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ITERATIVE MULTILEVEL PLANNING WITH PRODUCTION TARGETS

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Drawing up a medium term economic plan usually involves a complicated interaction between the planning ministry and representatives of the various industries, firms, or departments. Each economic agent works in his own environment with at best incomplete information about the other agents. Yet somehow the economic system as a whole is typically able to move toward an operational plan which is satisfactory even when judged by the criterion of complete information. This paper examines the properties of one particular theoretical model of economic planning in which the center transmits information via a system of quotas.

1. INTRODUCTION

BECAUSE MODERN technological processes are so intricate, it is usually expedient, in a large centrally planned economic organization, to delegate responsibility. Formal mathematical models have been constructed which verify the intuitive notion that under certain assumptions the need for completely centralized knowledge can be obviated. Convergence to overall optimality can be achieved, these models show, if an abbreviated amount of information is iteratively calculated by each economic agent and transmitted to the others in the form of a suitable index. In many theoretical procedures the prospective indices sent out by the center are prices, while those received by it are in the form of quantities. Here the reverse order will be incorporated into an algorithm of the simplex family.

Hopefully such a procedure may be useful as a computational device for dealing with large scale mathematical programming problems. While it is not really a difference of substance, here a somewhat greater emphasis will be placed on the role of this model as an abstract description of multilevel economic planning. In this context an algorithm which revolves around centrally prepared production quotas might be considered advantageous because they may be more appealing than price directives from a practical standpoint.

2. MOTIVATION

For many centrally planned organizations economic plans are prepared in accordance with the following rough format. As a result of past experience and a backlog of statistical information, the central planners possess an approximate but workable notion of the technological possibilities confronting the various individual production units of the economy. Combining this knowledge with their own planners' preferences, highly tentative sets of roughly consistent control

[1] Most of the basic ideas reported here were worked out during the summer of 1967 while I was attending the Ford Foundation sponsored Berkeley Summer Workshop on Analytic Techniques for the Comparison of Economic Systems. T. C. Koopmans, E. Malinvaud, and R. Radner were especially helpful in criticizing an early draft. At Yale this work was supported by a grant from the National Science Foundation.

figures are prepared for key economic sectors. The control figures gradually seep down to the lowest economic echelons in the form of specific production quotas.

Individual economic units will then typically propose quota changes to their immediate superiors. Ostensibly, the basis for a proposed change in a quota is its alleged infeasibility. Economic units will attempt to convince their superiors that technological considerations preclude the fulfillment of their assigned target. In the process, they will usually try to impart to the higher ups some kind of an aggregated version of the technological constraints binding them. This information indirectly serves to indicate the direction in which a new quota must move if it is to be producible ("we need at least *this* much coal to produce *that* much steel..."). Soon after they are distributed, therefore, production figures start working their way back up the planning hierarchy so that inconsistencies can be resolved. This is sometimes called "counter planning." New targets are then reassigned on the basis of the increasingly accurate picture of overall production possibilities being continually revealed to the authorities by the planning process itself, and the planning cycle begins anew.

Especially at the highest level, target reassignment can be a complicated process, involving as it does the interaction of planners' preferences with intricate and continually changing reallocation possibilities and problems of balancing materials. Eventually, when most of the quotas are neither overtight nor too slack, the plan will have converged to an operational stage and is ready to be implemented.

The principal aim of this study is to present a formal mathematical version of some aspects of the planning procedure just outlined and to examine its properties. Needless to say, a theoretical study of this sort cannot purport to reflect planning as it is practiced in any real economic organization. The aspect of reality most critically examined here, to the neglect of several others, is the learning game whereby the center iteratively comes closer and closer to knowing the relevant production possibilities as a result of the planning process itself.

3. A MODEL OF AN ECONOMY[2]

The hypothetical economy studied here deals with n distinct and homogeneous commodities, identified by the subscript i taking the values 1 to n. Production is carried out by m distinct productive units or firms, indexed by the subscript k running from 1 to m. The n commodities under consideration refer only to items centrally traded and do not include commodities specific to any firm.

The net output of commodity i produced by firm k is denoted y_{ik}. It is negative if in fact firm k consumes this good. Firms transform inputs into outputs by laws of production which involve the activities available to them. An activity will be understood here in its most general sense to be merely a designation of one of

[2] This section describes some general concepts used in the modern theory of resource allocation and is necessarily brief. Fortunately some excellent references can be consulted. The entire framework including, whenever possible, the notation, has been adopted from Malinvaud [**9**], which provides a general methodology for analyzing decentralized planning procedures and from which much of the inspiration for the present study has been derived. A comprehensive survey of the theory of resource allocation proper is contained in Koopmans [**6**].

the decision variables of the firm's production plan. For example, using a drill press to bore a particular block of steel in a specific way might be an activity. The level of the jth activity undertaken by firm k is denoted v_{jk} $(j = 1, \ldots, J_k)$. In our previous example, the activity level would be the number of such borings performed.

Production possibilities for firm k are limited by a scarcity of fixed factors (e.g., drill presses) and other restraints (e.g., nonnegativity). These are reflected by the set of inequalities

$$f_{\ell k}(v_k, y_k) \leqslant 0 \qquad (\ell = 1, \ldots, L_k). \tag{1}$$

Although we have chosen to represent them in mathematical form, the activity constraints are probably difficult to quantify and would at best be familiar only for "customary" activity levels.

The production set of all net outputs producible by firm k is denoted by Y_k and is formally defined as $Y_k \equiv \{y_k | \exists v_k \text{ with } f_{\ell k}(v_k, y_k) \leqslant 0 \text{ for } \ell = 1, \ldots, L_k\}$. It is assumed that[3] (i) Y_k is closed and convex; (ii) Y_k is bounded from above; and (iii) if $y_k \in Y_k$ and $\hat{y}_k \leqslant y_k$, then $\hat{y}_k \in Y_k$.

Final net output of commodity i is denoted by x_i. The final net output vector x, which includes both consumption and investment goods, is feasible from the viewpoint of the planners if it belongs to a set X given a priori and assumed to be closed. The ordering of social preference is represented by a welfare or utility function,[4] assumed to be continuous and defined for all $x \in X$.[5] In addition, it is assumed that if $x \in X$ and $x \geqslant \hat{x}$, then $U(x) \geqslant U(\hat{x})$.

The resource stock of commodity i initially available to the economy is denoted ω_i.

The problem confronting the central planning agency[6] is to maximize

$$U(x) \tag{2}$$

[3] Assumption (i) is familiar from resource allocation theory and Koopmans [**6**] should be consulted for an adequate discussion of its significance. We note here only that we are *not* requiring every operation performed within the firm to conform to the laws of decreasing or constant returns. We are merely presupposing that, together with possible decreasing returns in some operations, the "convexifying" effects of scarce fixed resources are strong enough to counteract the "deconvexifying" effects, if they are present, of increasing returns in other operations. Thus, the set of all vectors satisfying (1) need not be convex (if it were, we would not have to additionally postulate Y_k convex). Assumption (ii) can be thought of as being due essentially to the finiteness of fixed factors specific to firm k (like bolted-down capital). Assumption (iii) merely permits free disposal of commodities. The last two assumptions could be weakened but it would complicate the exposition without adding, in my opinion, much of economic significance.

[4] It is obviously beyond the scope of this study to examine the conditions under which collective choices can be properly quantified. In this paper it will simply be postulated that social choices are representable by a welfare function that the planners know. Other important difficulties, including the problems of intertemporal choice, aggregation, veracity, and implementation are likewise being ignored here.

[5] Interestingly enough, this algorithm does not require that the welfare function $U(\,)$ be concave or that the set X be convex. I do not understand the practical implications for economic planning of this unorthodox feature; perhaps there are none.

[6] For the problem under consideration to be interesting we can neither assume a time period so short that the possibilities for substitution are negligible nor one so long as to warrant an explicit treatment of capital formation. An intermediate term plan, say of about five years' duration, is what we have in mind. This issue is discussed by Porwit [**11**, pp. 8–9]. For many East European socialist countries the outline of Section 2 would actually be more appropriate as a description of short term planning; the intermediate term plan is often just a rough guideline and does not have the force of an operational document.

subject to

(3) $x \in X,$

(4) $y_k \in Y_k$ for $k = 1, \ldots, m,$

(5) $x \leqslant \sum_{k=1}^{m} y_k + \omega.$

The program $[x, y_1, \ldots, y_m]$ is called *feasible* if it satisfies the constraints (3), (4), (5). The program $[x^*, y_1^*, \ldots, y_m^*]$ is called *optimal* if it is feasible and if, for any other feasible program $[x, y_1, \ldots, y_m]$, $U(x^*) \geqslant U(x)$. Under the assumptions made so far, the problem (2), (3), (4), (5) will possess an optimal solution with maximum attainable utility $U^* \equiv U(x^*)$.

While the problem (2), (3), (4), (5) has been cast in a national planning setting, it should be clear that other interpretations are possible. In fact, many other important problems can be so structured. Even within the national planning framework, the concept of a firm is meant to be quite general. International trade, for example, could be accommodated by postulating two extra firms or departments. One would be in charge of exports, "producing" foreign exchange by "consuming" commodities sold abroad. The other, in charge of imports, "consumes" foreign exchange to "produce" commodities purchased from abroad. "Laws of production" for such firms would reflect supply and demand conditions on world markets.

4. THE IMPORTANT CONCEPT OF INCOMPLETE INFORMATION

Managers specialize in handling their own firm's problems and as such are likely to be ignorant of the exact situation prevailing in other firms, of society's total available resources, or of the planners' preferences among net output possibilities. Nor can it be presumed that the managers of firm k are explicitly aware of the set Y_k. It should not be forgotten that the production set or production function is an economist's concept of little or no direct relevance to managers or engineers.[7] For the purposes of this paper, the difference is more than semantic. Going from the activity constraints (1) to an efficient boundary point of Y_k involves the solution to a more or less difficult optimization problem. Under the circumstances it is hardly reasonable to suppose that even those closest to the operations of a firm know more than a small subset of efficient production points a priori. Nevertheless, in the sense that they could map out the relevant sections of Y_k if they were asked to do so in an operationally meaningful way, the managers of firm k might be said to know it implicitly.

An analogous situation prevails at the level of the central planning agency. While the central planners can be considered to know explicitly the vector of available resources ω and the set of acceptable consumption vectors X, they are not likely to be acquainted with social welfare in the same way. However, it is

[7] On this point see Dorfman, Samuelson, and Solow [**2**, Sections 6-1 (p. 130) and 8-6 (p. 201)].

assumed that, perhaps after some introspection, they can operationally choose unambiguously among various alternatives of social net output. In this sense, the planners can be thought of as implicitly possessing a utility function, even though such a function probably could not be explicitly displayed a priori.

When it comes to any aspect of the activities $\{v_k\}$ or the functions $\{f_{\ell k}(\ )\}$ specific to the firms, the center is considered to be completely ignorant. It would be futile to try to solve (2), (3), (4), (5) directly by having each firm transmit the activity constraints (1). Even if it could be done, the resulting central problem would be of such overwhelming magnitude in the number of constraints and unknowns as to be essentially unsolvable. Nor would it do to have the firms report the more abbreviated production sets $\{Y_k\}$. As we have noted, these are probably not known explicitly.

Despite their lack of precise knowledge, it would be unfair to characterize the central planners as being *completely* ignorant of the production sets. After all, they are aware of past performances, and having kept up with economic changes members of the central planning agency are more than likely to be acquainted with at least a broad picture of current possibilities. We denote by Y_k^o the planners' estimate of the production set Y_k. The elements of Y_k^o are all those production possibilities that are not patently unrealistic but whose feasibility cannot be ascertained in advance of consulting the managers of firm k.[8]

Formally, we assume that Y_k^o is closed, bounded from above[9] and that $Y_k \subseteq Y_k^o$. If, for some reason, literally nothing were known about Y_k, the planners could always choose Y_k^o by fixing arbitrarily large positive bounds on the components of y_k.

5. A DECENTRALIZED PLANNING PROCEDURE

From what has just been said, it should be obvious that a workable planning algorithm cannot impose excessive informational requirements on any single economic agent. The approach taken here views the planning procedure as a learning process whereby the center iteratively comes to understand more and more exactly the relevant parts of the production possibilities sets without ever requiring any firm to transmit the entire set.

Suppose at stage s the planners know of a closed, bounded from above production set Y_k^s such that $Y_k \subseteq Y_k^s$. At $s = 0$, Y_k^s is given; later it will become clear how a set with the required properties is recursively generated for other values of s. So far as the planners are aware, the set Y_k^s genuinely represents the technological options available to firm k. The members of the central planning agency are therefore in a position to determine what they believe to be an optimal program $[x^s, q_1^s, \ldots, q_m^s]$ by solving the following master problem. Maximize

(6) $\qquad U(x)$

subject to

(7) $\qquad x \in X,$

[8] Kornai [7, p. 416] calls Y_k^o the "set of possible programs."

[9] Once again, the assumption of boundedness is excessively strong, but is retained for convenience.

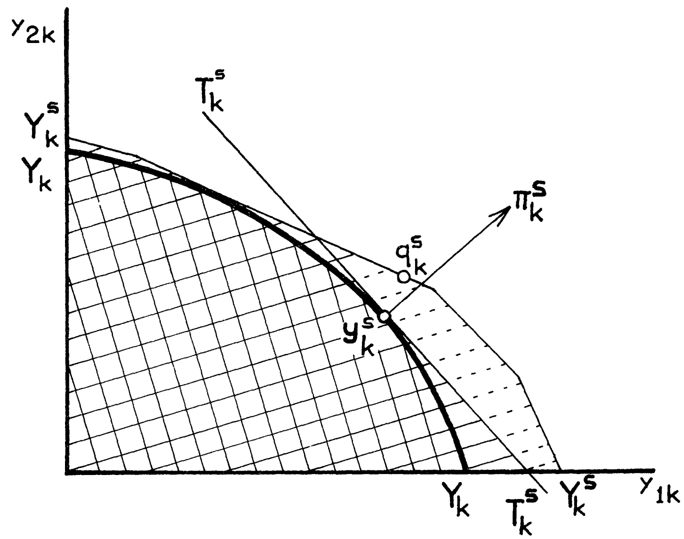
(8) $q_k \in Y_k^s$,

(9) $x \leqslant \sum_{k=1}^{m} q_k + \omega.$

Under the assumptions, this is a well defined problem with maximum utility $U^s \equiv U(x^s)$. The center now tries to impose the pseudo-optimal program $[x^s, q_1^s, \ldots, q_m^s]$ by assigning the vector q_k^s as a quota or target to firm k for $k = 1, \ldots, m$. If for each firm k the quota q_k^s is producible ($q_k^s \in Y_k$), the program $[x^s, q_1^s, \ldots, q_m^s]$ is also optimal for the center's original planning problem (2), (3), (4), (5). The planning agency thus has an easy way of identifying an optimal program when it has been attained.

If firm k cannot meet its assigned target ($q_k^s \notin Y_k$), a temporary impasse has been reached. It is now incumbent upon the managers of firm k to demonstrate a feasible alternative which in some sense is the best they can do but still fails to attain the assigned target. By educating the planners as to the true technological situation prevailing in the neighborhood of this "second best" production point, the managers can hope to induce the center to reissue a new, hopefully feasible, quota and to prevent the previous infeasible target from being reassigned.

The process can be visualized with the aid of Figure 1. Leaving in temporary abeyance its exact meaning or the question of how it is chosen, y_k^s is taken to represent a "second best" production point. One way of formalizing the notion that the managers of firm k, in order to show that q_k^s cannot be produced, impart to the planners a knowledge of the more modest production alternatives *really*



KEY: Y_k^s is the area encompassing all positively sloped lines. Y_k^{s+1} is the area encompassing all *solid* positively sloped lines. Y_k is the cross-hatched area.

FIGURE 1.—A geometric representation of the production target procedure.

available to them is to say that the managers select a hyperplane T_k^s tangent to Y_k at the point y_k^s which separates q_k^s from Y_k. Such a hyperplane is completely determined by specifying the point of tangency, y_k^s, and the normal to it at that point, π_k^s, from the following definition:

$$T_k^s \equiv \{y | \pi_k^s y = \pi_k^s y_k^s\}. \tag{10}$$

Let H_k^s stand for the closed half space defined by the tangent hyperplane T_k^s, i.e.,

$$H_k^s \equiv \{q | \pi_k^s q \leqslant \pi_k^s y_k^s\}.$$

By convexity, the planners know that Y_k must be contained within H_k^s, as well as within Y_k^s. We define Y_k^{s+1} as

$$Y_k^{s+1} \equiv Y_k^s \cap H_k^s.$$

If $q_k^s \in Y_k$, $Y_k^{s+1} \equiv Y_k^s$.

In general it will be true that

$$Y_k \subseteq Y_k^{s+1} \subseteq Y_k^s \ldots \subseteq Y_k^1 \subseteq Y_k^o,$$

and that

$$U^* \leqslant U^{s+1} \leqslant U^s \ldots \leqslant U^1 \leqslant U^o.$$

The remainder of this paper exposits a method for automatically generating meaningful y_k^s and π_k^s, called the "production target procedure."[10]

6. THE PRODUCTION TARGET PROCEDURE

Suppose that the production quota q_k cannot be produced by firm k. Let Q_k be defined as $Q_k \equiv \{y | y \leqslant q_k\}$. We say that a production point $\hat{y}_k$ is *efficient with respect to the quota* q_k, or q_k-efficient, if $\hat{y}_k \in Y_k \cap Q_k$ and if there exists a *positive* row vector p_k such that $y \in Y_k \cap Q_k$ implies $p_k y \leqslant p_k \hat{y}_k$.

Roughly speaking, q_k-efficiency can be interpreted as one way of formalizing the notion of a production combination for which the managers have gone as far as possible toward achieving their assigned but unattainable quota. It is a sort of Pareto optimality with respect to the production target. If $\hat{y}_k$ is a q_k-efficient point it has the property that, if it is to remain producible, any component of $\hat{y}_k$ strictly

[10] We say "a method" rather than "the method" because other approaches are certainly possible. Having experimented with some other methods, I can report that it would suffice to form a separating hyperplane from the optimal dual prices associated with minimizing any one of a variety of bona fide infeasibility forms or distance measures (distance, that is, from the infeasible point to points in the production set), of which the one selected for detailed study turns out to be a special case. Unfortunately, some otherwise plausible distance measures would probably not fare well as devices of administrative control. For example, it might be needlessly difficult for the manager of a firm to choose a production point which minimizes the Euclidean distance from the assigned quota simply because he has little understanding of what it means. This is why mathematical generality is abandoned at this time in favor of a particular idea which seems somewhat more plausible from an organizational viewpoint.

less than the corresponding component of q_k cannot be increased without decreasing at least one other component of $\hat{y}_k$.[11]

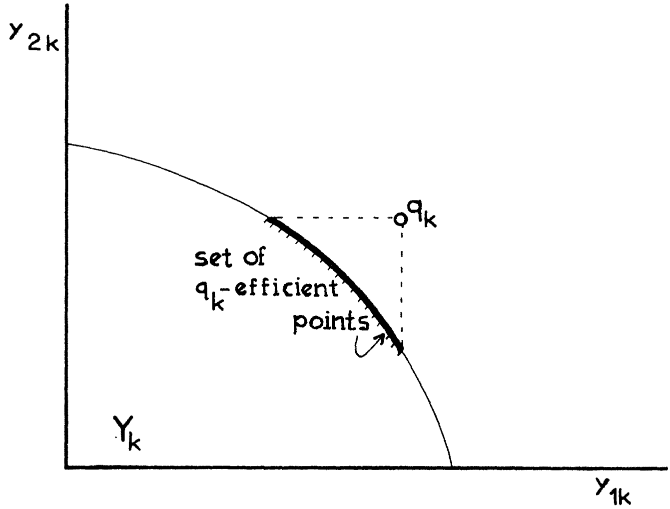


FIGURE 2.—A geometric representation of the set of q_k-efficient points.

A set of q_k-efficient points is portrayed in Figure 2. For the purposes of this paper, q_k-efficiency is an important concept because it can serve as an intuitively appealing way of automatically generating a separating hyperplane which will cause the algorithm to converge.

PROPOSITION 1. *Let $\hat{y}_k$ be any q_k-efficient point. There exists a hyperplane with normal π_k passing through y_k and possessing the following properties: (i) $\pi_k \geqslant 0$; (ii) if $q_{ik} > \hat{y}_{ik}$, then $\pi_{ik} > 0$; (iii) $y \in Y_k$ implies that $\pi_k y \leqslant \pi_k \hat{y}_k$; (iv) $\pi_k \hat{y}_k < \pi_k q_k$.*

Properties (iii) and (iv) mean that the hyperplane separates q_k from Y_k.

PROOF: For some $p_k > 0$, $\hat{y}_k$ is a solution of the problem: maximize

(11) $\quad p_k y_k$

subject to

(12) $\quad y_k \in Y_k$,

(13) $\quad y_k \leqslant q_k$.

[11] This property is also a *sufficient* condition for q_k-efficiency in the case of $Y_k \cap Q_k$ polyhedral; cf. Koopmans [5, Theorem 4.3, p. 61]. In the more general case all we can say is that sets of points obeying the two descriptions differ at most on a set of measure zero. The only difference arises from those exceptional efficient production points where the tangent hyperplane is parallel to one of the coordinate axes.

It follows that the Lagrangian expression $p_k y_k + \psi_k(q_k - y_k)$ possesses a saddle point at $y_k = \hat{y}_k$, $\psi_k = \hat{\psi}_k \geqslant 0$.[12] That is,

$$p_k y_k + \hat{\psi}_k(q_k - y_k) \leqslant p_k\hat{y}_k + \hat{\psi}_k(q_k - \hat{y}_k) \leqslant p_k\hat{y}_k + \psi_k(q_k - \hat{y}_k)$$

for all $y_k \in Y_k$, $\psi_k \geqslant 0$.

The saddle point property will only be true if $p_k \geqslant \hat{\psi}_k$, $\hat{\psi}_k(q_k - \hat{y}_k) = 0$, and $(p_k - \hat{\psi}_k)y_k \leqslant (p_k - \hat{\psi}_k)\hat{y}_k$ for all $y_k \in Y_k$. Defining $\pi_k \equiv p_k - \hat{\psi}_k$, conditions (i), (ii), and (iii) follow immediately.

Because $q_k \geqslant \hat{y}_k$, $q_k \neq \hat{y}_k$, and $p_k > 0$,

$$\pi_k q_k = (p_k - \hat{\psi}_k)q_k = p_k q_k - \hat{\psi}_k\hat{y}_k$$

is greater than $p_k\hat{y}_k - \hat{\psi}_k\hat{y}_k = \pi_k\hat{y}_k$, demonstrating (iv) and completing the proof.

Thinking of (11), (12), (13) as a mathematical programming problem, the vector $\hat{\psi}_k$ is a set of prices dual to equation (13). If the functions $f_{\ell k}()$ of equation (1) were linear (plus a constant), (11), (12), (13) would be a standard linear programming problem and $\hat{\psi}$ could be routinely obtained from the optimal simplex tableau.

An economic interpretation of $\pi_k = p_k - \hat{\psi}_k$ is facilitated by considering the closely related problem of finding y_k and *nonnegative* z_k to minimize

(14) $\quad p_k z_k$

subject to

(15) $\quad y_k \in Y_k$,

(16) $\quad y_k + z_k \geqslant q_k$.

It is not difficult to show that the vector π_k from problem (11), (12), (13) is a set of prices dual to equation (16). An interpretation of (14), (15), (16) is as follows. Suppose firm k can purchase commodities at fixed positive transfer prices p_k to help meet its assigned quota. The problem is to schedule production and arrange purchases so as to minimize the total "penalty cost" of fulfilling the target. Given this objective, π_{ik} represents the worth or marginal product to firm k of an extra unit of commodity i.

The production target procedure as it pertains to the firms can now be precisely defined. If at stage s the quota q_k^s is not producible, firm k reports back *any* q_k^s-efficient point y_k^s and a price vector π_k^s satisfying (i), (ii), (iii), (iv), whose existence is guaranteed by Proposition 1. In such situations we can, without loss of generality, normalize π_k^s so that $\Sigma_{i=1}^n \pi_{ik}^s = 1$ (at least one component of π_k^s must be positive from condition (ii)). The remainder of the production target procedure has already been formally described.

The procedure under study would not be of much interest if it did not, in some sense, move closer and closer to an optimum. While the convergence of each quota

[12] See Hurwicz [**3**, Theorem V.3.1, p. 91].

to a unique point is not to be expected under the circumstances, it is sufficient to require that each quota converges to its respective production set. Convergence of q_k^s to the production set Y_k means that if $\bar{q}_k$ is a limit point of the sequence $\{q_k^s\}$, then $\bar{q}_k \in Y_k$.

PROPOSITION 2: *Each quota of the production target procedure converges to its respective production set and* $\lim_{s \to \infty} U^s = U^*$.

PROOF: Let $\bar{q}_k$ be a limit point of the infinite sequence $\{q_k^s\}$ (at least one exists because each member of $\{q_k^s\}$ belongs to a closed bounded set). Suppose $\bar{q}_k \notin Y_k$. Then every member of a subsequence of $\{q_k^s\}$ converging to $\bar{q}_k$ must lie outside the closed set Y_k after at most a finite number of terms. For each of the infinite number of terms outside Y_k, the vectors π_k^s and y_k^s would be well defined. Since π_k^s and y_k^s also belong to bounded sets, there exists a subset of indices of s, denoted S, such that the subsequences of $\{q_k^s\}$, $\{y_k^s\}$, and $\{\pi_k^s\}$ corresponding to this subset are well defined and converge simultaneously to limits, respectively, of $\bar{q}_k$, $\bar{y}_k$, and $\bar{\pi}_k \geqslant 0$.

The production target procedure implies that for all $t \geqslant 1$ and $s, s + t \in S$,

$$\pi_k^s q_k^{s+t} \leqslant \pi_k^s y_k^s,$$

$$y_k^s \leqslant q_k^s,$$

$$y \in Y_k \quad \text{implies} \quad \pi_k^s y \leqslant \pi_k^s y_k^s.$$

It follows by passing to the limit first for $t \to \infty$, then for $s \to \infty (s, s + t \in S)$ that

$$\pi_k^s \bar{q}_k \leqslant \pi_k^s y_k^s, \tag{17}$$

$$\bar{\pi}_k \bar{q}_k \leqslant \bar{\pi}_k \bar{y}_k, \tag{18}$$

$$\bar{y}_k \leqslant \bar{q}_k, \tag{19}$$

$$y \in Y_k \quad \text{implies} \quad \bar{\pi}_k y \leqslant \bar{\pi}_k \bar{y}_k. \tag{20}$$

Since $\bar{q}_k \notin Y_k$, $\bar{q}_k \neq \bar{y}_k$. Without loss of generality suppose that $\bar{q}_{1k} > \bar{y}_{1k}$. Because $\bar{\pi}_k \geqslant 0$, it follows from (18) and (19) that

$$\bar{\pi}_{1k} = 0. \tag{21}$$

There must exist an integer N and a number $\delta > 0$ such that

$$\bar{q}_{1k} \geqslant y_{1k}^s + \delta \tag{22}$$

for all $s \geqslant N$, $s \in S$.

The condition (17) can be rewritten as

$$\pi_{1k}^s(\bar{q}_{1k} - y_{1k}^s) \leqslant \sum_{i=2}^{n} \pi_{ik}^s(y_{ik}^s - \bar{q}_{ik}). \tag{23}$$

From (22) and property (ii) of Proposition 1, $\pi_{1k}^s > 0$ for all $s \geqslant N, s \in S$. Dividing (23) by π_{1k}^s, substituting in (22) and rearranging, we have

$$
\begin{aligned}
0 < \delta &\leqslant \frac{\sum_{i=2}^{n} \pi_{ik}^s (y_{ik}^s - \bar{q}_{ik})}{\pi_{1k}^s} \\
&\leqslant \frac{\sum_{i=2}^{n} \pi_{ik}^s (y_{ik}^s - \bar{y}_{ik})}{\pi_{1k}^s} \qquad \text{(from (19))} \\
&= \frac{\sum_{i=2}^{n} (\pi_{ik}^s - \bar{\pi}_{ik})(y_{ik}^s - \bar{y}_{ik})}{\pi_{1k}^s} + \frac{\sum_{i=2}^{n} \bar{\pi}_{ik}(y_{ik}^s - \bar{y}_{ik})}{\pi_{1k}^s} \\
&\leqslant \frac{\sum_{i=2}^{n} (\pi_{ik}^s - \bar{\pi}_{ik})(y_{ik}^s - \bar{y}_{ik})}{1 - \sum_{i=2}^{n} \pi_{ik}^s} + \frac{\bar{\pi}_k (y_k^s - \bar{y}_k)}{\pi_{1k}^s} \qquad \text{(from (21)).}
\end{aligned}
$$

From (20), the second term is always nonpositive, whereas the first goes to zero in the limit as $s \to \infty, s \in S$. This forces the conclusion $\delta \leqslant 0$, contradicting $\bar{q}_{1k} > \bar{y}_{1k}$ and establishing that $\bar{q}_k \in Y_k$.

Since every limit point of $\{q_k^s\}$ must belong to Y_k and $x^s \in X$, there exists a subsequence each member of which converges to a feasible plan $(\bar{x}, \bar{q}_1, \ldots, \bar{q}_m)$. Because U^s is monotonically decreasing and $U^s \geqslant U^*$ for each s, $U(\bar{x}) = \lim_{s \to \infty} U^s \geqslant U^*$. But $\bar{x}$ is producible, implying $U(\bar{x}) \leqslant U^*$. Thus, $\lim_{s \to \infty} U^s = U^*$.

PROPOSITION 3: *If, in addition to the other assumptions previously made, the production set Y_k is assumed polyhedral, the production target procedure converges in a finite number of steps.*

The premises of this proposition would be fulfilled if the functions $f_{\ell k}()$ of equation (1) were all linear (plus a constant). In this case the firms possess what is often called a "linear programming technology."

PROOF: From Proposition 1, T_k^s as defined by equation (10) is a hyperplane tangent to Y_k. At every stage s for which the algorithm has not yet converged, there is at least one firm k such that $q_k^s \notin Y_k$. We now show that the facet $T_k^s \cap Y_k$ is different from the facet $T_k^r \cap Y_k$ for all $r < s$. Since $\pi_k^r y_k^s \leqslant \pi_k^r q_k^s$, two cases can be distinguished.

(i) $\pi_k^r y_k^s < \pi_k^r q_k^s$.

Then $\pi_k^r y_k^s < \pi_k^r q_k^s \leqslant \pi_k^r y_k^r$. Since y_k^s belongs to the set $T_k^s \cap Y_k$ but not to T_k^r, it must be that $T_k^s \cap Y_k \neq T_k^r \cap Y_k$.

(ii) $\pi_k^r y_k^s = \pi_k^r q_k^s$.

Because $\pi_k^s y_k^s < \pi_k^s q_k^s$, the following must hold simultaneously for at least one component i: $\pi_{ik}^r = 0$, $y_{ik}^s < q_{ik}^s$, $\pi_{ik}^s > 0$. Let u_i be an n-vector with the ith component positive and every other component equal to zero. By the assumption of free disposal, the vector $(y_k^r - u_i)$ belongs to $T_k^r \cap Y_k$, but not to T_k^s, because $\pi_k^s(y_k^r - u_i) < \pi_k^s y_k^s$, and a fortiori not to $T_k^s \cap Y_k$.

Since there are only a finite number of facets for each production set and every stage calls forth at least one new facet, the procedure must terminate after a finite number of stages.[13]

7. SOME CONCLUDING REMARKS ON THE PRODUCTION TARGET PROCEDURE

So far no mention has been made of how best to select a separating hyperplane. Yet some choices will undoubtedly result in quicker convergence than others. Suppose that instead of in effect allowing the firms to choose at each stage *any* positive value for p_k in equations (11), (12), (13), it is instead selected for them by the center.

A natural choice might be the dual prices associated with equation (9) of the master program. Choosing p in this manner probably does not involve any extra work on the part of the center since the dual vector to equation (9) is likely to be automatically available as a by-product of the solution to the master problem (6), (7), (8), (9). The price received by all the firms would then be identical for a given commodity, reflecting marginal conditions throughout the economy in the limit as s approaches infinity, and approximating them before the limit is reached. Such prices would presumably help guide infeasible quotas toward feasibility in a way which would do minimal damage to overall utility, and for this reason the algorithm might be expected to be efficient. From a strictly computational or algorithmic viewpoint, the firms do not have to perform a more difficult calculation since they are already, in effect, optimizing for a value of p given implicitly.

Other possibilities readily suggest themselves. In a one product firm, the center might fix inputs at the quota level (by implicitly setting the prices of purchased inputs at very high values) and ask for the maximum attainable output. The opposite case is also conceivable—fix output at the quota level (by setting its price at an arbitrarily high value) and ask for that combination of inputs which minimizes the total cost of inputs over and above the alloted quota. Or, one could envision a procedure that assigned fixed quotas for some commodities (perhaps allocatable primary resources like labor) by implicitly setting high administered prices and allowed the firms themselves to choose all other purchases by minimizing costs of fictitiously imported commodities. The common denominator of all these variants is the use, whether explicit or implicit, of a price p_k^s which is applied to excess demands over a target q_k^s in order to elicit a marginal productivity assessment π_k^s from the firms.[14]

It may be of interest to contrast the production target approach with another model of decentralized planning which has been discussed in the literature. An algorithm first proposed by Dantzig and Wolfe [**1**] which was applied to an economic planning setting by Malinvaud [**9**, Section V] is an example of a type of procedure whereby the center approximates a production set by building it up from

[13] If at stage s, $U^s < U^{s-1}$, nonactive constraints can be dropped from the master program without affecting the property of finite convergence—they will be regenerated later if they are needed.

[14] If $z_{ik}^s > 0$, then $p_{ik}^s = \pi_{ik}^s$. Thus, if the center supplies p_k^s it is not necessary to have the firms report back the marginal productivity of commodities which are "purchased" in positive amounts.

the *inside*, taking convex combinations of those feasible points which are recursively generated as part of the algorithm. The procedure presented here is dual to the Dantzig-Wolfe-Malinvaud (D-W-M) approach in several respects. Here the production set is reconstructed via tangent hyperplanes (rather than boundary points as with D-W-M) and the center becomes progressively less (rather than more) optimistic about attainable utility because the production possibilities sets revealed to it are continually being narrowed down (rather than expanded out). In the D-W-M procedure, the center announces prices and the firms respond with quantities; the reverse sequence is more nearly the case with the procedure presented here.[15]

While a polyhedral production set can be described either as the intersection of half spaces formed by tangent hyperplanes or as the convex combination of extreme points, in more than two dimensions typically far fewer tangent hyperplanes than extreme points would be required. For this reason, at least in the case of polyhedral production sets, it might be hoped that the procedure presented here would converge in fewer stages than the D-W-M approach. From a programming point of view, however, the subproblem and perhaps also the master may be more difficult to solve in the production target procedure.[16]

Although the production target algorithm has been shown to converge in the limit as the number of stages goes to infinity, any real life planning procedure must cease after a finite number of stages. In practice, the central planning agency could

[15] It is possible to construct a "primal-dual" type of planning algorithm which combines certain features of the procedure reported on in this paper with some other characteristics of the D-W-M model. For each firm the center's initial approximating set would neither be required to *contain* the true production set (as with the current procedure) nor to be *contained by* the true production set (as with the D-W-M algorithm). Instead, there might be any kind of an arbitrary relationship between the two. The method for rectifying unproducible central quotas would be exactly the same as that of the present paper, and it would also lead to a contraction of the central estimating set via the addition of a constraint hyperplane. If, on the other hand, it turned out that the center had announced a producible quota, the firms would be instructed to report back a feasible profit maximizing production point (a "better" alternative) given the dual master prices of the current iteration. Just as with the D-W-M algorithm this would result in an expansion of the center's estimating set via the inclusion of all convex combinations of the new point with the old estimating set. With each iteration the approximation sets, over which the center maximizes utility, would converge ever closer to the true production sets in the relevant planning region. Such a primal-dual algorithm is more general than either the D-W-M procedure or the one reported on in this paper, reducing to one or the other in special cases. Nevertheless, it was not presented in fuller detail because of a belief that, as with the D-W-M approach, unrealistic reliance is thrown on profit maximizing behavior with prices as a basic instrument of central control. This kind of message sequence simply does not occur in real world plan compilation. In less formal language, sub-units do not typically volunteer information on the direction of better feasible alternatives. (Indeed, this is one reason why we had no hesitation in assuming $Y_k \subseteq Y_k^o$; if the "true" physical production set is $\overline{Y}_k \nsubseteq Y_k^o$, then $Y_k \equiv \overline{Y}_k \cap Y_k^o \subseteq Y_k^o$ is all that the central planners could ever hope to learn is attainable anyway, since in our view firms are reluctant to make things more difficult for themselves by informing the center of inefficient targets.) The present "pressure" system, whereby the center distributes overoptimistic targets which the firms progressively whittle down to feasibility is believed to be a more relevant model for the institutional setting under consideration.

[16] While both share in common a rough similarity in the message sequencing—quantities from the center and marginal products from the firms—this algorithm differs significantly from that proposed by Kornai and Liptak [**8**]. Their algorithm is based on the method of fictitious play, a successive approximations approach, whereas the production target procedure is based on programming considerations not unlike those underlying the simplex method. Also, the K-L approach works only for an objective function which is separable among the firms.

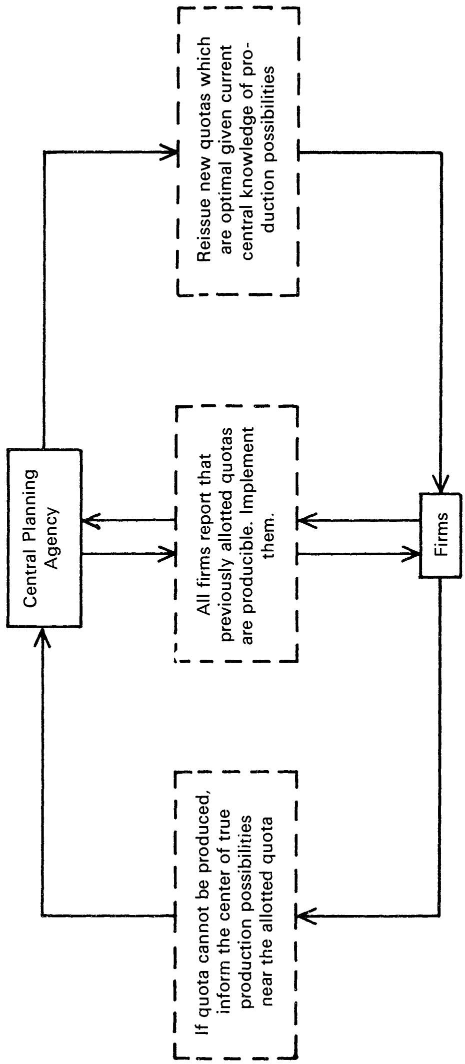


FIGURE 3.—The flow of information in the production target procedure.

probably call a halt to the proceedings whenever quotas were no longer overtight. For all practical purposes, this would undoubtedly be sufficient because in the real world the boundary of a production set is hardly an exact entity anyway. As far as the mechanics of the algorithm are concerned, the center could terminate at any stage by taking the best convex combination of previously proposed production points somewhat in the manner of the D-W-M approach. This would be the only time such a master problem would have to be solved. So long as at least one combination of previously proposed production points was feasible (which would, incidentally, also have to be the case for the proper operation of D-W-M), the utility attained as a result of solving the "termination problem" would have to increase monotonically with the number of stages. In the sense that realizable utility monotonically increases, the production target algorithm, with the termination modification just described, could be thought of as having one of the advantages usually attributed to a primal algorithm.

In an institutional setting, we would dispense with such an exact formalization as has been postulated here. The basic idea is that the firms must correct the center's exaggerated notion of their technology sets in a way that leads to convergence. Whether this is done by relaying one separating hyperplane or several, formal curvilinear surfaces or mere verbal descriptions, is not important so long as it achieves the desired effect of transmitting the true "terms on which alternatives are offered." The relevant feedback mechanism for the general case is flow-charted in Figure 3.

Finally we note that although everything in this paper has been presented in terms of but *two* levels of organization, represented symbolically by the center and the firms, generalization to three or more levels is certainly possible. While it is not examined in the present paper, such an extension contains an interesting interpretation in terms of a quota system with telescoped command levels

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