A MATHEMATICAL MODEL OF ENCLOSURES

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Introduction

Enclosing of land in England altered economic and social life in a fundamental way. The transition from communal to private regulation of property has stirred up a whole series of controversial issues and posed a number of important questions for the economic historian.

Did enclosures increase economic efficiency and raise national income? At the same time did they contribute to immiserization of the working population? When land was enclosed, did it result in less labor intensive production and cause depopulation? Were rents simultaneously pushed up? Did enclosures stimulate the cultivation of new lands, the growth of industry, the transfer of labor out of agriculture?

Conspicuously absent from the literature is a coherent framework for analyzing the economic effects of enclosures. The explicit model developed in this paper will provide such a framework. It will allow us to give definite affirmative answers to each of the above questions.

It is obvious that no model can describe completely the nature of so complex an historical phenomenon as the enclosure movement. Such descriptive analysis is the task of the historian. Our intention is to focus sharply on those basic economic features of enclosures which underlie the whole movement and for this purpose the model building approach is the appropriate one.

The model

We treat the countryside as if it is split up into n non-over lapping localities or villages. The term "village" is used in an extended sense because it really refers not to a collection of dwellings but rather to the land area

under control of its inhabitants. A village (with no inhabitants) is imputed to be located even in unsettled areas.

For analytical convenience, the land of any given village is considered to be of uniform quality, although it may differ from one village to another. Alternatively, we can think of the land within a village as being non-homogeneous but distributed in such a way that every villager's holdings are divided in the *same* fixed proportions between lands of varying quality. So long as each man allocates his time between his holdings to maximize his total output and there is relative income equality, for our purposes the results will be the same as if village land were of uniform quality. We also treat labor as if it is homogeneous and uniform. This kind of an assumption is needed to even begin to talk about whether laborers as a whole are made better or worse off by different institutional arrangements.

The index i running from 1 to n will stand for village i. Let x_i denote the number of people working village i. This results in total agricultural product y_i given by the production function

$$y_i = f_i(x_i).$$

It is important to be clear about just what is meant in the present context by the above functional relation. The production function $f_i(x_i)$ gives the maximum value of aggregate agricultural output in village i as a function of the number of people working in that village. When x_i is large, aggregate output is maximized by economizing on land and producing relatively labor intensively. In the range where x_i is small, the greatest value of output would be realized by using land-intensive, labor-saving techniques. Naturally the appropriate price-weights to use for aggregating outputs might shift over time, but for any given moment the interpretation of a village production function should be clear. In what follows we assume that as a first approximation village production functions are stable over time.

A typical village production function is depicted in Figure 1.

Production functions will normally differ from village to village due to varying conditions of soil fertility, climate, drainage, etc. The following three features are taken as standard for any village.

(1) $f_i(0) = 0$ (No output without labor).

(2) If $x_i < x_i'$, then $f_i(x_i) \leq f_i(x_i')$ (The more labor, the more output).

(3) If $x_i < x_i'$, then $\dfrac{f_i(x_i)}{x_i} > \dfrac{f_i(x_i')}{x_i'}$ (The more labor, the less output per laborer).

Purely for expository convenience, we assume that the derivative $f_i'(x)$ exists for all $x \geq 0$ (at $x = 0$ existence of the right hand derivative suffices).

To get a feeling for what the allocation of resources is like when there is communal ownership and for how it differs under private ownership, we at first co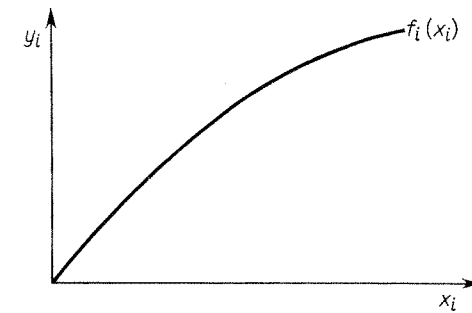mpare the polar situation where all property is strictly communally regulated with the opposite extreme where it is all private. These

Fig. 1.

are unrealistic pure cases but they will serve to focus attention on the essential economic contrasts between communal and private property and to indicate what happens when an economic system makes the transition from one to the other. Later we consider more complicated "mixed" systems, but the results will be similar.

Communal ownership is one of the most ancient forms of land management. It is the predominant system for almost all primitive hunting, fishing, trapping and gathering societies and is common to many primitive agricultural communities. We start by considering the pure economic theory of such an arrangement.

The essence of the communal ownership system in its pure form is that society as a whole denies to any individual or group the prerogative to block the usage of commonly owned property. In the terminology of our model, anyone willing to work an equal amount as the other villagers can enter a village and is entitled to an equal share of the output.

What will be the distribution of labor on purely communally held land? Suppose that the prevailing return per man throughout the economy is $\tilde{w}$, with agricultural output as numeraire. Men will enter the i-th village if the average product in that village is greater than $\tilde{w}$ and leave it if the average product is less than $\tilde{w}$. In equilibrium, $\tilde{x}_i$ men will settle in village i, where

$$\frac{f_i(\tilde{x}_i)}{\tilde{x}_i} = \tilde{w}.$$

This point is shown in Figure 2. If $f_i'(0) \leq \tilde{w}$, no one will be induced to enter village i because he can fare better elsewhere and the land of village i, which is of inferior quality, will not be cultivated. Such a situation is depicted in Figure 3.

If people are mobile they will gravitate to that village offering them the most product per person. An equilibrium is reached only when the average
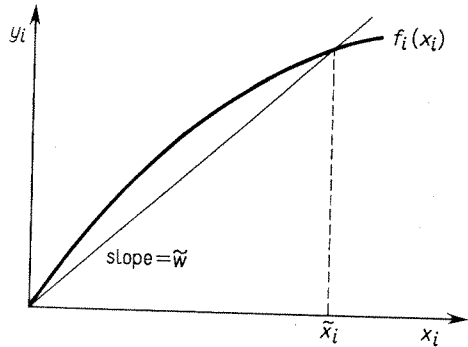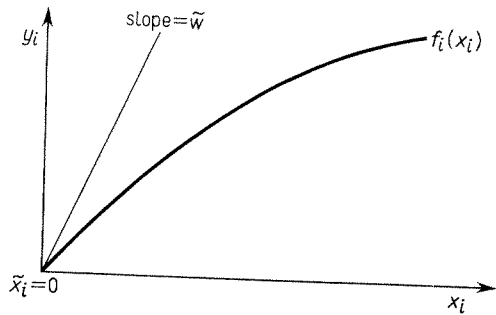

Fig. 2.

Fig. .3

product per person is equalized on all lands in use. In this case, there is no incentive for further movement. The equilibrium is stable because once out of it, forces are set in motion (via resettlement) which bring the system back to it.

The allocation system described above is denoted C (for communal) and the equilibrium values of variables in it are capped by a tilde.

Let there be a total of L laborers. The equilibrium values of $\{\tilde{x}_i\}$ and $\tilde{w}$ are determined as solutions to the following equations:

$$\tilde{x}_i > 0: \frac{f_i(\tilde{x}_i)}{\tilde{x}_i} = \tilde{w}, \tag{4}$$

$(C) \qquad \tilde{x}_i = 0: f_i'(0) \leq \tilde{w}, \tag{5}$

$$\sum_1^n \tilde{x}_i = L. \tag{6}$$

The opposite extreme to C is the case of pure private property embedded in a market economy. This kind of an economic system sanctions the property rights of a certain class or group who owns the land and determines its use. The services of land and labor can be bought and sold freely in the market, an operation which had no meaning under C. With competition, labor can be hired at a common wage, and income seeking land owners will hire that amount which maximizes their profits. When $\hat{w}$ is the prevailing wage rate in terms of output as numeraire, $\hat{x}_i$ workers will be hired in village i, where

$$f_i(\hat{x}_i) - \hat{w}\hat{x}_i = \max_{x \geq 0} f_i(x) - \hat{w}x.$$

If $\hat{x}_i > 0$, then

$$f_i'(\hat{x}_i) = \hat{w}.$$

The marginal product of the last worker hired equals the wage rate. In Figure 4 this point is illustrated.
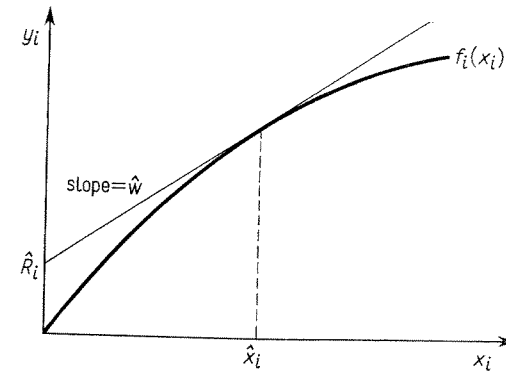

Fig. 4.

The rental income earned on land i, which will also be the competitive value of its service in exchange, is

$$\hat{R}_i \equiv f_i(\hat{x}_i) - \hat{w}\hat{x}_i.$$

A lower rent would not be income maximizing, whereas a higher rent would

find no takers. If $f_i'(0) \leqslant \hat{w}$, it is not worthwile to cultivate the land, no laborers will be hired, and the rental is zero. Such a situation is depicted in Figure 3 with $\hat{w}$ replacing $\tilde{w}$.

Note that formally it is the same thing to a profit maximizing land owner whether he hires the optimal number of laborers at the going wage rate or charges a competitive rent for the use of his land and allows laborers to allocate themselves after paying the rent. With the one case we think of land as hiring labor, with the other of labor hiring land. In theory the resulting allocation of resources is the same with either case.

Our ultimate aim is to compare the pure private property market system denoted P with the pure communal property system C previously described. In order to be able to make a meaningful comparison, we assume that the physical basis of the economy—the production functions $f_i(x_i)$ and the total number of laborers L—is the same in both systems.

To close our model of this system we have to specify what the landlords do with the rental surplus they obtain. For analytical simplicity we assume that all rental income is spent on non-agricultural "S-goods" which are manufactures, construction and services financed out of the surplus. To a first approximation, S-goods are produced by labor alone, without the help of fixed capital or the use of agricultural commodities as intermediate materials.

The assumption that landlords buy only non-agricultural goods while workers just consume agricultural produce is an analytically inessential expository simplification which captures the essence of the situation. Every result we are going to obtain could be gotten with the much weaker condition that the workers' propensity to consume agricultural goods is at least as high as the rentiers'. Adopting the weaker assumption would just complicate the exposition and proofs without otherwise adding anything of substance to the model.

Let the variable $\hat{x}_s$ denote the total amount of S-goods produced. An S-good is measured by its direct and indirect labor content. With competition, the price of a unit of S-goods would have to be $\hat{w}$, the prevailing wage rate. The total amount of S-goods that could be purchased with rental income $\sum_1^n \hat{R}_i$ would be $\sum_1^n \hat{R}_i/\hat{w}$. Exactly as many workers (Quesnay's "classe sterile") will be drawn out of agricultural production to satisfy this demand.

The P (private property) equilibrium system we have been describing,

whose variables are capped with a circumflex, satisfies the following three equations:

$$\hat{R}_i \equiv f_i(\hat{x}_i) - \hat{w}\hat{x}_i = \max_{x \geq 0} f_i(x) - \hat{w}x, \tag{7}$$

$$(P) \qquad \sum_1^n \hat{R}_i = \hat{w}\hat{x}_s, \tag{8}$$

$$\sum_1^n \hat{x}_i + \hat{x}_s = L. \tag{9}$$

In the long course of historical development, economic societies can be viewed as moving in a general way from C to P (and then beyond when capital starts to be accumulated). Suppose the transformation from C to P took place instantaneously. What kinds of differences would we expect to find between the C and P economies? Of course such a transformation did *not* take place quickly, but this sharp way of posing the question will highlight such differences as there may be.

The most immediately striking difference between the two systems is that in the P system there is an agricultural surplus of magnitude $\sum_1^n \hat{R}_i$ which is extracted by the landowners. This surplus manifests itself by causing a net transfer of workers out of agriculture to satisfy the demand for non-agricultural goods which it creates. In C, on the other hand, villagers themselves consume everything that they produce. There are no surplus activities (like the production of S-goods), there is no reason to transfer workers out of agriculture as there is in P, and there is no potential for accumulation. The accumulation of capital, largely financed at first out of rental income, eventually carries the P system forward into a new stage of development where the prime mover becomes the accumulation of capital which raises labor productivity wherever it is employed. But this comes later, after the transition from C to P which is the subject of our present analysis.

A difference between the two systems which is not so apparent is that P is efficient whereas C is not. The peasants in C are inefficiently distributed on the land. By reshuffling the existing amount of labor, greater total output could be achieved.

This phenomenon can be illustrated by the following numerical example. Suppose there are two villages, A and B, and a total of 5 laborers. The production function for village a is

$$v_a = 12x_a - 2x_a^2,$$

and for village b it is

$$y_b = 7x_b - \tfrac{1}{2}x_b^2.$$

Production in village a is more productive for small numbers of men, but the production function runs into diminishing returns more rapidly.

For C, the distribution of men on land obeys the equations

$$\frac{12x_a - 2x_a^2}{x_a} = \frac{7x_b - \tfrac{1}{2}x_b^2}{x_b},$$

$$x_a + x_b = 5,$$

which has the solution $x_a = 3$, $x_b = 2$.

Each man gets 6 units of output. But this is inefficient. To maximize total output it is the *marginal* product of labor that should be equalized among different lands, not its *average* product. In the example above, the marginal product of labor in village A is lower than it is in village B.

The distribution of the five laborers which would maximize total output equalizes marginal products and therefore obeys the equations

$$12 - 4x_a = 7 - x_b,$$

$$x_a + x_b = 5.$$

This has the solution $x_a = 2$, $x_b = 3$. Output per man is $(16 + 16\tfrac{1}{2})/5 = 6\tfrac{1}{2}$, higher than that obtainable under C.

The preceding example could easily be generalized. Under average product equalizing, the better properties are always overworked and the maximum output is not realized. It would be better to transfer people from the overcrowded lands of better quality where their marginal product is lower to those of poorer quality where their marginal product is higher. In the P system marginal products are automatically equalized due to profit maximization with a uniform wage, and it is impossible to increase the output of agricultural produce given the number of laborers working in agriculture. This is *not* true for C, which is an inefficient system yielding less than its maximum possible agricultural product.

There is an alternative way of looking at the efficiency issue which may be just as instructive. Although the composition of output is different in the two systems, national product as conventionally measured (with agricultural goods as numeraire) will have to be higher in P than in C. The national product in P is

$$\sum_1^n f_i(\hat{x}_i) + \hat{w}\hat{x}_s,$$

while in C it is

$$\sum_1^n f_i(\tilde{x}_i).$$

From (7),

$$f_i(\hat{x}_i) - \hat{w}\hat{x}_i \geqslant f_i(\tilde{x}_i) - \hat{w}\tilde{x}_i \quad \text{for all } i$$

with strict inequality if $\hat{x}_i \neq \tilde{x}_i$. Summing the above inequality,

$$\sum_1^n f_i(\hat{x}_i) - \hat{w}\left(\sum_1^n \hat{x}_i\right) > \sum_1^n f_i(\tilde{x}_i) - \hat{w}\left(\sum_1^n \tilde{x}_i\right),$$

which can be rewritten as

$$\hat{w}\hat{x}_s + \sum_1^n f_i(\hat{x}_i) - \hat{w}\left(\sum_1^n \hat{x}_i + \hat{x}_s\right) > \sum_1^n f_i(\tilde{x}_i) - \hat{w}\left(\sum_1^n \tilde{x}_i\right).$$

Using (6) and (9), we have

$$\sum_1^n f_i(\hat{x}_i) + \hat{w}\hat{x} > \sum_1^n f_i(\tilde{x}_i) \tag{10}$$

which is the result to be proved.

We now ask a fundamental question. The answer will have far reaching consequences for our analysis. Under which system do the peasant-laborers fare better? P is an efficient system and in it the pie of national income is higher than in C. But the workers in P are getting only a slice of the pie, and they are getting all of it in C. So the answer would appear to be ambiguous.

Nevertheless, we can prove that in all cases the working population must be better off under C than under P. That is,

$$\tilde{w} > \hat{w}. \tag{11}$$

This theorem is proved as follows:

Suppose first there is an integer j with $\hat{x}_j > 0$ and $\hat{x}_j \geq \tilde{x}_j$. Then

$$\tilde{w} \geq \frac{f_j(\tilde{x}_j)}{\tilde{x}_j} \geq \frac{f_j(\hat{x}_j)}{\hat{x}_j} > f_j'(\hat{x}_j) = \hat{w},$$

where

$$\frac{f_j(0)}{0} \equiv f_j'(0).$$

If $\hat{x}_i < \tilde{x}_i$ for all $\hat{x}_i > 0$, then $\sum_1^n f_i(\hat{x}_i) < \sum_1^n f_i(\tilde{x}_i)$. This implies that

$$\hat{w}L = \sum_{1}^{n} \hat{w}\hat{x}_i + \hat{w}x_s = \sum_{1}^{n} f_i(\hat{x}_i) - \sum_{1}^{n} \hat{R}_i + \sum_{1}^{n} \hat{R}_i$$

$$= \sum_{1}^{n} f_i(\hat{x}_i) < \sum_{1}^{n} f_i(\tilde{x}_i) = \sum_{1}^{n} \tilde{w}\tilde{x}_i = \tilde{w}L.$$

In either case we have the result to be proved.

The implications of (11) are important. While P is efficient and has a higher national income than C, the workers don't share in the increased benefits.

Because the price of non-agricultural commodities is primarily determined by labor costs, in the P system by comparison with C the terms of trade have moved in favor of agricultural goods and against manufactures, construction, and services.

What are the effects on land usage of changing from C to P. As we have already noted, under P the number of agricultural workers will decline by $\hat{x}_s$. Peasants move off the land, seeking work in manufacturing and other trades which satisfy the growing demand for non-agricultural goods arising out of increased rental income. The better pieces of land will have less workers on them under P than under C (this notion could be quantified and proved as a theorem). On the other hand new lands which were marginally undesirable under C will now be brought into cultivation for the first time. If a village i is such that $\hat{w} < f_i'(0) < \tilde{w}$, it will be settled under P whereas it was unpopulated under C. The transition from C to P thus evens out settlement patterns, moving people off the better lands, into the cities and onto newly cultivated areas.

So far we have been carrying out the analysis as if the transformation from communal to private property occurred instantaneously. That assumption has given us some valuable insights. Now it is time to analyze in the context of a more realistic model of the economy what happens when an individual village is turned into private property or "enclosed". Although the analysis is more complicated, we will see that all the basic conclusions of the simpler story are borne out.

Suppose that at a given time the total of n villages are divided into m open-field, traditionally organized villages and $n-m$ enclosed villages run along profit maximizing lines. In each of the m open-field villages peasants pay implicit or explicit obligations for their use of the land. Whether they are in the form of a head tax or a land rent, the essence of traditional feudal obligations is that they are relatively fixed by law or custom (possibly

at zero in certain cases) and that they yield the lord less than a profit maximizing income.

What will be the equilibrium number of workers in open-field village i, where $1 \leqslant i \leqslant m$? Suppose that $\bar{w}$ is the prevailing standard of living throughout the economy. Let $\bar{R}_i$ be the fixed feudal rent on the land of village i, and $\bar{\tau}_i$ the fixed feudal head tax. These are a stylized representation of medieval economic obligations. If there are $x_i > 0$ villagers in i, the net return per man will be

$$\frac{f_i(x_i)}{x_i} - \left(\frac{R_i}{x_i} + \bar{\tau}_i\right).$$

In equilibrium this must equal $\bar{w}$. If $\bar{x}_i$ is the equilibrium number of workers in village i, then

$$f_i(\bar{x}_i) = \bar{R}_i + (\bar{w} + \bar{\tau}_i)\bar{x}_i. \qquad (12)$$

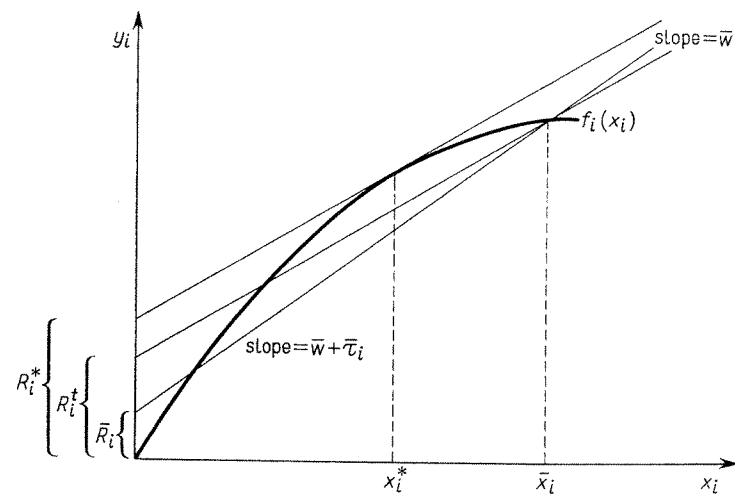This solution is depicted graphically in Figure 5.

Fig. 5.

At the point $x_i = \bar{x}_i$, (12) is satisfied and each peasant receives $\bar{w}$. The total return which the feudal landlord receives is

$$R_i^t \equiv \bar{R}_i + \bar{\tau}_i\bar{x}_i = f_i(\bar{x}_i) - \bar{w}\bar{x}_i.$$

This is less than the profit maximizing return of R_i^*, obtained by employing only x_i^* laborers, at the point where $f_i'(x_i^*) = \bar{w}$.

What happens when village i is enclosed? As the lord begins to view open fields primarily as a source of income and sees the prevailing standard of living as a potential wage rate, he starts to fight for complete control of the land. If he could enclose, the lord begins to perceive, he could exclude $\bar{x}_i - x_i^*$ unnecessary workers. This would reduce agricultural output, but it would cut labor costs even more. Were he to pay those who remained at the same prevailing standard $\bar{w}$, the lord could increase his income from R_i^f to R_i^*. Thus begins the struggle for property rights and for the clear definition of who owns land in the modern sense.

The main thrust of this analysis, which is completely missed by all the writers on enclosures, is that by its very economic nature *the act of enclosing land must lead to its depopulation*. That is why enclosing is undertaken by a would-be landlord in the first place. Profits cannot be increased unless peasants are squeezed off the land. Of course, the lord would not need to exclude peasants if they collectively agreed to pay the new rent R_i^*, but this they will refuse to do. They could not raise a rent of R_i^* and yet have all $\bar{x}_i$ of them simultaneously remain on the land of village i without depressing their standard of living below the prevailing rate of $\bar{w}$.

When property i is enclosed, the profit maximizing landowner chooses to operate at a less labor-intensive point on the production function. To some of the $x_i^* - \bar{x}_i$ dispossessed laborers, it appears as if they are being displaced for technological reasons because new, less labor-intensive techniques or commodities have been introduced by the landlord after enclosure.

The equilibria before and after village m is enclosed are as follows:

Before m is enclosed   *After m is enclosed*

$\bar{x}_i > 0: f_i(\bar{x}_i) = \bar{R}_i + (\bar{w} + \bar{\tau}_i)\bar{x}_i$

$\bar{x}_i = 0: f_i'(0) \leq \bar{w}, \bar{R}_i = 0, \bar{\tau}_i = 0$

$i = 1, ..., m$ $i = 1, ..., m-1$

$R_i^* \equiv f_i(x_i^*) - \bar{w}x_i^* = \max_{x \geq 0} f_i(x) - \bar{w}x$

$i = m+1, ..., n$ $i = m, ..., n$

$\bar{w}x_s = \sum_1^m (\bar{R}_i + \bar{\tau}_i\bar{x}_i) + \sum_{m+1}^n R_i^*$ $\bar{w}x_s = \sum_1^{m-1} (\bar{R}_i + \bar{\tau}_i\bar{x}_i) + \sum_m^n R_i^*$

$\sum_1^m \bar{x}_i + \sum_{m+1}^n x_i^* + \bar{x}_s = L$ $\sum_1^{m-1} \bar{x}_i + \sum_m^n x_i^* + \bar{x}_s = L$

Note that before a village is enclosed, the feudal obligations $\bar{R}_i$ and $\bar{\tau}_i$ are treated as fixed and exogenously given. After enclosure the profit

maximizing rental R_i^* is endogenously determined (as are $\bar{w}$, all the $\bar{x}_i$, and all the x_i^*).

The results of changing a single village from open-field control to private property are just a scaled down version of what happens when the whole system changes from pure communal to pure private property. The return to labor is lowered, the surplus goes up; land previously untilled gets drawn into cultivation, the terms of trade turn in favor of foodstuffs, national income rises, there is a net flow of workers out of agriculture, etc.

As we have seen, when an open-field village is enclosed, it must be depopulated. Where do the displaced laborers from the enclosed village go? Since each enclosure nudges the prevailing wage rate down, profit maximizing landlords on previously enclosed land will want to hire more wage laborers. Some displaced workers will crowd into the remaining unenclosed villages because the return obtainable there looks more attractive when the general standard of living is lower. Virgin land will be called into cultivation and a fraction of the displaced people will move there. Finally, as rents rise, the demand for non-agricultural goods financed out of the surplus increases, causing centers of manufacturing and other non-agricultural trades to attract some newly displaced immigrants. Thus, as enclosures proceed, they give rise to a lot of population movement and resettlement, accompanied at all times by a decline in the standard of living of the working population.

In summary, other things being equal, when village land is enclosed:
1) Tenants are displaced from the newly enclosed land.
2) The standard of living of the working population declines.
3) Rents rise and the surplus increases.
4) Less labor-intensive techniques are used on the newly enclosed land.
5) The population of other villages rises.
6) Surplus supported economic activities increase.
7) New lands are settled.
8) National income is higher.
9) Agricultural output is produced more efficiently.
10) The terms of trade move in favor of agriculture against industry.
11) There is a net flow of labor out of agriculture.