

## The linearised Hamiltonian as comprehensive NDP

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**ABSTRACT.** For guidance in determining which items should be included in comprehensive NDP (net domestic product) and how they should be included, reference is often made to the linearised Hamiltonian from an optimal growth problem. The paper gives a rigorous interpretation of this procedure in terms of a money-metric utility function linked to familiar elements of standard welfare theory. A key insight is that the *Hamiltonian itself is a quasilinear utility function*, so imposing the money-metric normalisation is simply equivalent to using Marshallian consumer surplus as the appropriate measure of welfare when there are no income effects. The twin concepts of the ‘sustainability-equivalence principle’ and the ‘dynamic welfare-comparison principle’ are explained, and it is indicated why these two principles are important for the theory of national income accounting.

### **The linearised quasilinear Hamiltonian with one consumption good**

In this first part of the paper, an attempt has been made to keep everything analytically sparse in order to focus as sharply as possible on the basic theoretical issue of how to interpret the linearised Hamiltonian from an optimal growth problem. In a later section the problem will be treated in much greater generality. To begin in familiar territory, here it is assumed that there is just one composite consumption good, denoted  $C(t)$  at time  $t$ . This first part of the paper thus follows the usual green-accounting literature by abstracting away from any problems that might be associated with constructing such an ‘ideal measure’ of aggregate consumption.<sup>1</sup>

Unlike consumption in this part of the paper, the notion of capital used throughout is from the beginning intended to be multidimensional. Furthermore, ‘capital’ is meant to be understood throughout the paper as a concept quite a bit more general than the traditionally produced means of production like equipment and structures. Most immediately, pools of

Thanks to Partha Dasgupta and Karl-Göran Mäler for getting me interested in this topic. By the end of the process I hope that more light than heat has been shed, but I apologise to them for the excessive heat I generated at the beginning.

<sup>1</sup> Nordhaus (1995), in his section entitled ‘What is Consumption?’, contains a relevant discussion of the basic conceptual issues involved. On the closely related dual issue of constructing an accurate measure of the cost of living, see Boskin *et al.* (1996).

natural resources are considered to be capital. Human capital should also be included, to the extent that we know how to measure and evaluate it. Under a very broad interpretation, environmental assets generally might be treated as a form of capital. In this context, environmental quality is viewed as a stock of capital that is depreciated by pollution and invested in by abatement.<sup>2</sup> The underlying ideal is to have the list of capital goods as comprehensive as possible, subject to the limitation that meaningful prices are available for evaluating the corresponding net investments.

Suppose that altogether there are  $n$  capital goods, including stocks of natural resources. The stock of capital of type  $j$  ( $1 \leq j \leq n$ ) in existence at time  $t$  is denoted  $K_j(t)$ , and its corresponding net investment flow is  $I_j(t) = \dot{K}_j(t)$ . The  $n$ -vector  $\mathbf{K} = \{K_j\}$  denotes all capital stocks, while  $\mathbf{I} = \{I_j\}$  stands for the corresponding  $n$ -vector of net investments. Note that the net investment flow of a natural capital asset like a timber reserve would be negative if the overall extraction rate exceeded the replacement rate. Generally speaking, net investment in environmental capital should be regarded as negative whenever the underlying asset is being depleted or run down more rapidly than it is being replaced or built up.

Again in the spirit of focusing sharply in this section on the basic issue that characterises the paper, it is assumed that the production system is time autonomous.<sup>3</sup> Let the (convex) attainable possibilities set be denoted here  $S(\mathbf{K})$ . Then the consumption–investment pair  $(C(t), \mathbf{I}(t))$  is attainable at time  $t$  if and only if

$$(C(t), \mathbf{I}(t)) \in S(\mathbf{K}(t)). \quad (1)$$

Consider a welfare functional of the familiar form

$$\int_0^{\infty} e^{-\rho t} U(C(t)) dt, \quad (2)$$

where  $U(C)$  is a concave, smoothly differentiable instantaneous utility function with positive first derivative, while  $\rho$  is the rate of pure time preference.<sup>4</sup>

As is well known, any instantaneous utility function that is a positive affine transformation of  $U(C)$  gives the same welfare ordering as (2). For this reason, the magnitude of (2) depends on how utility is scaled. Typically, nothing much is made of this observation because we do not usually attribute importance to such a magnitude. However, in this paper it will play a critical role.

Consider next the standard optimal growth problem: Maximise (2) subject to constraints (1) and  $\dot{K}(t) = \mathbf{I}(t)$ , and obeying the initial condition  $\mathbf{K}(0) = \mathbf{K}_0$ , where  $\mathbf{K}_0$  is given.

<sup>2</sup> Mäler (1991) includes a good discussion of some of the relevant issues here.

<sup>3</sup> For some treatment of the time-dependent case, see Weitzman (1997) or Weitzman and Löfgren (1997) and the further references cited there. Time dependence introduces a host of unpretty complications, but a modified (and messy) version of the result presented here will hold.

<sup>4</sup> This particular functional form can be defended on (what to me is) a reasonable axiomatic basis. See, e.g., Koopmans (1960).

In what follows, it is assumed for simplicity that an optimal solution exists and is unique. Let  $\{C^*(t)\}$  represent the optimal consumption trajectory. Let  $\Psi(0)$  represent the dual vector of shadow investment prices at time zero, relative to utility being the numeraire.

Applying the maximum principle of control theory to the above optimisation problem, the ‘current value Hamiltonian’ at present time  $t = 0$  is of the quasilinear form

$$H(0) = U(C^*(0)) + \Psi(0) \cdot \mathbf{I}^*(0) = \underset{(C, \mathbf{I}) \in S(\mathbf{K}(0))}{\text{maximum}} [U(C) + \Psi(0) \cdot \mathbf{I}]. \quad (3)$$

There are also some other important duality conditions, which we ignore here because they do not play a significant role in what follows.

What is the interpretation of  $H(0)$ ? The quasilinear expression (3) looks like what might be called ‘utility NDP’ because it is a NDP-like index where all values are being expressed relative to the utility function  $U(C)$ , which serves as numeraire. From a previous result on valuation in such time-independent systems,<sup>5</sup> we have

$$\int_0^\infty e^{-\rho t} H(0) dt = \int_0^\infty e^{-\rho t} U(C^*(t)) dt. \quad (4)$$

Condition (4) means that, under the ideal circumstances of the problem, ‘utility NDP’ is the ‘stationary equivalent’ or ‘sustainable equivalent’ or ‘annuity equivalent’ of the welfare that is actually attained. Unfortunately, this welfare interpretation may not really be so useful or operational because everything is being expressed in terms of utility rather than consumption.

As an aside relevant to empirical work, if we are granted an economy moving along an efficient growth trajectory and having a constant own rate of interest on consumption, then the situation is *as if* utility is linear, while  $\rho$  under this interpretation parameterises the constant rate of return. This is probably a decent approximation for many practical purposes because it accords with a well-known stylised fact that the real rate of interest has been essentially trendless over time throughout the past.<sup>6</sup> One could argue that the measurable entity corresponding most closely to this concept is the annual after-tax real return on capital (because it approximately defines the relevant intertemporal consumption tradeoff faced by the average citizen in deciding how much to save). As a rough approximation, a trendless round figure of 5 per cent per year might then be used for this real interest rate in the post-war period. Still, over and above any

<sup>5</sup> See Weitzman (1970) for utility (in the general multi-consumption-good case), and Weitzman (1976) for the special case of a single aggregate consumption good.

<sup>6</sup> Nordhaus (1994), in his section entitled ‘Empirical Evidence on the Return on Capital’ summarises a number of studies that are consistent with a trendless interpretation. Indeed, this is one of Kaldor’s famous ‘stylised facts’ about the growth of advanced industrial economies. (For a discussion, see Solow (1970), p. 3.) Nordhaus (1995), Jorgenson (1994), or Feldstein (1997) could each be cited to justify a lack of persistent trend fluctuating around 5 per cent per year.

practical applications, we would like to understand theoretically what happens with a curved utility function.

A seemingly possible way out of the dilemma of interpreting (3) as a real-world NDP measure is to focus instead on a 'linearised Hamiltonian' of the form

$$Y(0) = C^*(0) + Q(0) \cdot I^*(0), \quad (5)$$

where  $Q(0) \equiv \Psi(0)/U'(C^*(0))$ .

Typically, expression (5) is justified as some transformed version of a first-order approximation or linear support of (3), normalised so that consumption is the numeraire. Let us sketch further the argument that NDP is a linearised version of the current value Hamiltonian.

Under the usual smoothness assumptions, one can write as a first-order approximation for small perturbations around (3) that

$$U(C) + \Psi(0) \cdot I \sim H(0) + U'(C^*(0))[C - C^*(0)] + \Psi(0) \cdot [I - I^*(0)]. \quad (6)$$

Then it is typically noted by practitioners that the *absolute* value of the Hamiltonian has little economic meaning in itself. Rather, it is *changes* in the Hamiltonian that are meaningful. Therefore, the only economically meaningful part of the right-hand side of (6) is the expression

$$U'(C^*(0)) C + \Psi(0) \cdot I. \quad (7)$$

To turn (7) into a more familiar form where aggregate consumption is the numeraire, it is then convenient to divide through by  $U'(C^*(0))$ , resulting in the expression

$$C + Q(0) \cdot I. \quad (8)$$

Since the utility Hamiltonian, which is the left-hand side of (6), is being maximised, this implies that (8) is also being maximised. Hence, the linearised-Hamiltonian argument concludes, we might as well work with the more amenable form (8), whose maximised value is (5).

The great practical advantage of being able to work with (5), rather than (3), is that expression (5) represents ideally measured comprehensive NDP in a directly recognisable form. One can then appeal to (5) for guidance in determining what items should be included in comprehensive NDP and how they should be included. In practice, this strategy has been used creatively to shed useful light on some important basic questions of green accounting.<sup>7</sup>

In this paper I attempt to give a rigorous interpretation for the 'linearised Hamiltonian' approach, which, I think, also provides the essential connection with mainstream welfare theory.

There is a central analogy here with the way economists conceptualise the idea of consumer surplus. 'Linearising the Hamiltonian' is analogous to the intuitive argument that 'adding up the area to the left of the demand curve' is a consumer-surplus-like welfare measure of the equivalent

<sup>7</sup> Some particularly valuable examples of an imaginative use of this approach include Mäler (1991), Dasgupta and Mäler (1991), Hartwick (1994), and Dasgupta, Kriström and Mäler (1997).

money value of price changes. A rigorous argument can be made, in both cases, if the utility function is quasilinear in income, as the Hamiltonian itself already is.

We have noted, and it is well known, that any positive affine transformation of the utility function  $U(C)$  induces the same welfare ordering by (2) as  $U(C)$  itself. In other words, the instantaneous utility function  $V(C) = aU(C) + b$  gives a welfare ordering that is independent of  $a$  or  $b$  (so long as  $a$  is positive) in the welfare expression

$$\int_0^{\infty} V(C(t)) e^{-\rho t} dt. \tag{9}$$

Thus, the optimal growth problem under consideration has the same solution no matter what values of  $a$  and  $b$  are selected above to calibrate  $V(C)$  as the instantaneous utility function.

Since the parameters  $a$  and  $b$  represent two degrees of freedom that can be pinned down in any way we want, it then becomes natural to ask whether there is any useful way we might wish to calibrate or normalise or scale the utility function  $V(C)$ . I think most economists would agree it is 'natural' to scale  $V(C)$  so that utility at time zero is commensurate with consumption at time zero, in the sense that  $V'(C^*(0)) = 1$  and  $V(C^*(0)) = C^*(0)$ . This is done by choosing  $a = 1/U'(C^*(0))$  and  $b = C^*(0) - U(C^*(0))/U'(C^*(0))$  in the expression  $V(C) = aU(C) + b$ . With this natural money-metric normalisation, the underlying standard is calibrated in terms of the value of consumption at time zero. Then the expression (9) stands for a measure of summed-up overall welfare *relative to* consumption at time zero.

The transformed version of condition (4) then becomes

$$\int_0^{\infty} e^{-\rho t} [C^*(0) + Q(0) \cdot I^*(0)] dt = \int_0^{\infty} e^{-\rho t} V(C^*(t)) dt. \tag{10}$$

If we interpret 'linearising the Hamiltonian' as money metricising the instantaneous utility function, then it does two things simultaneously to equation (4). On the left-hand side, it converts the 'utility Hamiltonian' (3) into the more amenable 'goods Hamiltonian' (5), with aggregate consumption as numeraire. On the right-hand side, it calibrates present discounted utility to make it commensurate with present consumption as the standardised unit of measurement.

Expression (10) then indicates how to construct a rigorous welfare interpretation for the linearised Hamiltonian as NDP in the simple case with just one consumption good. When all units are expressed relative to initial consumption, then comprehensive NDP is the 'annuity equivalent' or 'stationary equivalent' or 'sustainable equivalent' of the utility that is actually delivered. Thus, at least in principle, NDP is a proxy for money-metricised welfare, which is a way of justifying why we turn to it for help when confronted with basic questions of green accounting.

### Generalisation and synthesis

This section begins by backing away from particular formulations, like consumption being one-dimensional or technology being independent of

time, to be in a better position later to stand back and reassemble all of the basic pieces of the puzzle together in a big-picture synthesis. For such a purpose we need to start by generalising—the better to see the forest instead of the trees. Consider, therefore, the following extremely general optimal growth problem: Maximise

$$\int_0^{\infty} U(\mathbf{C}(t)) e^{-\rho t} dt \quad (11)$$

subject to the constraints:

$$(\mathbf{C}(t), \mathbf{I}(t)) \in S(\mathbf{K}(t); t), \quad (12)$$

$$\dot{\mathbf{K}}(t) = \mathbf{I}(t), \quad (13)$$

and obeying the initial condition  $\mathbf{K}(0) = \mathbf{K}_0$ , where  $\mathbf{K}_0$  is the initially given capital stocks.

Previous work on the welfare significance of NDP, including the set-up in the previous sections of this paper, has effectively postulated a single homogeneous ‘aggregate consumption’ good, while allowing multiple capital goods. Here we are covering in full generality the case of heterogeneous consumption *and* investment.<sup>8</sup>

The vector  $\mathbf{C}$  represents an  $m$ -dimensional fully disaggregated bundle of consumption flows, which is here conceptualised as being a complete list containing everything that influences current well being, including environmental amenities and other externalities. Consumption here would ideally include all components that influence the true ‘standard of living’—not just the goods we buy in stores and the government services ‘purchased’ with our taxes, but also non-market commodities, such as those produced at home, and environmental services, such as those rendered by natural capital, like forests and clean air. For the sake of developing the core theory,  $\mathbf{C}(0)$  is fully observable and we also know the associated  $m$ -vector of market-like relative prices.

The convex set  $S(\mathbf{K}(t); t)$  appearing on the right-hand side of (12) represents for the general case the  $(m+n)$ -dimensional attainable-possibilities set at time  $t$ , while  $U(\mathbf{C}(t))$  is the relevant smoothly differentiable instantaneous utility function and  $\rho$  is the rate of pure time preference. In what follows it is assumed, purely for ease of exposition, that an optimal solution of (11)–(13) not only exists, but is unique. Let  $\{\mathbf{C}^*(t), \mathbf{I}^*(t), \mathbf{K}^*(t)\}$  represent the optimal trajectory.

As is well known, the solution of the optimisation problem (11)–(13) can be envisioned as a decentralised competitive equilibrium for an economy with a single representative agent. In such an economy, let  $\mathbf{p}(t)$  represent the competitive  $m$ -vector of consumption goods prices at time  $t$ , while  $\mathbf{q}(t)$  represents the corresponding competitive  $n$ -vector of investment goods prices.

Consider now the following four basic concepts:

<sup>8</sup> The full mathematical details are rigorously treated in my paper ‘Comprehensive NDP and the Sustainability-Equivalence Principle’, available on request. In the present paper, I am trying to apply and interpret this theoretical apparatus.

**Concept 1:** *Comprehensive NDP* (at time zero) is defined to be

$$\mathbf{p}(0) \cdot \mathbf{C}^*(0) + \mathbf{q}(0) \cdot \mathbf{I}^*(0). \tag{14}$$

**Concept 2:** *Sustainable-equivalent utility* (at time zero) is defined to be the hypothetical value  $U^*$  satisfying the condition

$$\int_0^\infty U^* e^{-\rho t} dt = \int_0^\infty U(\mathbf{C}^*(t)) e^{-\rho t} dt. \tag{15}$$

**Concept 3:** A national income accounting system is defined to be *perfectly complete* whenever all sources of future growth have been completely ‘accounted for’ by capital accumulation, meaning that there is no ‘residual’ dependence upon time and one can write

$$S(\mathbf{K}(t);t) = S(\mathbf{K}(t)). \tag{16}$$

**Concept 4:** The direct utility function  $U(\mathbf{C})$  is defined to be *money-metricised* (at time zero) whenever it has been normalised by the unique positive affine transformation that makes

$$U'(\mathbf{C}^*(0)) = \mathbf{p}(0) \tag{17}$$

and

$$U'(\mathbf{C}^*(0)) = \mathbf{p}(0) \cdot \mathbf{C}^*(0). \tag{18}$$

The above money-metric utility function has a simple, natural interpretation as representing exactly *the expenditure required at base prices  $\mathbf{p}(0)$  to attain the same utility level as is reached by consuming the bundle  $\mathbf{C}$* . Conditions (17) and (18) are equivalent here to interpreting the ‘area under the demand curve’ as the relevant measure of welfare relative to an initial position, whenever  $U(\mathbf{C})$  has been normalised so that the ‘demand curve’  $\mathbf{D}(\mathbf{p})$  is defined implicitly by the conditions  $U'(\mathbf{D}(\mathbf{p})) = \mathbf{p}$  and the ‘initial position’ is  $\mathbf{D}(\mathbf{p}(0)) = \mathbf{C}^*(0)$ . What follows next is just a more formal exposition of this basic, intuitive idea.

The key starting point here is the insight that *the Hamiltonian itself is a quasilinear utility function*—which means it is an objective function having several very important special properties. To begin with, the demand for consumption goods is independent of the level of income. For the representative agent acting as a consumer, the act of ‘maximising the Hamiltonian’ translates into solving a decentralised problem of the reduced form: Maximise

$$U(\mathbf{C}) + \lambda Z, \tag{19}$$

subject to

$$\mathbf{p} \cdot \mathbf{C} + Z = Y, \tag{20}$$

where  $Z (= \mathbf{q} \cdot \mathbf{I})$  symbolises aggregate investment, while  $Y$  represents the national-income budget, and  $\lambda$  is the representative agent’s marginal utility of income.

What is the *observed consumer-demand function* in the representative-agent competitive- equilibrium interpretation? It is the implicit solution  $\mathbf{C} = \mathbf{D}(\mathbf{p})$  of the above problem (19), (20), which therefore satisfies, for all parametrically given hypothetical values of  $\mathbf{p}$ , the condition



$$U(\mathbf{D}(\mathbf{p})) + \lambda[Y - \mathbf{p} \cdot \mathbf{D}(\mathbf{p})] = \underset{\mathbf{p} \cdot \mathbf{C} + Z = Y}{\text{maximum}} = \{U(\mathbf{C}) + \lambda Z\}. \quad (21)$$

In all that follows here, the guiding precept of the sustainability-equivalence framework is to express each entity in terms of currently observable variables on the right-hand side of the equation, with all values normalised relative to existing base-economy prices as the benchmark.

For the quasilinear objective function (19), the *representative consumer's surplus* is

$$A(\mathbf{p}(0), \mathbf{p}) \equiv \int_{\mathbf{p}}^{\mathbf{p}(0)} \mathbf{D}(\mathbf{p}) \cdot d\mathbf{p}. \quad (22)$$

The *money-metric indirect utility/expenditure function*<sup>9</sup>  $\mu(\mathbf{p}(0); \mathbf{p}, y)$  measures how much money the representative consumer would need at prices  $\mathbf{p}(0)$  to be as well off as when facing prices  $\mathbf{p}$  and having income  $y$ . With the quasilinear form (21), it is readily shown that

$$\mu(\mathbf{p}(0); \mathbf{p}, y) = y + A(\mathbf{p}(0), \mathbf{p}). \quad (23)$$

Viewed from this representative-consumer demand-theory perspective, imposing the 'money-metricising' scaling operation (17), (18) is then exactly equivalent to having the equation

$$U(\mathbf{D}(\mathbf{p})) \equiv \mu(\mathbf{p}(0); \mathbf{p}, \mathbf{p} \cdot \mathbf{D}(\mathbf{p})) \quad (24)$$

hold as an identity for all possible consumption prices  $\mathbf{p}$ . Imposing the normalisation (24) on the direct utility function is, of course, a time-tested approach to welfare evaluation, used routinely to turn the idea of consumer surplus into an important applied tool of economic analysis. With a quasilinear Hamiltonian, income effects are absent from the representative-consumer's demand function, and the welfare analysis is the same whether based upon compensating or equivalent variation, money-metric direct or indirect utility, Hicksian or Marshallian consumer surplus.

Having explored the concept of 'money-metricised utility' in the context of a competitive dynamic equilibrium, the following result<sup>10</sup> then links together all four basic concepts.

**Theorem (Sustainability-equivalence principle):** When the national income accounting system is perfectly complete and the utility function has been money-metricised, then

$$U^* = \mathbf{p}(0) \cdot \mathbf{C}^*(0) + \mathbf{q}(0) \cdot \mathbf{I}^*(0). \quad (25)$$

The sustainability-equivalence principle is theoretically and practically important for national income accounting because it synthesises the above four basic concepts by tying them together tightly in the form of one tidy conceptual package. The point is that when all units are expressed relative

<sup>9</sup> See, e.g., Varian (1992), section 10.4 and *passim*.

<sup>10</sup> A rigorous statement and proof is contained in Weitzman (1998), but is too complicated to be reproduced here. Conceptually, the sustainability-equivalence principle is a straightforward generalisation of (10) from the case of  $m = 1$  to the case of  $m > 1$  consumption goods.



to initial income, so that the utility of consumption at time zero is 'money metricised' to the money value of consumption at time zero, then comprehensive NDP becomes the 'sustainable equivalent' of that utility stream which is actually able to be delivered over time.

Of course, we do not know *directly* what is the value of sustainable-equivalent money-metricised utility, because we cannot directly calculate the integral on the right-hand side of equation (15). But we can readily imagine how the present-discounted value of a representative consumer's 'area under the demand curve' might, in principle, be evaluated. If we can conceptualise a future unfolding in the generic *form* of a dynamically competitive representative-agent equilibrium, then we are entitled also to conceptualise the economy as if maximising an objective having the *form* of (11), even when we do not directly know  $U(C)$  or  $\rho$ . And then the basic principle holds: a truly comprehensive measure of NDP reflects the representative agent's sustainable-equivalent utility and/or consumer's surplus exactly. Put slightly differently, present comprehensive NDP is probably the closest we can actually come to measuring the sustainable-equivalent welfare of the future development path we have embarked upon. In a sense, the principle is saying that most of what we care about in the future will be picked up and measured by present NDP, if only this current-income measure can be made sufficiently comprehensive.

The ultimate welfare justification for NDP from within the optimal-growth/dynamic-competitive-equilibrium paradigm is the idea that, with complete accounting, comprehensive NDP is an exact proxy for the appropriately weighted measure of sustainable-equivalent money-metricised utility that is implicit in the entire framework. With complete accounting, *present* comprehensive NDP, which is in principle observable, reflects exactly *future* welfare. For this reason, the sustainability-equivalence principle provides a powerful organising framework for conceptualising *which* items to include in comprehensive NDP, and *how* to include them.

Intuitively, an accounting system is *complete* when its coverage of capital goods is so comprehensive that *all* sources of growth have been identified as investments able to be evaluated at proper efficiency prices. Under these circumstances, as has been noted, the striking welfare characterisation of the sustainability-equivalence principle is possible. Unfortunately, however, we do not now happen to live in a world where national accounting systems are perfectly complete. Completeness is perhaps best envisioned as a limiting case, which some accounting systems approach but few attain. In our actual world there tend to exist 'atmospheric' sources of positive or negative growth, which we cannot or do not include in NDP. These omitted 'atmospheric' contributions are identified primarily as a time-dependent residual, which is obtained by subtracting off from actual growth the effects of all known, properly identified sources of growth.

One example here might be the positive investment of human capital formation or carbon sequestration by newly planted forests. Another example might be the negative investment of natural resource depletion or some forms of crowding externalities. A third example could be a stock revaluation effect from a time-autonomous change in the terms of trade faced by a small natural-resource-exporting country. In all such cases the

traditional accounting system is ‘incomplete’ because we cannot or do not include fully these changes as properly identified investments evaluated at proper efficiency prices. Typically, but not always, the reason such items are omitted from net investments is that we lack the necessary data to be able to include them.

Even if it were admitted that we live in a world whose accounting is incomplete, it would still be indispensable to understand fully the pure theory of perfectly complete accounting if for no other reason than as a base case, or reference, or starting point for a more complete analysis. More significantly, the most important practical reason for studying the pure theory of complete accounting is that it can suggest what things to include, and how best to include them, to ‘green up’ NDP—meaning to make it a more complete aggregate reflecting more accurately what the future portends relative to the present. So the pure theory is useful in a world of incomplete accounting *precisely because* it suggests the best way to make the accounting more complete.

### How to make rigorous dynamic welfare comparisons

As we have seen, the sustainability-equivalence principle informs us that current fully comprehensive NDP is a forward-looking leading indicator of future sustainable-equivalent welfare *relative to* the money value of current consumption. But the sustainability-equivalence *framework* is more general than the sustainability-equivalence principle, and it is readily applied to making powerful, general welfare comparisons across very different temporal or spatial situations.

In what follows, let the ‘base’ economy be described by the notation used in the previous section of this paper. Let the appropriate overall measure of welfare in the ‘base’ economy be the sustainable-equivalent utility expression

$$U^* \equiv \rho \int_0^{\infty} U(C^*(t)) e^{-\rho t} dt, \quad (26)$$

where the utility function  $U(C)$  is money-metricised by the normalisation (17), (18).

Consider a comparison ‘hat’ economy sharing exactly the same instantaneous utility function  $U(C)$  (naturally normalised by the same ‘base’ metric (17), (18)), and having exactly the same rate of pure time preference  $\rho$ . The comparable measure of welfare in the ‘hat’ economy is

$$\hat{U}^* \equiv \rho \int_0^{\infty} U(\hat{C}^*(t)) e^{-\rho t} dt. \quad (27)$$

The task before us now is to ascertain the difference between (27) and (26), which at first glance might appear to involve an incredibly difficult comparison of two complicated wealth-like indices. In this context it must be fully appreciated that we are allowing arbitrarily different endowments and technology between the ‘base’ and ‘hat’ economies. We assume no relationship whatsoever between  $K(0)$  and  $\hat{K}(0)$ , or between  $S(K)$  and  $\hat{S}(\hat{K})$ . In other words, only ‘*tastes*’ between the two comparison economies are presumed identical, while ‘*technology*’ and ‘*endowments*’ can vary at will and are fully allowed to be completely unrelated.

Understanding implicitly in the shorthand notation used here that all prices and quantities are being evaluated at time zero, comprehensive money NDP in the 'base' economy is

$$y \equiv \mathbf{p} \cdot \mathbf{C}^* + \mathbf{q} \cdot \mathbf{I}^*, \tag{28}$$

while in the 'hat' economy comprehensive money NDP is

$$\hat{y} \equiv \hat{\mathbf{p}} \cdot \hat{\mathbf{C}}^* + \hat{\mathbf{q}} \cdot \hat{\mathbf{I}}^*. \tag{29}$$

What scaling factor is appropriate for converting hat-economy money prices  $(\hat{\mathbf{p}}, \hat{\mathbf{q}})$  into base-economy-benchmark prices of equivalent purchasing power? While underlying technologies and endowments may be extremely different, the representative consumers in the 'base' and 'hat' economies share the same reduced form of maximising a quasilinear Hamiltonian objective function of type (19) subject to a budget constraint of type (20). Therefore, *hat-economy money prices  $\hat{\mathbf{p}}$  must be strictly proportional to the base-economy inverse-demand vector  $\mathbf{D}^{-1}(\mathbf{C}^*)$* . Without loss of generality, the scalar constant of proportionality may be normalised to unity, the economic meaning of which is that *the implicit price deflator for the 'hat' economy reflects 'true' purchasing-power parity in terms of the benchmark 'base' economy*. Thus, we are assuming here that the absolute scalar level of hat-economy money prices has been ideally deflated to conform with the correspondingly observed base-economy real consumer demands, so that

$$\mathbf{D}(\hat{\mathbf{p}}) = \hat{\mathbf{C}}^*. \tag{30}$$

The following proposition then links the dynamic sustainability-equivalence methodology strongly to the traditional methodology of classical static welfare analysis.

**Theorem (Dynamic Welfare-Comparison Principle):** Within the framework of the paper, the following isomorphism holds between dynamic and static welfare comparisons

$$\hat{U}^* - U^* = \hat{y} - y + A(\mathbf{p}, \hat{\mathbf{p}}). \tag{31}$$

Note the astonishingly simple form of the isomorphism parable being told by (31). The difference in sustainable-equivalent utility between any two comparison economies is just exactly the answer to the following standard question of classical static welfare analysis. 'By how much extra money must a consumer facing prices  $\mathbf{p}$  with income  $y$  be compensated to be equally as well off as when facing prices  $\hat{\mathbf{p}}$  with income  $\hat{y}$ ?' Because the Hamiltonian is a quasilinear objective function, (31) can be restated in an *equivalent-variation* form, independent of how  $(\hat{\mathbf{p}}, \hat{y})$  is scaled

$$\hat{U}^* - U^* = M(\mathbf{p}; \hat{\mathbf{p}}, \hat{y}) - y, \tag{32}$$

or, alternatively, in a *compensating-variation* form, which is dependent on scaling  $(\hat{\mathbf{p}}, \hat{y})$  by (30)

$$\hat{U}^* - U^* = \hat{y} - M(\hat{\mathbf{p}}; \mathbf{p}, y), \tag{33}$$

It should be appreciated that the dynamic welfare-comparison principle is an extremely powerful result because a seemingly very complicated welfare expression on the left-hand side of (31), which involves comparing

the differences over an indefinite future of wealth-like present discounted utilities (arising from arbitrarily different technologies and endowments), reduces to the familiar user-friendly static expression on the right-hand side of (31), which involves only comparing presently observable prices and quantities along the existing demand function. Thus, a relatively straightforward, simple-minded shorthand application of static consumer-welfare theory gives the 'correct answers' to some incredibly complicated questions, the longhand version of which must intrinsically involve comparing wealth-like 'true indicators' of dynamic welfare.

To see the interplay of these various issues isolated and expressed sharply, consider the very simple example of a pure-Hotelling-world economy. The only capital good is oil, which is also the only consumption good. The initial stock of oil is  $K_0$  and extraction costs are zero. There is no technological progress or any other form of time dependence, so that the national accounting system is perfectly complete.

The relevant optimal control problem is the following: *Maximise (2) subject to the constraints  $C(t)+I(t)=0$  and  $\dot{K}(t)=I(t)$ , with  $K(t)\geq 0$ , and obeying the initial condition  $K(0)=K_0$ .*

A little reflection reveals that comprehensive NDP must be zero in this pure-Hotelling-world economy. Applying the sustainability-equivalence principle and integrating by parts yields

$$p(0) C^*(0) = - \int_0^{\infty} p(t) \dot{C}^*(t) e^{-\rho t} dt. \quad (34)$$

In words, equation (34) can be translated into the following statement. The fact that comprehensive NDP is zero for the simple Hotelling economy here is a red flag, warning us now that a very serious decline of future consumption, towards zero, is underway, which over time will quantitatively undo in welfare terms the current value of consumption itself. The sustainability-equivalence principle tells us in this Hotelling economy that the present discounted loss of consumer surplus over time will exactly offset today's level of consumption.

Of course, this does *not* mean that two Hotelling economies otherwise identical except for differing stocks of oil are equally well off just because comprehensive NDP is identically zero in both cases. If Hotelling-twin economy #1 has reserves of  $K_1$ , while Hotelling-twin economy #2 has reserves of  $K_2$ , where  $K_2 > K_1$ , then Hotelling-twin economies #1 and #2 face equally devastating declines in their consumer-surplus loss of consumption valued *relative to their current consumption*. But since  $C_2^*(0) > C_1^*(0)$ , Hotelling-twin economy #2 is in a better position *because it is starting from a higher level, and lower price, of current consumption*.

In order for it to make sense as a forward-looking welfare measure of *what the future portends relative to the present*, the sustainability-equivalence principle requires a continuous recalibration over time so that the reference point is always advanced to current consumption. This calibration operation is the dynamic analogue of choosing base-period weights for a series of chain-linked welfare comparisons, when the base period itself is changing.

If we want to make a true welfare comparison of Hotelling-twin economies #1 and #2, we should set  $y = \hat{y} = 0$  in equation (31), and the dynamic welfare-comparison principle then reduces to

$$\hat{U}^* - U^* = \int_{p_2^{(0)}}^{p_1^{(0)}} D(p) dp, \quad (35)$$

where the consumption-demand function  $D(p)$  is the implicit solution of  $U'(D(p)) = p$ .

### Summary

Using a 'sustainability-equivalence framework', this paper has explained how there is an important connection between the following five core concepts: comprehensive NDP, sustainable-equivalent utility, the completeness of an accounting system, money-metricised utility, dynamic welfare comparisons. One underlying connection is the 'sustainability-equivalence principle', a result which states that comprehensive NDP represents sustainable-equivalent utility whenever, on the one side, the accounting system is perfectly complete, and, on the other side, the utility function has been money metricised to be comparable with national income. The required money-metric normalisation involves exactly the same operations as what is required in static welfare theory to interpret consumer surplus as a base-dollar-denominated true measure of welfare.

The sustainability-equivalence principle is of core importance in the theory of national income accounting because it identifies a strong connection between a comprehensive production-based index of present NDP and a future-oriented sustainability-like measure of overall welfare relative to present consumption. This principle provides a rigorously correct way to understand the heuristic shortcut of 'linearising the Hamiltonian' as representing an intuitive measure of comprehensive NDP—just like using quasilinear utility and a money-metric indirect utility function is a rigorously correct way to understand the heuristic shortcut of 'adding up the area to the left of the demand curve' as an intuitive measure of the welfare effects of a price change.

The sustainability-equivalence principle is a fundamental organising principle for conceptualising the construction of comprehensive national-income accounts. The underlying conceptual linkages, which hold for an ideal world of perfectly complete accounting, are useful as a guide in the real world of incomplete accounting precisely because they indicate *which* items should be included in NDP, and *how* they should be included, if the national accounting system is to be moved in the direction of being a more complete measure of what the future portends.

A second basic result of the sustainability-equivalence framework is the 'dynamic welfare-comparison principle', which lets us compare welfare situations rigorously, yet relying only on currently observable prices and quantities along the present demand function. This isomorphism assures us that it is 'OK' to translate dynamic welfare comparisons into a simple as-if static story told in terms of conventional consumer-welfare theory. The simple-minded story gives the correct answers to complicated questions that intrinsically involve comparing wealth-like dynamic welfare measures across any two economies differing arbitrarily in technologies or endowments.

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