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# A Contribution to the Theory of Welfare Accounting

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## Abstract

A kind of "unified theory" is proposed as a dynamic generalization of the standard consumer-surplus methodology for evaluating welfare changes. The "unified theory" allows rigorous dynamic welfare comparisons to be inferred between any two economic situations—from just knowing current incomes and observing a short-run market demand schedule. Essentially, the change in present discounted future utility is exactly captured by the formula: *difference in current income plus consumer surplus*. This well-known formula is thereby shown to cover a far wider class of welfare comparisons than is customarily treated in the textbook static case.

*Keywords*: National income accounting; dynamic welfare

*JEL classification*: *C*43; *D*6; *D*9

## I. Introduction

The contribution in the paper's title refers to a kind of "dynamic welfare-comparison principle," which extends into the dynamic realm the standard static methodology of using currently observable incomes and market demand schedules to infer welfare differences among economic situations. The theory might be considered "unified" not just in the sense that the standard static methodology becomes a special case of the more general dynamic theory, but also because some basic unifying connections can be made with consumer-surplus theory and index-number theory. The connection with consumer-surplus theory is especially striking, because this familiarly useful methodology is shown to possess a far wider domain of applicability and a much more rigorous underpinning than had previously been suspected. There are several possible motivations for the paper, then, but I think that the best introduction is by way of seeing the contribution placed initially in the context of the national-income-accounting literature.

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Recent times have witnessed a greatly heightened awareness of the interactions between economic, social and environmental issues. People throughout the world have become much more sensitive to the important possible links between their own human societies and the natural environmental surroundings within which these societies may thrive or fail. Terms like "green accounting" and "sustainability" have found their way into the lexicon of popular jargon. There has appeared a widespread interest in the idea of extending the concepts and measurement of national income to include important non-market activities in related areas that bear on welfare and productivity—in particular environmental goods and services (including natural resources), but also human capital formation, unpaid home production (possibly including leisure-time activities), the services of consumer-owned durables, near-market research and development, and so on.[1]

Many questions have been raised about augmented national income accounting, ranging from broad concerns, posed at a high level of abstraction, about its welfare foundations, through basic issues touching on the design of green national income accounts, down to narrow advice on which particular activities to include and how to include them. In response, as if wanting to be able to answer such questions, has arisen a branch or application of economic analysis that might be called the "pure theory of comprehensive national income accounting." Through the core of this theory runs a common strand attempting to connect a currently observable index of comprehensive net national income or product with some appropriate but not-currently observable welfare measure of future power to consume, which typically has a "sustainability-like" flavor or undertone.

We seem presently to have created at least a partially successful body of theory.[2] However, some big pieces of the conceptual puzzle do not yet fit snugly into a fully coherent overall picture. One piece is obvious because the existing theory is typically built around a fictitious entity of "aggregate consumption," while what we really want is a general theory that incorporates heterogeneous consumption as seamlessly as it weaves in heterogeneous capital.[3] Another piece of the puzzle is a seeming disconnection between the idea that, to be observable in the first place, net national product

[1]For more background and details, set in a practical context, see National Academy Panel on Integrated Environmental and Economic Accounts (1999).

[2]For a good overview, with an extensive bibliography, see Aronsson, Johansson and Löfgren (1997); see also Asheim (2000).

[3]The failure to treat seriously heterogeneous consumption is actually a much more significant shortcoming than is commonly realized, because many of the key concepts used in the green-accounting literature, such as the various "welfare-equivalent-consumption" measures, are not even well defined for the general case. The essential problem here is that with multiple consumption goods, the distinction between "aggregate consumption" and "utility" becomes untenable at the most fundamental conceptual level.

or income must essentially take the form of a production-based index of the money value of current aggregate output, whereas the concepts that show up naturally in optimal growth theory, such as the Hamiltonian, are essentially utility-based measures.

To my mind, the biggest and most critical piece of the puzzle not yet fitting neatly into the existing body of theory concerns an answer to the following question. At least in principle, how are we actually supposed to *use* national income statistics and other currently observable market information to make rigorous welfare comparisons among different economies, or the same economy over time? Taken seriously, such inferences would appear to require the computation of inherently dynamic, wealth-like, present-discounted-utility magnitudes.[4] Is there a way to circumvent these daunting calculations, or at least to relate such wealth-like welfare-stock magnitudes to some simpler, and more readily observable, static income-like surrogates located within the national income statistician's "production boundary"?

Posed this way, the paper addresses the dynamic version of a fundamental question of welfare economics. When we have two inherently dynamic situations whose welfare we wish to compare, then in theory we should directly evaluate the two conceptually correct wealth-like present-discounted-utility magnitudes. But such welfare-stock measures seem very remote from anything out there that is actually observed, or that is even observable in principle. Meanwhile, within the "production boundary" of observable statistics we have some flow information about current prices and quantities (including price–quantity pairs observed along the current consumer-demand curves). The fundamental question is this. What relationship connects the currently observable incomes (and consumer demands) with the not-directly observable difference in dynamic welfare between the two situations we wish to compare?

From a national-income perspective the paper is aimed primarily at showing how to make rigorous dynamic welfare-stock comparisons based only on directly observable market information about current income flows. The theory treats fully disaggregated consumption as a natural formulation, and also shows implicitly how a money-valued production-based national product can be reconciled with utility-based welfare. Last but not least, when static consumer-welfare theory is correctly placed in its proper dynamic setting, the analysis actually becomes much simpler—and considerably more revealing. By embedding short-run consumer behavior within the unified theory of an optimal growth framework, the paper casts new light on some old but important controversies in consumer-surplus theory and index-number theory.

[4]This point was first made forcefully by Samuelson (1961).

It should be understood that while the motivation has thus far been framed in terms of the theory of national income accounting, the essential contribution of the paper is to provide a proper dynamic generalization of the standard static formula for the welfare evaluation of economic changes. As a contribution to the theory of consumer surplus and cost–benefit analysis, the paper therefore also has potential applications in many other areas of economics.

## II. The Setting of the Model

It is important to state clearly at the outset that the results obtained in the paper do not depend on any tricky or unorthodox assumptions. The assumptions are the usual familiar ingredients of the conventional multi-sector optimal growth model. It is possible, of course, to criticize these assumptions, or this model. For better or for worse, however, on no substantive point does the formulation here deviate from the standard representative-agent "consensus" version of intertemporal optimization with multiple consumption and investment goods.

Let the vector $\boldsymbol{C}$ represent an $m$-dimensional fully disaggregated consumption bundle. (More specifically, component $i$ of $\boldsymbol{C}(t)$ measures the *instantaneous flow of consumption services* from consuming at the rate of $C_i(t)$ units of commodity $i$ per unit time at time instant $t$, for $i = 1, 2, \ldots, m$.) The consumption vector $\boldsymbol{C}$ is conceptualized as a complete list containing everything that influences current well-being, including environmental amenities and other externalities. Consumption here would ideally include all components that influence the true "standard of living"—not just the goods we buy in stores and the government services "purchased" with our taxes, but also non-market commodities, such as those produced at home, and environmental services, such as those rendered by natural capital like forests and clean air. For the sake of developing the core theory, initial consumption $\boldsymbol{C}(0)$ is presumed to be fully observable, along with its associated $m$-vector of competitive or efficiency prices. We also presume to know or be able to observe the relevant short-run market demand function in the domain over which consumption comparisons are to be made.

For any consumption-flow time series $\{\boldsymbol{C}(t)\}$, it is supposed that it is meaningful to measure overall intertemporal well-being by the familiar expression:

$$W(\{\boldsymbol{C}(t)\}) \equiv \int_0^\infty e^{-\rho t} U(\boldsymbol{C}(t)) \mathrm{d}t, \tag{1}$$

where $U(\boldsymbol{C})$ is some given concave, non-decreasing, instantaneous utility function with continuous second derivatives defined over all non-negative consumption flows $\boldsymbol{C}$, while $\rho$ is some given rate of pure time preference. As practically every economist will attest, for better or for worse, formula (1) is the standard workhorse objective function used widely in economics as a maximand in intertemporal optimization problems. Also for what it is worth, a linear functional taking the form of $W(\{\boldsymbol{C}(t)\})$ in (1) can be given an axiomatic justification as representing the appropriate dynamic preference ordering whenever independence, stationarity, continuity and a few other seemingly reasonable (to me) conditions are postulated.[5]

The notion of "capital" used in the model is intended to be quite a bit more general than the traditionally produced means of production like equipment and structures. Most immediately, subsoil mineral resources are unquestionably considered to be forms of capital. Forms of human capital, such as education, should in principle be included, as well as the knowledge capital accumulated from R&D-like activities. Generally speaking, every possible type of capital ought to be included—to the extent that we know how to measure and evaluate the associated net investment flows. Under a broad interpretation, renewable resources in particular and environmental assets more generally should be treated as forms of capital. From this perspective, environmental quality would be viewed as a stock of capital that is depreciated by pollution and invested in by abatement.[6] The underlying ideal is for the list of capital goods to be as comprehensive as possible, subject to the practical limitation that meaningful competitive-market-like efficiency prices are available for evaluating the corresponding net investments.

Suppose that altogether there are $n$ capital goods, including stocks of natural resources and other non-orthodox forms. The stock of capital of type $i$ ($1 \leqslant i \leqslant n$) in existence at time $t$ is denoted $K_i(t)$, and its corresponding net investment flow is $I_i(t) = \dot{K}_i(t)$. The $n$-vector $\boldsymbol{K} = \{K_i\}$ denotes all capital stocks, while $\boldsymbol{I} = \{I_i\}$ stands for the corresponding $n$-vector of net investments. Note that the net investment flow of a natural capital asset like a timber reserve would be negative if the overall extraction rate exceeds the replacement rate. Generally speaking, net investment in environmental capital should be regarded as negative whenever the underlying asset is being depleted or run down more rapidly than it is replaced or built up.

Again in the spirit of focusing sharply for the sake of developing the core theory, we assume the "attainable possibilities" of the production–

[5]See e.g. Koopmans (1960). This is not the place to get embroiled in controversy, but I believe a majority of economists would agree that the critics of the utilitarian form (1) have yet to deliver a workable alternative objective function.

[6]Mäler (1991) contains a good discussion of some of the relevant issues here.

distribution system are time autonomous.[7] For theoretical purposes, we thus imagine an idealized world where the coverage of capital goods is so comprehensive, and the national accounting system is so complete, that there remain no unaccounted-for residual "atmospheric" growth factors. In the paper, *all* sources of future growth have been attributed as proper investments, which are fully "accounted for" by being valued at their proper efficiency prices and included in the national product.

Unfortunately, we do not now live in a world where national income accounting is complete, even though our theoretical models typically assume this feature. Completeness is perhaps best envisioned as a limiting case, which some real-world accounting systems approach in coverage but few attain. In our actual world we cannot measure all investments accurately, many externalities are not internalized, it is often difficult to impute market-like prices for non-market goods, there are various "atmospheric" sources of positive or negative growth, which we cannot or do not include in net national product, etc. (The omitted "atmospheric" contributions are identified primarily as a residual, which is obtained by subtracting off from actual growth the effects of all known, properly attributed, sources of growth.)

The justification traditionally given for studying the pure theory of complete accounting in a real world of incomplete accounting is that the pure theory can serve as a beacon guiding the way toward greater completeness—by suggesting what activities to include, and how best to include them, to "green up" national income into a more comprehensive aggregate reflecting more accurately what the future portends relative to the present. The motivation here has a slightly different nuance. For this paper, the pure theory of complete accounting is important because it indicates how to *use* current income-like data to make rigorous dynamic welfare comparisons—at least in principle.

In a setting where the comparisons take the form of a hypothetical cost–benefit evaluation of the welfare difference between two economic situations (i.e., attainable possibilities "with" and "without" the proposed project), the assumption of accounting completeness may not be so much of a practical constraint. Loosely speaking, in such a context it matters only that the accounting be complete for the relevant subset of goods that are changed between the two situations.

Mathematically, the national-income accounting system is complete or comprehensive if the attainable-possibilities set at any time $t$ can be

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[7] For some treatments of the time-dependent case, see Nordhaus (1995), Weitzman (1997), or Weitzman and Löfgren (1997), and the references cited therein. Time dependence introduces a host of messy complications, but a modified (and much less pretty) version of the result presented here can usually be found, contingent on some simplifying assumptions about the particular form of time dependency.

described in reduced form as a function only of the capital stocks $\boldsymbol{K}(t)$ existing at that time. Therefore, by making this assumption, we are allowed to denote the $(m+n)$-dimensional attainable-possibilities set here as $S(\boldsymbol{K})$. Then the consumption–investment pair $(\boldsymbol{C}(t), \boldsymbol{I}(t))$ is attainable at time $t$ if and only if

$$(\boldsymbol{C}(t), \boldsymbol{I}(t)) \in S(\boldsymbol{K}(t)). \tag{2}$$

As usual, the set of attainable possibilities $S(\boldsymbol{K})$ is presumed to be convex. This completes the background description required to formulate the basic problem of the paper.

## III. A Tale of Two Economies

Suppose we are interested in comparing the dynamic welfare achievable by two different economies across space or two different economic situations over time. The formulation here is intended to be quite general, in principle covering actual real-world welfare comparisons across space and over time, as well as "with project" and "without project" hypothetical benefit–cost evaluations. (Benefit–cost evaluations are done prospectively "with project" and "without project" by comparing the welfare attainable from a hypothetical "after-the-project-is-included" attainable-possibilities set with the welfare delivered by the existing "sans-project" status quo attainable-possibilities set.) In what follows, let the economy "type" or "role" be indexed by the superscript indicator variable $j$. The index value $j=1$ indicates the given *base* economy. The index value $j=2$ indicates some particular *comparison* economy. Both economies share the *same preferences*, but they may have *arbitrarily different endowments* and/or *arbitrarily different attainable possibilities*. The main contribution of this paper is to compare (1) across the two economies *relying only on currently observable market information*.

Both economies or economic situations $j=1$ and $j=2$ are postulated to exhibit dynamic behavior as if they are solutions, respectively, to a pair of optimal growth problems of the form:
maximize

$$\int_0^\infty U(\boldsymbol{C}^j(t))e^{-\rho t}\,\mathrm{d}t, \tag{3}$$

subject to the constraints

$$(\boldsymbol{C}^j(t), \boldsymbol{I}^j(t)) \in S^j(\boldsymbol{K}^j(t)), \tag{4}$$

and the differential equations

$$\dot{\boldsymbol{K}}^j(t) = \boldsymbol{I}^j(t), \tag{5}$$

and obeying the initial conditions

$$\boldsymbol{K}^j(0) = \boldsymbol{K}_0^j, \tag{6}$$

where $\boldsymbol{K}_0^j$ is the initially given capital stocks—all of the above holding for $j = 1$ and $j = 2$.

Concerning the above formulation (3)–(6), note that the "attainable-possibilities sets" (or "technologies") $S^j(\boldsymbol{K})$ in (4) and the "endowments" $\boldsymbol{K}_0^j$ in (6) are allowed to differ arbitrarily between the base economy ($j = 1$) and the comparison economy ($j = 2$), while "preferences" are identical,[8] as indicated by the shared objective (3). The goal of the paper is to infer the difference in the value of the optimized objective function (3) between the two economies from currently observable market information alone—without actually having to solve the pair of optimal growth problems (3)–(6). This might appear to be a formidable task since no additional structure is being imposed on the technologies or endowments of the two economies.

In what follows, it is assumed, purely for ease of exposition, that the two optimal solutions of (3)–(6) corresponding to $j = 1$ and $j = 2$ not only exist, but are unique. Let $\{\boldsymbol{C}^{*j}(t), \boldsymbol{I}^{*j}(t), \boldsymbol{K}^{*j}(t)\}$ represent the optimal trajectory for economy $j$. As is well known from duality theory, the solutions of (3)–(6) for both economies will generate corresponding dynamic competitive prices, denoted here by the $m$-vector time series $\{\boldsymbol{p}^{*j}(t)\}$ for consumption-goods (money) prices, and by the $n$-vector time series $\{\boldsymbol{q}^{*j}(t)\}$ for investment-goods (money) prices. Then (money) national income or product for economy $j$ at time $t$ is

$$Y^{*j}(t) \equiv \boldsymbol{p}^{*j}(t) \cdot \boldsymbol{C}^{*j}(t) + \boldsymbol{q}^{*j}(t) \cdot \boldsymbol{I}^{*j}(t). \tag{7}$$

Let $\lambda^j(t)$ represent the non-observable (to an outsider) marginal utility of money income along an optimal trajectory in economy $j$ (= 1, 2) at time $t$. The investment-goods price $n$-vector, expressed in *real* current-value utility terms for economy $j$ (= 1, 2) at time $t$ is then

$$\lambda^j(t)\boldsymbol{q}^{*j}(t), \tag{8}$$

[8]Unless preferences are postulated to be comparable in some way across any two situations, it is impossible to make rigorous general welfare comparisons. The standard static framework yields a bona fide welfare-change indicator only by assuming that, in essence, the *same* consumer faces two different price–income situations.

while the corresponding consumption-goods price $m$-vector, expressed in *real* current-value utility terms for economy $j$ (= 1, 2) at time $t$ is

$$\lambda^j(t)\mathbf{p}^{*j}(t). \tag{9}$$

In the model, $\{\lambda^j(t)\}$ may be chosen arbitrarily because it represents an extra degree of freedom that merely parameterizes the marginal utility of money income, which can be given a life of its own, related behind the scenes of the real economy to the money supply and other background, purely monetary, factors that determine the price level. What matters for the allocation of resources in the *real* economy—through the classical-dichotomy veil of arbitrary $\{\lambda^j(t)\}$, so to speak—are the *real* prices (8) and (9), which are denominated in terms of the contemporaneous value of utility serving as numeraire, and are therefore invariant to $\{\lambda^j(t)\}$. In other words, changing the exogenous specification of $\{\lambda^j(t)\}$ would merely induce inversely proportional changes in $\{\mathbf{q}^{*j}(t)\}$ and $\{\mathbf{p}^{*j}(t)\}$ without altering (8) or (9). (Typically, a paper on optimal growth theory specifies, without ceremony, all prices to be expressed in "real" utility-valued units. The reason we have to deal carefully with the issues raised by arbitrary $\{\lambda^j(t)\}$ in this paper is that the ultimate goal here is to translate observable market values, denominated in the arbitrary monetary units of the two different economies, into a statement about their real welfare difference, expressed, ultimately, in utiles.)

As is well known, the duality conditions corresponding to (3)–(6) can be given an interpretation as if describing a decentralized perfectly competitive economy in dynamic equilibrium with a single representative agent having the preference ordering (1). We emphasize this decentralized market interpretation throughout the paper, concentrating especially on how the observable short-run market demand function of the representative consumer–agent can be used to reveal critical aspects of the agent's underlying preferences.

Define the maximized current-value Hamiltonian expression

$$\tilde{\mathbf{H}}^j(\mathbf{K};\, \mathbf{q}, \lambda) \equiv \max_{(\mathbf{C},\mathbf{I})\in S^j(\mathbf{K})} \{U(\mathbf{C}) + \lambda \mathbf{q}\cdot \mathbf{I}\}. \tag{10}$$

The first type of optimality condition requires that the Hamiltonian expression (10) should actually attain its maximum everywhere along an optimal trajectory. In the representative–agent interpretation, maximizing the Hamiltonian is equivalent to the combination of a condition describing the representative *consumer*'s decentralized behavior in choosing among consumption goods $\mathbf{C}$ and aggregate net savings or investment $Z$:

$$U(C^{*j}(t)) + \lambda^j(t)q^{*j}(t) \cdot I^{*j}_{\cdot}(t) = \max_{p^{*j}(t)\cdot C + Z = Y^{*j}(t)} \{U(C) + \lambda^j(t)Z\}, \quad (11)$$

along with a condition describing the representative *producer*'s decentralized static-equilibrium behavior:

$$p^{*j}(t) \cdot C^{*j}(t) + q^{*j}(t) \cdot I^{*j}(t) = \max_{(C,I)\in S^j(K^{*j}(t))} \{p^{*j}(t) \cdot C + q^{*j}(t) \cdot I\}. \quad (12)$$

A second set of optimality conditions can be translated as describing a perfect capital/stock market in dynamic competitive equilibrium:

$$\frac{d}{dt}[\lambda^j(t)q^{*j}(t)] - \rho[\lambda^j(t)q^{*j}(t)] = -\frac{\partial \tilde{H}^j}{\partial K}\bigg|_{*j(t)}, \quad (13)$$

where the notation $|_{*j(t)}$ means evaluation along the optimal trajectory of economy $j$ at time $t$.

Finally, the third optimality condition here is the transversality requirement

$$\lim_{t\to\infty} e^{-\rho t}\lambda^j(t)q^{*j}(t) \cdot K^{*j}(t) = 0. \quad (14)$$

If conditions (13) or (14) did *not* hold, then pure positive profits could be made by intertemporal arbitrage operations, which would induce a change in (13), (14)—meaning these equations could *not* have been describing a dynamic competitive equilibrium in the first place.

Because of the underlying convexity of problem (3)–(6), the duality conditions (11)–(14) are both necessary and sufficient for an optimal solution.[9] Thus, whenever we postulate or observe here a dynamic competitive equilibrium of the form (11)–(14), it is as if we are postulating or observing the solution to an optimal growth problem of the form (3)–(6).

## IV. Current Directly Observable Market Information

From this point on, the paper deals with market-behavior observations made *only* at the present time $t = 0$. More precisely, we take on faith that the dynamic optimality-equilibrium conditions describing the coupled system (3)–(6), (10)–(14) will hold over all future time. But, aside from this general

[9]This aspect, along with the representative–agent dynamic-competitive-equilibrium interpretation of duality, is discussed in several advanced theory treatises. For an exposition whose notation is very close to this paper, see Weitzman (1970) and/or Weitzman (1973).

knowledge, everything we are permitted to know or infer at the present time $t = 0$ must be based solely on what is, at least in principle, the *current directly observable market behavior* of the representative consumer. In keeping with this restriction on knowable information, the symbol $X^{*j}(0)$ (for all pertinent variables $X$) is hereafter referenced simply by the symbol $X^j$.

Consistent with long-standing economic usage, all consumption enumerations, including the current consumption vector $\boldsymbol{C}^j$, are conceptualized as *flows of services*. At least in theory, this means the "short run" is envisioned as a period of arbitrarily short duration, with the corresponding prices of consumer durables, like owner-occupied houses (or automobiles, or refrigerators), imputed as competitive-market-equivalent rental flow rates.[10] Practically, for most commodities and for most applications, it probably suffices to think in terms of short-run consumption as occurring over a period of, say, a year, or, for the most extreme cases, maybe a month. Theoretically, however, we are eliminating time aggregation altogether by going to the limit in distinguishing among commodities consumed at each instant of time.

The current short-run consumer demand function in economy $j$ is the representative consumer–agent's response to the following counterfactual question. At what rate would you choose to consume (throughout a vanishingly short time interval starting now) if the instantaneous rental prices of consumption service flows (during this interval) were $\boldsymbol{p}$ (but the rest of the price path does not change)? The traditional way of formalizing this question is to represent consumption choices over time by an intertemporal budget constraint of the form

$$\int_0^{\delta} e^{-\rho t}\lambda^j(t)\boldsymbol{p}\cdot\boldsymbol{C}\,\mathrm{d}t + \int_{\delta}^{\infty} e^{-\rho t}\lambda^j(t)\boldsymbol{p}^{*j}(t)\cdot\boldsymbol{C}(t)\mathrm{d}t = \int_0^{\infty} e^{-\rho t}\lambda^j(t)\boldsymbol{p}^{*j}(t)\cdot\boldsymbol{C}^{*j}(t)\mathrm{d}t \quad (15)$$

for some "vanishingly small" $\delta$. The short-run consumer demand function in economy $j$ is the limiting optimized value of $\boldsymbol{C}$ $(= \boldsymbol{C}(0))$, expressed parametrically as a function of $\boldsymbol{p}$ $(= \boldsymbol{p}(0))$, which maximizes the intertemporal utility function subject to budget constraint (15), as $\rightarrow 0^+$.

An equivalent (but much neater) description of the short-run consumer demand function comes straight out of the maximum principle of optimal

[10]See e.g. Boskin, Dulberger, Gordon, Griliches, and Jorgenson (1998) for a discussion of imputed rents aimed at applications.

control theory. The act of "maximizing the Hamiltonian" translates behaviorally from (11) into having the representative consumer–agent in situation $j$ (= 1, 2) solve a decentralized problem of the reduced form:
maximize

$$U(\boldsymbol{C}) + \lambda^j Z, \tag{16}$$

subject to the budget constraint

$$\boldsymbol{p} \cdot \boldsymbol{C} + Z = Y^j, \tag{17}$$

where $\boldsymbol{p}$ stands for the counterfactual parametrically fixed short-run money consumption prices, $Y^j$ represents the given as-if-fixed national-income budget, $\lambda^j$ is the (not observable to an outsider) given as-if-fixed marginal utility of income, and $Z$ symbolizes aggregate net savings or investment, to be chosen along with $\boldsymbol{C} \geqslant \boldsymbol{0}$ by the representative consumer in $j$.

The short-run demand function is simply the optimized value of $\boldsymbol{C}$ in (16), (17) expressed parametrically as a function of $\boldsymbol{p}$. The important implication here for consumer demand theory is that *the Hamiltonian is a quasi-linear utility function*. Intuitively, this quasi-linear Hamiltonian objective form (16) is inherent in a continuous-time formulation because the consumer can fully offset, via changes in savings behavior, any and all possible income effects of a short-run price change—merely by shifting the tiniest bit of investment income across time. It is for this reason that (16), (17) with $\lambda^j$ constant describes the *same* short-run instantaneous demand function of $\boldsymbol{p}$ as would a rigorous limiting argument when $\delta \to 0^+$ in the intertemporal budget constraint (15).[11]

We write the directly observable *short-run consumer-demand function* in economy $j$ (= 1, 2) as $\boldsymbol{D}^j(\boldsymbol{p})$. The vector function $\boldsymbol{D}^j(\boldsymbol{p})$ is the implicit non-negative solution of the above problem (16), (17), which therefore satisfies, for all parametrically given hypothetical values of $\boldsymbol{p} \geqslant \boldsymbol{0}$, the standard duality conditions

$$\boldsymbol{U}'(\boldsymbol{D}^j(\boldsymbol{p})) \leqslant \lambda^j \boldsymbol{p}, \tag{18}$$

and

$$[\lambda^j \boldsymbol{p} - \boldsymbol{U}'(\boldsymbol{D}^j(\boldsymbol{p}))] \cdot \boldsymbol{D}^j(\boldsymbol{p}) = 0. \tag{19}$$

Reflecting the fact that there are no income effects as $\boldsymbol{p}$ is varied hypothetically in the short run, the representative consumer of economy $j$

[11] A proposition very close to this is proved rigorously in Bewley (1977).

regards $\lambda^j$ as fixed in (16)–(19); i.e., $\lambda^j$ is implicitly contained as a parameter in the demand function $\boldsymbol{D}^j(\boldsymbol{p})$. The ratio of marginal utilities $\lambda^2/\lambda^1$ is not directly observable to an outsider, but rather must be inferred indirectly from observing the ratio of inverse-demand functions. Here, the relevant inverse-demand ratio takes the form of an "ideal price deflator," to which we now turn.

## V. The Ideal Market-basket Price Index

The ultimate goal of the paper is to be able to make rigorous welfare comparisons between dynamic economic situations, based only on currently observable market behavior. Since observable prices are always denominated in arbitrary monetary units, the first item on this agenda is to deflate current money-price levels to a common standard. The natural common standard to use here is the price level of the base economy. Thus, the immediate task in this section is to express "the price level" of the comparison economy relative to "the price level" of the base economy. We proceed to derive a price index unified with our theory of a dynamic competitive equilibrium as follows.

Over all consumption flows $\boldsymbol{C} \geqslant \boldsymbol{0}$, define the directly observable *short-run inverse-demand function* in economy $j$ $(= 1, 2)$, denoted $\boldsymbol{P}^j(\boldsymbol{C})$, to be (any) solution of the equation:

$$\boldsymbol{D}^j(\boldsymbol{P}^j(\boldsymbol{C})) = \boldsymbol{C}. \tag{20}$$

The corresponding short-run *consumer-expenditure function* in economy $j$ $(= 1, 2)$ is

$$E^j(\boldsymbol{C}) \equiv \boldsymbol{P}^j(\boldsymbol{C}) \cdot \boldsymbol{C}. \tag{21}$$

The expenditure formula (21) describes *the expense to consumers in economy j of purchasing the fixed market basket of consumption goods* $\boldsymbol{C}$. In familiar terms, expression (21) is just exactly the short-run "revenue function" from basic economics, which a hypothetical monopolist would face in economy $j$.

The concept, now introduced here, of an *ideal "market-basket price index"* is intended as an abstraction or idealization of a *consumer price index* (CPI) or a *purchasing power parity* (PPP) deflator, which uses, instead of the *actual* existing prices $\boldsymbol{p}^j$ $(= \boldsymbol{P}^j(\boldsymbol{C}^j))$, the *imputed* market-clearing prices $\boldsymbol{P}^j(\boldsymbol{C})$ that *would be* observed in economy $j$ $(= 1, 2)$ *if* the market basket being consumed in $j$ were the quasi-fixed benchmark basket $\boldsymbol{C}$.

*Definition*. An ideal "market-basket price index" for deflating the current prices of comparison economy 2 into the current prices of base economy 1, evaluated at the fixed-benchmark market basket of consumption goods $\boldsymbol{C}$, is defined as the expenditure ratio

$$\theta(\boldsymbol{C}) \equiv \frac{E^1(\boldsymbol{C})}{E^2(\boldsymbol{C})}. \tag{22}$$

Expression (22) may be seen, perhaps, as an abstraction representing that "ideal measure" toward which the makers of a CPI or PPP-type price index implicitly strive when they try to select judiciously a "representative" market basket straddling the two economies. The intent of the index makers is to choose a benchmark basket representing, at least conceptually, consumption goods and services "*of the same quantity and quality*"[12] in both market-like situations across which the comparison-pricing imputation exercises are performed.

The index $\theta(\boldsymbol{C})$ is a *local* measure of the price level in economy 1 relative to the price level in economy 2—in the vicinity of the fixed market basket $\boldsymbol{C}$. That there may be some kind of an imputation issue involved in calculating (22) should perhaps come as no more of a surprise here than the idea that the appropriate "price" of owner-occupied housing needs to be imputed as what "would be" the observed rental price in the economy at some given level of housing consumption-flow services. The appropriate prices to use in (21) and (22) are the counterfactual, other-things-being-equal *imputed prices that would be observed in the marketplace* of each economy ($j = 1$ and $j = 2$), *if the consumption-flow basket being purchased were the benchmark* $\boldsymbol{C}$.

In practice, this imputation is usually not difficult to make for economies that are structurally very similar, like the US and Canada, or like the US from one year to the next, because the index number comparison implicit in (22) then typically reduces to attributing the existing observed market prices $\boldsymbol{p}^j$ to the given well-specified representative market basket of consumer goods $\boldsymbol{C}$. Even so, in any actual real-world comparison-pricing exercise, surprisingly many imputations are required to deal with so-called "comparison-resistant" items "of the same quantity and quality" as the particular consumption market basket chosen to be representative in the comparisons. And there is absolutely no way of escaping the central necessity to make some genuine price imputations in constructing a market-basket price index when the two comparison economies differ substantially in structure—so that, for example, one economy may have commodities in its marketplace that are not purchased at all in the marketplace of the other economy. The

[12]Summers and Heston (1991 p. 329, emphasis added). This paper contains a good practical overview, using standard terminology, and also contains some further references.

concept defined below may help to shed some analytic light on this important set of issues.

*Definition.* An ideal market-basket price index is called "benchmark invariant" if $\theta(\boldsymbol{C})$ defined by (22) is independent of the market basket of consumption goods $\boldsymbol{C}$ chosen as benchmark, so that we are permitted to write as an identity

$$\theta(\boldsymbol{C}) \equiv \theta \tag{23}$$

holding for all possible $\boldsymbol{C} \geqslant \boldsymbol{0}$.

The following result is of prime theoretical importance for this paper. It might also have some implications for index-number theory more generally.

**Lemma.** *Under the assumptions of the model, the ideal market-basket price index (22) is benchmark invariant, meaning (23) holds here as an identity.*

*Proof*: See Appendix.

As lemma (23) permits it, we henceforth *replace the symbol $\theta(\boldsymbol{C})$ by the symbol $\theta$*. Because the expenditure ratio (22) resulting from the exercise of pricing-out "the same quantity and quality" of consumption flows in both economies always turns out to be the identical constant (*independent* of the market basket $\boldsymbol{C}$ chosen as benchmark), the expenditure index $\theta$ in (23) may be conceptualized as a truly *global deflator* for converting "*the* price level" of comparison economy 2 into "*the* price level" of base economy 1.

An old-fashioned intuitive way of relating price levels across two situations is to compare the cost or expenditure required to attain "the same quantity and quality" as some quasi-fixed representative market basket of consumption. The treatment here, based on (22), is in the spirit of this formerly favored approach. A utility-theoretic approach, currently more favored, is fashioned in a somewhat different spirit, being also based on a ratio of expenditures, but with the main conceptual difference that, in (22), a quasi-fixed representative utility level $u$ takes the place of a quasi-fixed representative market basket $\boldsymbol{C}$. This paper legitimizes the old-fashioned concept of a market-basket-type price deflator, insofar as the next section shows rigorously that an index-number theory based on (22) is intrinsically unified in a desirable way with the underlying optimal-growth/dynamic-competitive-equilibrium framework.

## VI. Dynamic Welfare Comparisons and Static Consumer Surplus

We come now to the basic result of the paper. With complete accounting, *all relevant information* for making dynamic welfare comparisons is contained in market behavior that is currently observable within the domain of the relevant current comparison. Equation (24) shows that the theoretically correct but non-observable dynamic welfare index on the LHS of the equality sign is exactly the familiar, even famous, currently observable static welfare formula on the RHS.

**Theorem (Dynamic Welfare-comparison Principle).** *Under the assumptions of the model,*

$$\frac{\rho}{\lambda^1}\left[\int_0^\infty U(\boldsymbol{C}^{*2}(t))e^{-\rho t}\,\mathrm{d}t - \int_0^\infty U(\boldsymbol{C}^{*1}(t))e^{-\rho t}\,\mathrm{d}t\right]$$
$$= \theta Y^2 - Y^1 + \int_{\theta \boldsymbol{p}^2}^{\boldsymbol{p}^1} \boldsymbol{D}^1(\boldsymbol{p})\cdot \mathrm{d}\boldsymbol{p}. \quad (24)$$

*Proof*: See Appendix.

Expression (24) can be conceptualized as "compressing" or "reducing" the wealth-like dynamic welfare ordering within square brackets on the LHS into the isomorphic income-like static welfare ordering on the RHS. A way to think about the theoretical equivalence of these two welfare orderings is to envision economic situations $j = 1$ and $j = 2$ as varying over all possible technologies and initial endowments. Then situation $j = 2$ will be "better" than situation $j = 1$ by welfare criterion (1) if and only if the RHS of equation (24) is positive. It follows that, for the purpose of making comparisons, the dynamic welfare ordering induced by $W(\{\boldsymbol{C}(t)\})$ in (1) is equivalent to the static welfare ordering induced by the expression

$$\theta Y^2 - Y^1 + \int_{\theta \boldsymbol{p}^2}^{\boldsymbol{p}^1} \boldsymbol{D}^1(\boldsymbol{p})\cdot \mathrm{d}\boldsymbol{p}. \quad (25)$$

The basic result (24) can thus be interpreted as proving that expressions (1) and (25) are here just *different representations of the same underlying dynamic welfare ordering*. The currently observable static expression (25) might even be called a *sufficient statistic* for comparisons based on the standard but not currently observable dynamic welfare criterion $W(\{\boldsymbol{C}(t)\})$—because expression (25) exhausts *all* of the relevant welfare-comparison information contained in (1). Equation (24) tells us that if we can envision or imagine the evaluation of a proposed project in terms of the familiar static formula (25), then we are entitled to envision or imagine that we have effectively calculated the overall amount by which present discounted utility will change as the project impacts the economy over time.

The unobservable "normalization constant"

$$\frac{\rho}{\lambda^1}, \tag{26}$$

which appears on the LHS of (24), involves a compounding of two "conversion coefficients." The pure-time-preference coefficient $\rho$ converts the utility *wealth-stock* expression within the square brackets of (24) into an annuitized *income-flow* of stationary-equivalent or sustainable-equivalent utility. The coefficient $1/\lambda^1$ represents an arbitrary and inessential scaling constant for converting from units of utility into units of current income in the base economy.

Without any loss of generality, by taking a positive affine transformation of $U(\boldsymbol{C})$, we are free here to impose the *base-economy money-metric normalization*:

$$\lambda^1 \equiv 1. \tag{27}$$

When scaled in dollar-utile units defined by (27), equation (24) may then be expressed more neatly in the equivalent form:

$$\int_0^\infty [U(\boldsymbol{C}^{*2}(t)) - U(\boldsymbol{C}^{*1}(t))]e^{-\rho t}\,\mathrm{d}t$$
$$= \int_0^\infty \left[\theta Y^2 - Y^1 + \int_{\theta \boldsymbol{p}^2}^{\boldsymbol{p}^1} \boldsymbol{D}^1(\boldsymbol{p}) \cdot \mathrm{d}\boldsymbol{p}\right] e^{-\rho t}\,\mathrm{d}t. \tag{28}$$

Note the very simple symmetry of the isomorphism parable told by (28). The difference in sustainable-equivalent utility between the comparison economy 2 and the base economy 1 (money-metricized at base-economy prices) is exactly the answer to the following standard question of classical static welfare analysis. How much extra money must the representative base-economy consumer facing prices $\boldsymbol{p}^1$ with income $Y^1$ be paid to be equally as well off as when facing prices $\theta \boldsymbol{p}^2$ with income $\theta Y^2$? The answer here is given by the well-known expression (25), where the "substitution term"

$$\int_{\theta \boldsymbol{p}^2}^{\boldsymbol{p}^1} \boldsymbol{D}^1(\boldsymbol{p}) \cdot \mathrm{d}\boldsymbol{p} \tag{29}$$

stands for the appropriate change in classical, old-fashioned, Marshall–Dupuit consumer surplus. It is because the Hamiltonian itself is in the form of a quasi-linear utility function that the answer to the narrow static question posed above (as well as to the broader dynamic-welfare question answered by (28)) is such a simple direct function of observable short-run market demands, entirely free of messy and extraneous income-effect corrections.

It would appear to follow from the general theory justifying (28) that there is no need to apologize *at all* for using the old-fashioned consumer surplus expression (29) routinely in welfare comparisons, whenever consumption is conceptualized as a short-run flow of services embedded within a larger dynamic competitive equilibrium—and this, it could further be argued, is the half-hidden backdrop implicit in most economic settings. The distinction between "compensated" and "market" demand functions appears in this light to be the extraneous residue from a fuzzy specification of the short run, since the magnitude of any income effect depends directly, and somewhat artificially, on the length of the time period over which the act of consumption is supposed to occur or be measured. Equation (28) seems to be telling us that when the full economic dynamics of a welfare comparison are properly specified, and the demand function is genuinely short run, then the rigorously correct term accounting for substitution effects is precisely the Marshall–Dupuit consumer surplus expression (29). For this reason, the standard optimal-growth/dynamic-competitive-equilibrium framework may be viewed as opening a door on rehabilitating old-fashioned intuitive consumer surplus as a useful apparatus of some quite general respectability.

To see more clearly the exact sense here in which static welfare comparisons can be viewed as a special case of dynamic welfare comparisons, define the (static) indirect utility function:

$$V(\boldsymbol{p}, \mu; y) \equiv \max\{U(\boldsymbol{C}) + x\} \tag{30}$$

subject to

$$\boldsymbol{p} \cdot \boldsymbol{C} + \mu x = y. \tag{31}$$

As is well known, with a quasi-linear utility function, the utility difference between any two static economic situations differing in income and prices can be measured by the static welfare formula of type (25)—consisting of the change in real income plus consumer surplus.[13] An exact translation to the notation of this paper would take the form

[13] See e.g. Varian (1992, Section 10.4).

$$V(\boldsymbol{p}^2, 1/\lambda^2; Y^2) - V(\boldsymbol{p}^1, 1; Y^1) = \theta Y^2 - Y^1 + \int_{\theta \boldsymbol{p}^2}^{\boldsymbol{p}^1} \boldsymbol{D}^1(\boldsymbol{p}) \cdot \mathrm{d}\boldsymbol{p}. \tag{32}$$

It is a welfare relation of the generic form (32) that is typically cited behind the scenes to justify applying a formula of type (25)—thereby easing the way for this kind of formula to have become, historically, a veritable workhorse of static partial-equilibrium welfare analysis.[14] Although the details are omitted here, it is readily demonstrated that the static equation (32) is just a special stationary case of the more general dynamic equation (28).

The reader should be able to see, or at least intuit, that using in (32) the ideal market-basket price deflator $\theta$ defined by (22), (23) is exactly equivalent here to making the quasi-linear good $x$ in (30), (31) serve as the measuring stick for welfare comparisons. In textbook quasi-linear settings, this particular normalization is imposed routinely by economists (typically without comment) because it facilitates enormously conceptualization of the analysis. Whenever we use the "area to the left of the demand curve" as a partial-equilibrium measure of a welfare change, then we are implicitly selecting $\theta = 1$, which may be rationalized from (22) on the grounds that hypothetical expenditures for *any fixed market basket* must be the same, in the same economy, before and after the change whose value is being read off the demand curve. Any *other* price deflator than $\theta$—based, say, on choosing one of the non-quasi-linear consumption goods $C_i$ as numeraire—will generally wreck the simplicity of formula (25) by introducing messy income effects. In the pure theory of dynamic competitive equilibrium, then, income effects essentially appear as an extraneous artifact of using either the "wrong" price deflator, or the "wrong" time period, or both.

As (32) is a special static case of (28), and as (32) has proved itself to be of great practical importance in many fields of applied economic analysis, it might be hoped that its dynamic generalization (28) may also find useful applications. Result (28) shows that expressions (1) and (25) are two operationally equivalent representation forms for the same underlying dynamic preference ordering. An economist is therefore free to choose whichever representation is more convenient to work with. For most economic applications, I believe, the income-like form (25) is probably simpler, more intuitive, and more useful than the equivalent wealth-like form (1), which is unlikely to be directly observable anyway.

To summarize, a relatively straightforward, simple-minded shorthand application of static consumer-welfare theory—which involves only comparing presently observable prices and quantities along the relevant parts of the

[14] For a recent survey of applications, see Hines (1999).

short-run consumer-demand function—gives the "correct answers" to some seemingly very complicated general questions, the longhand versions of which must intrinsically involve comparing wealth-like "true indicators" of dynamic welfare. Put slightly differently, every time we perform a familiar consumer-surplus-like economic analysis of the welfare difference between two static situations, we are implicitly answering a dynamic question posed in terms of an underlying dynamic welfare comparison.

## VII. Conclusion

We have derived a kind of "dynamic welfare-comparison principle," which lets us compare dynamic welfare between situations rigorously, yet relies only on currently observable prices and quantities evaluated along the current short-run consumer-demand function within the current consumption-comparison domain. The underlying isomorphism assures us that it is "OK" to translate dynamic welfare comparisons into a simple as-if-static story told in terms of conventional, old-fashioned consumer welfare theory. The simple-minded parable gives the correct answers to complicated questions that intrinsically involve comparing wealth-like dynamic welfare measures across any two economic situations differing arbitrarily in technologies or endowments. It is to be hoped that there may be useful applications of a welfare-comparison principle having this kind of simplicity and generality.

### Appendix

*Proof of Lemma*
From (18), (19) and the definition (20), it follows that the equation

$$\boldsymbol{U}'(\boldsymbol{C}) = \lambda^j \boldsymbol{P}^j(\boldsymbol{C}) \tag{A1}$$

holds for $j = 1, j = 2$, all $\boldsymbol{C} \geqslant \mathbf{0}$.

An immediate consequence of comparing (A1) with the definitions (21), (22) is that

$$\theta(\boldsymbol{C}) = \theta \equiv \frac{\lambda^2}{\lambda^1} \tag{A2}$$

for all $\boldsymbol{C} \geqslant \mathbf{0}$. ∎

Equation (A2) represents a stronger-than-required form of the conclusion to be proved in the statement of the lemma, because $\theta$ here is not just "a constant," but actually equals $\lambda^2/\lambda^1$. The strong form (A2) is needed in the following proof of the theorem.

*Proof of Theorem*:
A basic result from Weitzman (1970, p. 15, eq. (16)), transposed to the notation of this paper, is

$$\rho \int_0^{\infty} U(\boldsymbol{C}^{*j}(t)) e^{-\rho t} \mathrm{d}t = U(\boldsymbol{C}^j) + \lambda^j \boldsymbol{q}^j \cdot \boldsymbol{I}^j. \tag{A3}$$

Taking the difference of (A3) between comparison and base economies gives

$$\rho \left[ \int_0^{\infty} U(\boldsymbol{C}^{*2}(t)) e^{-\rho t} \, \mathrm{d}t - \int_0^{\infty} U(\boldsymbol{C}^{*1}(t)) e^{-\rho t} \, \mathrm{d}t \right]$$
$$= U(\boldsymbol{C}^2) + \lambda^2 \boldsymbol{q}^2 \cdot \boldsymbol{I}^2 - U(\boldsymbol{C}^1) - \lambda^1 \boldsymbol{q}^1 \cdot \boldsymbol{I}^1. \tag{A4}$$

Now just using basic mathematical considerations arising from smooth differentiability of the function $U(\boldsymbol{C})$, we have

$$\int_{\boldsymbol{C}^1}^{\boldsymbol{C}^2} \boldsymbol{U}'(\boldsymbol{C}) \cdot \mathrm{d}\boldsymbol{C} = U(\boldsymbol{C}^2) - U(\boldsymbol{C}^1), \tag{A5}$$

where the LHS integral of (A5) is path independent because the second mixed partial derivatives of $U(\boldsymbol{C})$ are equal by the assumption of continuous second derivatives.[15]

Now (A1) implies directly that

$$\int_{\boldsymbol{C}^1}^{\boldsymbol{C}^2} \boldsymbol{U}'(\boldsymbol{C}) \cdot \mathrm{d}\boldsymbol{C} = \lambda^1 \int_{\boldsymbol{C}^1}^{\boldsymbol{C}^2} \boldsymbol{P}^1(\boldsymbol{C}) \cdot \mathrm{d}\boldsymbol{C}. \tag{A6}$$

Because $\{\boldsymbol{P}^1(\cdot)\}$ and $\boldsymbol{D}^1(\cdot)\}$ from (18), (19), (20) are inverse functions to each other, integration by parts along any continuous connecting path yields the equation

$$\int_{\boldsymbol{C}_1}^{\boldsymbol{C}^2} \boldsymbol{P}^1(\boldsymbol{C}) \cdot \mathrm{d}\boldsymbol{C} = \boldsymbol{P}^1(\boldsymbol{C}^2) \cdot \boldsymbol{C}^2 - \boldsymbol{P}^1(\boldsymbol{C}^1) \cdot \boldsymbol{C}^1 - \int_{\boldsymbol{P}^1(\boldsymbol{C}^1)}^{\boldsymbol{P}^1(\boldsymbol{C}^2)} \boldsymbol{D}^1(\boldsymbol{p}) \cdot \mathrm{d}\boldsymbol{p}. \tag{A7}$$

Selecting $\boldsymbol{C} = \boldsymbol{C}^2$ in (A1) for $j = 1$ and for $j = 2$, and then comparing the resulting expression with (A2) implies

$$\boldsymbol{P}^1(\boldsymbol{C}^2) = \theta \boldsymbol{P}^2(\boldsymbol{C}^2). \tag{A8}$$

[15] Actually, because the function $U(\boldsymbol{C})$ is concave, the assumption of differentiability is not even required here, since the singular points where the second derivatives fail to exist or are not continuous have measure zero in the relevant domain. However, the slight gain in generality of recasting the paper without any differentiability assumptions is not worth the messy and excessively mathematical notation that is thereby required. But it could be done!

Now, by the definition (20),

$$P^j(C^j) = p^j. \tag{A9}$$

Making use of (A8) and (A9), expression (A7) can be transformed into the equivalent form

$$\int_{C^1}^{C^2} P^1(C) \cdot \mathrm{d}C = \theta p^2 \cdot C^2 - p^1 \cdot C^1 - \int_{p^1}^{\theta p^2} D^1(p) \cdot \mathrm{d}p \tag{A10}$$

Next, substitute (A10) into (A6) into (A5) to yield the equation

$$U(C^2) - U(C^1) = \lambda^1 \left[ \theta p^2 \cdot C^2 - p^1 \cdot C^1 - \int_{p^1}^{\theta p^2} D^1(p) \cdot \mathrm{d}p \right]. \tag{A11}$$

Finally, substitute (A11) into the RHS of equation (A4) and use (A2) to obtain the expression

$$\rho \left[ \int_0^\infty U(C^{*2}(t)) e^{-\rho t} \, \mathrm{d}t - \int_0^\infty U(C^{*1}(t)) e^{-\rho t} \, \mathrm{d}t \right]$$

$$= \lambda^1 \left[ \theta p^2 \cdot C^2 + \theta q^2 \cdot I^2 - p^1 \cdot C^1 - q^1 \cdot I^1 - \int_{p^1}^{\theta p^2} D^1(p) \cdot \mathrm{d}p \right] \tag{A12}$$

Using (7) to abbreviate (A12) and rearranging terms, we have, at last, equation (24), which is the result desired to be proved. ■

## References


Aronsson, T., Johansson, P.-O. and Löfgren, K.-G. (1997), *Welfare Measurement, Sustainability and Green National Accounting*, Edward Elgar, Cheltenham, UK.

Asheim, G. B. (2000), Green National Accounting: Why and How?, *Environment and Development Economics 5*, 25–48.

Bewley, T. (1977), The Permanent Income Hypothesis: A Theoretical Formulation, *Journal of Economic Theory 16*, 252–292.

Boskin, M. J., Dulberger, E. R., Gordon, R. J., Griliches, Z., and Jorgenson, D.W. (1998), Consumer Prices, the Consumer Price Index, and the Cost of Living, *The Journal of Economic Perspectives 12*, 3–36.

Hines, J. R. Jr. (1999), Three Sides of Harberger Triangles, *Journal of Economic Perspectives 13*, 167–188.

Koopmans, T. C. (1960), Stationary Ordinal Utility and Impatience, *Econometrica 28*, 287–309.

Mäler, K.-G. (1991), National Accounts and Environmental Resources, *Environmental and Resource Economics 1*, 1–15.

National Academy Panel on Integrated Environmental and Economic Accounts (1999),

*Nature's Numbers: Expanding the National Economic Accounts to Include the Environment*, National Academy Press, Washington, DC.
Nordhaus, W. D. (1995), How Should We Measure Sustainable Income?, mimeo, New Haven, CT.
Samuelson, P. A. (1961), The Evaluation of "Social Income": Capital Formulation and Wealth, in F. A. Lutz and D. C. Hague (eds.), *The Theory of Capital*, Macmillan, London, UK.
Summers, R., and Heston, A. (1991), The Penn World Table (Mark 5): An Expanded Set of International Comparisons, 1950–1988, *Quarterly Journal of Economics CVI*, 327–368.
Varian, H. R. (1992), *Microeconomic Analysis*, 3rd ed. W. W. Norton, New York, NY.
Weitzman, M. L. (1970), Aggregation and Disaggregation in the Pure Theory of Capital and Growth: A New Parable, Cowles Foundation Discussion Paper no. 292, New Haven, CT.
Weitzman, M. L. (1973), Duality Theory for Infinite Horizon Convex Models, *Management Science XIX*, 783–789.
Weitzman, M. L. (1997), Sustainability and Technical Progress, *Scandinavian Journal of Economics 99*, 1–13.
Weitzman, M. L. and Löfgren, K.-G. (1997), On the Welfare Significance of Green Accounting as Taught by Parable, *Journal of Environmental Economics and Management 32*, 139–153.

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