obviously intended to be a practical approximation—but precisely what kind of an approximation, to what, and valid under exactly what circumstances?

\*I would like to thank Professors W. E. Diewert and A. Bergson for several useful comments on a previous version of this paper. Professor Diewert saved me from making a serious error. They should be absolved from any remaining errors or inaccuracies.

The issue has hardly gone unnoticed. A large number of excellent articles and books have been written about placing in proper context the familiar consumer's surplus triangle-and-rectangle methodology. (A very few of the many important contributions are listed in the reference section.) There seems to be some general agreement about what ideally should be measured. Yet, throughout all of the voluminous literature, I have not quite found the basic approximation issue posed and resolved in just the manner of the present paper. The new wrinkle is that the extra degree of freedom implicit in any price normalization rule can be exploited to yield a relatively neat approximation formula. Modern duality theory is used to show a reasonable, operational sense in which consumer's surplus triangles and rectangles represent exact approximations to welfare changes when an appropriate price deflator is chosen. My hope is that the approach taken here gives a few fresh insights into what is already a venerable subject skillfully treated by many others.

#### THE MODEL

The model economy has  $n$  goods, whose quantities are represented by the column  $n$ -vector  $Q$ . Let  $P$  represent the associated row  $n$ -vector of prices and  $Y = PQ$  the corresponding income.

The economy starts off in the base period at the initial condition,

$$(3) \quad P^0, Q^0, Y^0 (= P^0 Q^0),$$

and, due to some variation, actual or conjectural, ends up in the changed state,

$$(4) \quad P^1, Q^1, Y^1 (= P^1 Q^1).$$

Since welfare is presumably unaffected by equiproportional changes in prices and income, state 1 is equivalent to

$$(5) \quad \frac{P^1}{\theta}, Q^1, \frac{Y^1}{\theta} \left( = \frac{P^1 Q^1}{\theta} \right),$$

where  $\theta > 0$  is interpretable as some price deflator.

If there is to be hope of making any sense at all out of a simple consumer's-surplus-type expression, like (1) plus (2), which is a weighted sum of various terms involving price changes, then presumably prices (and incomes) must be normalized so that they are somehow compatible between the two periods or states. The litera-

ture typically sidesteps this problem, implicitly assuming that  $\theta = 1$  represents a reasonable price normalization rule. The present paper attempts to meet the problem head on by adopting the convention of viewing all price changes as economically consistent from the perspective of the initial period. Posed generally, it seems intuitive that, in some sense, the price deflator  $\theta$  should be selected to make the value of an extra dollar of income in period 1 prices equal to the value of an extra dollar of income in the prices of the base period. Thus viewed, it turns out that the issue of representing welfare changes by a simple consumer's-surplus-like formula is intimately related to the issue of choosing an appropriate price deflator that preserves the purchasing power value of an extra dollar across the two periods. We return to this basic theme presently.

With  $\theta$  as a price deflator, period 1 normalized prices can be rewritten as

$$(6) \quad P' \equiv P^1/\theta,$$

while period 1 income becomes

$$(7) \quad Y' \equiv Y^1/\theta (= P'Q').$$

Define

$$(8) \quad \Delta Q \equiv Q^1 - Q^0,$$

$$(9) \quad \Delta P \equiv P' - P^0,$$

$$(10) \quad \Delta Y \equiv Y' - Y^0.$$

A suggestive expression of the welfare change from period 0 to period 1 might be

$$(11) \quad P^0\Delta Q + \frac{1}{2}\Delta P\Delta Q.$$

Formulas like (11) have repeatedly been proposed as practical criteria for evaluating the difference between alternative economic situations.<sup>1</sup> It will be the task of this paper to prove rigorously that

1. As just one example, see Harberger [1971] and the references cited there. Diewert [1976] proved that Harberger's indicator (11) is consistent with revealed preference theory and is ordinally correct for certain classes of quadratic preferences. In more recent work, Diewert [1986] has shown that a family of expressions like (11) can be given an exact global interpretation as a welfare change indicator, provided that the expenditure function is of a certain quadratic functional form. Since the proposed quadratic form has enough free parameters to provide a second-order approximation at a single point to any smoothly differentiable expenditure function, by analogy with the superlative index number literature Diewert calls such performance indicators "superlative welfare gain measures." Diewert's indirect approach is different from the one taken here, which involves making a direct Taylor series approximation of the welfare change itself.

(11) can be interpreted as representing a well-defined approximation to a meaningful measure of welfare change.

Throughout the paper, all evaluation is performed as if from the perspective of base period prices. An analogous treatment from the viewpoint of the final state yields symmetric results.

In a way, the consistent treatment of price changes, as if viewed from a single time perspective (the status-quo base period), is the key to getting neat results that make sense out of formula (11). Most attempts to explain rigorously the concept of consumer's surplus in a general setting start off by arbitrarily designating one of the  $n$  goods as numeraire or else implicitly impose some other arbitrary price normalization rule. Then it is usually discovered that an expression like (11) is a messy or inexact approximation. The present paper shows that when an appropriate price deflator is chosen, one close to what the economist would ideally want to use anyway to normalize price changes from a base period, then things fall neatly into place and expression (11) can be rigorously justified. In other words, the "right" price deflator automatically makes the compensating corrections that legitimize formula (11). When prices are correctly deflated, at least in principle consumer's surplus can be justified as an exact approximation to the change in consumer's welfare.

#### A PRELIMINARY RESULT

To obtain sharp answers to sharply posed questions, I assume that the quantity data are generated by a representative consumer. The problems posed by aggregating over different tastes and incomes are quite formidable and belong, really, to a far more forbidding arena of discourse.<sup>2</sup>

Suppose, then, that quantities  $Q(P, Y)$ , which correspond to prices  $P$  and income  $Y$ , uniquely maximize the nicely behaved utility function  $U(Q)$  subject to the budget constraint  $PQ \leq Y$ . In other words,  $Q(P, Y)$  uniquely solves

$$(12) \quad U(Q(P, Y)) = \underset{Q: PQ \leq Y}{\text{maximum}} U(Q).$$

Throughout the paper it is assumed that  $Q(P, Y)$  and all other functions are nicely behaved.

2. In practice, I believe the issues raised by heterogeneous tastes and income represent a quite serious obstacle for formal analysis. Some statements can be made if a condition analogous to (34) holds "on average" for all individuals, but this theme is not developed further in the present paper.

As usual, the expenditure function is defined by

$$(13) \quad E(P, V) = \underset{Q: U(Q) \geq V}{\text{minimum}} PQ.$$

The money metric utility function (calibrated in base period prices) is

$$(14) \quad W(Q; P^0) \equiv E(P^0, U(Q)).$$

The money metric utility function measures the minimum income required at base-period prices  $P^0$  to achieve the level of utility  $U(Q)$ .

A rigorous expression of willingness to pay for the welfare change from state 0 to state 1 is the equivalent variation,

$$(15) \quad \Delta W \equiv W(Q^1; P^0) - W(Q^0; P^0).$$

The equivalent variation measure (15) automatically quantifies the benefit of a proposed project or policy in a money metric naturally commensurate with its base-period current-price cost. More generally, the equivalent variation is an easily interpretable cardinal measure of any welfare change. Accepting  $\Delta W$  as an appropriate quantifier of changes in welfare, this paper's main goal will be to provide a meaningful exact approximation for (15). (It should be noted that an analogous treatment using the compensating variation, based on an evaluation in period 1 prices, would yield results symmetric to those derived here below.)

Before proceeding to the main task of developing an exact approximation for (15), a necessary preliminary is to note some properties of the indirect compensation function:

$$(16) \quad M(P^0; P, Y) \equiv E(P^0, U(Q(P, Y))).$$

Expression (16) stands for the minimum income required to make the consumer as well off in base period prices as he would be with income  $Y$  when facing prices  $P$ .

The following properties of the indirect compensation function (16) are important and will be used in what follows:<sup>3</sup>

$$(17) \quad \frac{\partial M}{\partial P_i} = - \frac{\partial M}{\partial Y} Q_i(P, Y),$$

$$(18) \quad M(P^0; P^0, Y) \equiv Y.$$

3. See Varian [1984; Ch. 3] for a contemporary reference.

Let

$$(19) \quad \lambda(P^0; P, Y) \equiv \frac{\partial M}{\partial Y}$$

represent the "marginal utility of income" (for the indirect utility function (16)) at prices  $P$  and income  $Y$ , assumed to be a well-behaved function.

An important and useful property relating (14) and (16) is

$$(20) \quad \frac{\partial W}{\partial Q_i} = \lambda P_i.$$

Condition (18) implies that

$$(21) \quad \lambda(P^0; P^0, Y^0) = 1$$

and

$$(22) \quad \left. \frac{\partial \lambda}{\partial Y} \right|_0 = 0.$$

By definition,

$$(23) \quad \Delta \lambda \equiv \lambda' - \lambda^0,$$

where

$$(24) \quad \lambda' \equiv \lambda(P^0; P', Y') = \lambda(P^0; P^1/\theta, Y^1/\theta)$$

and

$$(25) \quad \lambda^0 \equiv \lambda(P^0; P^0, Y^0) = 1.$$

Returning now to the main task, I seek to approximate (15) by the first two terms of a Taylor series expansion around base period values of the relevant variables.<sup>4</sup> The following proposition holds for any  $\theta$ .

PROPOSITION 1.

$$(26) \quad \Delta W = P^0 \Delta Q + \frac{1}{2} \Delta P \Delta Q + \frac{1}{2} \Delta \lambda P^0 \Delta Q + 0(\Delta^3).$$

The expression

$$(27) \quad 0(\Delta^3)$$

4. Several other authors have used a Taylor series expansion as a quadratic approximation to measure utility changes; see especially Hotelling [1938; section II], Hicks [1946; 331-33], and Harberger [1971]. The present paper provides a rigorous proof for the theoretically appropriate money metric utility function and pushes the analysis for this specification further by determining the appropriate price deflator that nullifies the awkward "change in the marginal utility of income" term spoiling the second-order part of the approximation.

stands for all polynomial terms of third order or higher.<sup>5</sup>

The proof of the proposition is as follows.

Expanding (15) yields this expression:

$$(28) \quad \Delta W = \sum \frac{\partial W}{\partial Q_i} \Delta Q_i + \frac{1}{2} \sum \sum \frac{\partial^2 W}{\partial Q_i \partial Q_j} \Delta Q_i \Delta Q_j + 0(\Delta^3).$$

All derivatives in (28) and what follows are understood as being evaluated at the base position,

$$(29) \quad Q^0 = Q(P^0, Y^0).$$

Plugging (20) into (28) yields

$$(30) \quad \Delta W = \lambda^0 \sum P_i^0 \Delta Q_i + \frac{1}{2} \sum \sum \frac{\partial}{\partial Q_j} (\lambda P_i) \Delta Q_i \Delta Q_j + 0(\Delta^3).$$

From the assumption that everything is smoothly differentiable,

$$(31) \quad \Delta(\lambda P_i) = \sum \frac{\partial}{\partial Q_j} (\lambda P_i) \Delta Q_j + 0(\Delta^2).$$

Substituting from (31) into (30) yields the following expression in vector notation:

$$(32) \quad \Delta W = \lambda^0 P^0 \Delta Q + \frac{1}{2} \Delta(\lambda P) \Delta Q + 0(\Delta^3).$$

Now,

$$(33) \quad \Delta(\lambda P) \Delta Q = \lambda^0 \Delta P \Delta Q + \Delta \lambda P^0 \Delta Q + 0(\Delta^3).$$

Plugging (33) into (32) and making use of (25) then yields

$$(34) \quad \Delta W = P^0 \Delta Q + \frac{1}{2} \Delta P \Delta Q + \frac{1}{2} \Delta \lambda P^0 \Delta Q + 0(\Delta^3),$$

the result to be proved. |

Q.E.D.

#### THE "APPROPRIATE" PRICE DEFLATOR

What would an "ideal" price deflator look like in the present context? In some sense an ideal price deflator should preserve the value of an extra dollar of income across both sets of prices.

5. Since all functions are assumed to be smoothly differentiable, it does not matter whether one thinks of the Taylor series expansion as involving  $Q$  directly, or indirectly—through the underlying variables  $P$  and  $Y$ .

A formal way of expressing this idea (evaluated, as usual, from the perspective of base period prices) is the following. Income and prices in the period 1 position should be jointly normalized so that it requires the same number of extra dollars to achieve an extra unit of utility with the normalized prices of period 1 as it would with the given prices of period 0.

Mathematically, this statement translates into the condition that  $\theta$  should be chosen to satisfy the equation,

$$(35) \quad \lambda(P^0; P', Y') = 1.$$

Equation (35) possesses a unique solution in  $\theta$ , since it follows from (19) and (24) that the marginal utility of income in state 1,  $\lambda'$ , is directly proportional to the price deflator  $\theta$ . Using (23)–(25), condition (35) is equivalent to

$$(36) \quad \Delta\lambda = 0.$$

Unfortunately, the ideal price deflator is not a very usable concept. Luckily for the purposes of this paper, it suffices to work with a first-order approximation of condition (36), which yields an operational formula.

Define an “appropriate” price deflator, denoted  $\theta^*$ , as that value of  $\theta$  which makes  $\lambda' = 1$  hold to a first-order approximation. In other words, when  $\theta = \theta^*$ ,

$$(37) \quad \Delta\lambda(\theta^*) = 0 + 0(\Delta^2).$$

The following result then obtains:

**PROPOSITION 2.** The appropriate price deflator  $\theta^*$  satisfies the equation,

$$(38) \quad \theta^* = \frac{\sum \omega_i P_i^1}{\sum \omega_i P_i^0},$$

where the weights  $\{\omega_i\}$  are defined by

$$(39) \quad \omega_i \equiv \left. \frac{\partial Q_i}{\partial Y} \right|_0.$$

The weights used to define the appropriate price deflator are the base period changes in quantities induced by a change in income.<sup>6</sup> What is relevant for an appropriate set of price-deflating weights is quantities on the margin, not on the average.

6. If all base weights are positive, because all goods are superior, then  $\theta^* > 0$ . Note that the denominator of (38) is identically equal to +1. By continuity, the numerator of (38), and hence  $\theta^*$ , must be also be positive for sufficiently small changes in relative prices, even when some of the base weights are negative.



Note that in the important special case where preferences are homothetic,  $\theta^*$  is equivalent to the familiar Laspeyres price index.

In the special case, often used for theoretical examples, where exactly one of the  $n$  goods happens to enter the utility function as a linearly additive term, formula (38), (39) reduces to choosing that one good as numeraire.

Proposition 2 is proved as follows.

From the assumption that (19) is smoothly differentiable,

$$(40) \quad \Delta\lambda = \sum \frac{\partial\lambda}{\partial P_i} \Delta P_i + \frac{\partial\lambda}{\partial Y} \Delta Y + 0(\Delta^2).$$

From (19),

$$(41) \quad \frac{\partial\lambda}{\partial P_i} = \frac{\partial^2 M}{\partial P_i \partial Y}.$$

From (17) and (19),

$$(42) \quad \frac{\partial^2 M}{\partial Y \partial P_i} = -\lambda \frac{\partial Q_i}{\partial Y} - \frac{\partial\lambda}{\partial Y} Q_i.$$

Combining (41) with (42),

$$(43) \quad \frac{\partial\lambda}{\partial P_i} = -\lambda \frac{\partial Q_i}{\partial Y} - \frac{\partial\lambda}{\partial Y} Q_i.$$

Plugging (43) into (40) and using (22), (25) to evaluate terms yields

$$(44) \quad \Delta\lambda = -\sum \frac{\partial Q_i}{\partial Y} \Delta P_i + 0(\Delta^2).$$

From (44) and (9), it follows that (37) holds if and only if

$$(45) \quad \sum \frac{\partial Q_i}{\partial Y} P_i = \sum \frac{\partial Q_i}{\partial Y} P_i^0.$$

But from (6), equation (45) holds only when  $\theta = \theta^*$ , where  $\theta^*$  is defined by conditions (38), (39).<sup>1</sup>

Q.E.D.

#### THE BASIC RESULT

The appropriate price index is very near to what an economist would like to use in practice as a price deflator, because it keeps the value of a marginal dollar approximately constant under both sets of prices. To what degree real world price indexes end up being

sufficiently "close" in value to (38), (39) for the results of this paper to be relevant is an empirical issue. (As preferences are close to homothetic, for example,  $\theta^*$  is close to a Laspeyres index.) But it is at least useful to be aware of the general principle that when the appropriate price deflator is used, consumer's surplus can be justified as an exact approximation. This statement is formalized by the following main result.

PROPOSITION 3. When prices are appropriately deflated by (38), (39), then

$$(46) \quad \Delta W = P^0 \Delta Q + \frac{1}{2} \Delta P \Delta Q + 0(\Delta^3).$$

The proof is almost immediate. Proposition 2 asserts that (37) holds when  $\theta = \theta^*$ . Applying (37) to (26) yields (46).<sup>1</sup>

Q.E.D.

#### CONCLUDING REMARKS

The basic result (46) represents an exact approximation theorem that can be used to justify the familiar consumer's surplus expression as being accurate to the second order. The sparsity of form of the consumer's surplus rectangle and triangle approximation (46) is notable. There are no mixed ( $i, j$ ) cross product terms. A simple economic interpretation is possible.

It is worth noting that for sufficiently small changes there is a well-defined sense in which the linear term  $P^0 \Delta Q$  is likely to dominate the quadratic term  $1/2 \Delta P \Delta Q$ . As a situation changes smoothly from one price-income configuration to another, the first-order rectangle term initially overwhelms the second-order triangle term.<sup>7</sup> Furthermore, the linear plus quadratic terms of (46) are likely to be a reasonably accurate approximation for small changes because only cubic and higher order terms are being neglected.

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#### REFERENCES

- Bergson, A., "A Note on Consumer's Surplus," *Journal of Economic Literature*, XII (1975), 38-44.  
 Diamond, P., and D. McFadden, "Some Uses of the Expenditure Function in Public Finance," *Journal of Public Economics*, III (1974), 3-21.

7. As Tobin [1977] has colorfully put the idea in a different context: "It takes a heap of Harberger triangles to fill an Okun gap."

- Diewert, W. E., "Harberger's Welfare Indicator and Revealed Preference Theory," *American Economic Review*, LXVI (1976), 143-52.
- , "Cost Functions," *The New Palgrave*, Vol. 1 (London: Macmillan, 1987), pp. 690-96.
- Dixit, A., and P. A. Weller, "The Three Consumer's Surpluses," *Economica*, XLVI (1979), 125-35.
- Harberger, A. C., "Three Basic Postulates for Applied Welfare Economics: An Interpretive Essay," *Journal of Economic Literature*, IX (1971), 785-97.
- Hausman, J. A., "Exact Consumer's Surplus and Deadweight Loss," *American Economic Review*, LXXI (1981), 662-76.
- Hicks, J. R., *Value and Capital*, 2nd ed. (Oxford: Clarendon Press, 1946).
- Hotelling, H., "The General Welfare in Relation to Problems of Taxation and of Railway and Utility Rates," *Econometrica*, VI (1938), 242-69.
- Tobin, J., "How Dead is Keynes?" *Economic Inquiry*, XV (1977), 459-68.
- Varian, H. R., *Microeconomic Analysis* (New York: Norton, 1984).
- Willig, R., "Consumer's Surplus without Apology," *American Economic Review*, LXVI (1976), 589-97.