10 On the Meaning of Comparative Factor Productivity

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The point of departure for this chapter is a 1975 article by Abram Bergson entitled "Index Numbers and the Computation of Factor Productivity." That essay, which has its immediate origins in previous work of Bergson and Moorsteen, was a wide-ranging analysis of the nature and meaning of index number formulas used in the calculation of factor productivity. What I would like to do here is look closely at one particular aspect of the problem—on the one hand, the relationship between a disaggregated input-output economy having alternative techniques with variable factor proportions and, on the other hand, the standard aggregate production function exercises used in comparing factor productivity.

The basic idea is that even in a quite complicated input-output economy with variable techniques, so long as for each technique capital and labor substitute for each other in fairly similar fashion, the aggregate production function approach can be justified. This idea is not so surprising, but I think a formal statement of it is useful, if only because it may give a clearer picture of what is involved or implied in aggregate production function analysis, and perhaps it may provide some rationale for such conventional practices as aggregating outputs arithmetically and inputs geometrically.

Our scenario is the following. There are two economic situations, α and β. They might represent the same economy at different times, or different economies at the same time, or different economies at different times. Suppose there are two factors of production (the analysis readily generalizes), labor, L, and capital, K. We are interested in the relative total factor productivity of α and β.

The usual approach (at least when it is told as a paradigm) goes something like this. Suppose α is a market economy in equilibrium. Let its national product be Y_α. Let the national product of β, *measured in the prices of* α, be Y_β. Suppose the aggregate production function (more on that —whatever it is —later) for α is of the functional form $f(K, L)$. Economy α's endowments of capital and labor are $\bar{K}_\alpha$ and $\bar{L}_\alpha$, whereas those of β are denoted $\bar{K}_\beta$, $\bar{L}_\beta$.

Total factor productivity is

$$\frac{Y}{f(K, L)}$$

(output, aggregated arithmetically, divided by combined inputs, typically aggregated geometrically corresponding to a Cobb-Douglas production function).

In such a situation *the factor productivity of* α *relative to* β might be defined as

$$\frac{Y_\alpha/f(\bar{K}_\alpha, \bar{L}_\alpha)}{Y_\beta/f(\bar{K}_\beta, \bar{L}_\beta)} \tag{1}$$

(Naturally the definition is entirely from the viewpoint of α, but some asymmetry of that sort is unavoidable; I want to sidestep the issue in this chapter.)

Formula (1) is typically interpreted as some measure of the relative power of the technology of economy α to produce β output mix with β factor endowments. If the technology of α were hypothetically applied to produce β factor endowments, (1) represents how much proportionately more (or less) output could be expected.

We know a fair bit about when such an interpretation of (1) makes sense, at least in simple situations. Indeed, a good part of Bergson's 1975 article is devoted to this issue. What I would like to do here is push the analysis a bit harder to see how far it can be made to go. Nothing revolutionary, or for that matter entirely satisfactory (or even exciting) seems to emerge. But I think it is worthwhile to probe this issue further in a formal theoretical direction because it gives some idea of what we are implicitly assuming when we work with formulas like (1) in a general context.

Any analysis must begin somewhere, and I will start with the following assumptions. Let there be n commodities in α, produced by an input-output technology with variable techniques. If one unit of commodity i is produced by technique j (sometimes denoted j_i when ambiguity must be avoided), it requires a^j_{hi} units of commodity h $(h = 1, \ldots, n)$ and c^j_i units of "composite factor." Composite factor is an amalgam of capital and labor whose substitution possibilities are given by the function $F^j_i(K^j_i, L^j_i)$. In other words, if one unit of commodity i is to be produced by technique j, we must have a combination of capital K and labor L satisfying $F^j_i(K, L) \geq c^j_i$. The equation $F^j_i(K, L) =$ constant merely expresses how capital and labor substitute for each other in producing commodity i with technique j.

We are thus making the special assumption that a technique specifies as fixed coefficients the flow of input materials but leaves some flexibility in the combination of capital and labor required to process the inputs. This is a restriction, though perhaps a fair one. Within each (different) technique for producing a commodity, capital and labor can substitute (differently for each technique) in processing materials, but the materials themselves are relatively fixed in proportions (different for each technique). There are exceptions; however, this formulation seems a fair generalization, and it will give a lot of analytic power.

All production is constant returns to scale (including the functions $F_i^j(\cdot)$).

As postulated, economy α is in market equilibrium with total supplies of capital $\bar{K}_\alpha$ and labor $\bar{L}_\alpha$. Let the equilibrium wage rate be w and the capital rental be r. The competitive price of commodity i is p_i. The following general equilibrium pricing equations must hold:

$$p_i = \min_{j,\, K_i^j,\, L_i^j} \{wL_i^j + rK_i^j + \sum_{h=1}^{n} p_h^j a_{ji}\} \tag{2}$$

$$F_i^j(K_i^j, L_i^j) = c_i^j.$$

The equation states that production is to be done at least cost, and price equals cost. Given w and r, equation (2) will have a solution $j(i)$, $K_i^{j(i)}$, $L_i^{j(i)}$, p_i for each i. Let the corresponding input-output matrix be $A = (a_{hi}^{j(i)})$ with composite factor row vector $c = (c_i^{j(i)})$.

On the demand side, suppose final net output is the column vector D^α. The corresponding vector of gross sectoral output levels is

$$X^\alpha = (I - A)^{-1} D^\alpha. \tag{3}$$

The final equilibrium condition is then

$$\sum X_i^\alpha L_i^{j(i)} = \bar{L}_\alpha, \tag{4}$$

$$\sum X_i^\alpha K_i^{j(i)} = \bar{K}_\alpha. \tag{5}$$

For the purpose of using α data to evaluate β, the only relevant thing about economy β is the net final output vector it produces, D^β, and its endowments of capital $\bar{K}_\beta$ and labor $\bar{L}_\beta$.

Now let us *define* the coefficient of comparative factor productivity as the proportion of β output that could be produced by α technology given β factor endowments. The coefficient of comparative factor productivity μ is then the solution (the maximum value of λ) of the problem:

$$\max \lambda \quad \text{subject to} \quad \sum_{j_i} X_i^{j_i} - \sum_h \sum_{j_h} a_{ih}^{j_h} X_h^{j_h} \geqq \lambda D_i^\beta, \tag{6}$$

$$F_i^j(K_i^j, L_i^j) \geq c_i^j, \tag{7}$$

$$\sum_i \sum_{j_i} K_i^{j_i} X_i^{j_i} \leq \bar{K}_\beta, \tag{8}$$

$$\sum_i \sum_{j_i} L_i^{j_i} X_i^{j_i} \leq \bar{L}_\beta, \tag{9}$$

$$X_i^{j_i},\ K_i^{j_i},\ L_i^{j_i} \geq 0. \tag{10}$$

In this problem, $X_i^{j_i}$ is the amount of commodity i produced by technique j_i, and $K_i^{j_i}(L_i^{j_i})$ is the amount of capital (labor) employed per unit output in technique j_i of sector i.

For the most general case it is difficult to characterize the solution of (6) through (10) and state with precision what μ depends upon. Fortunately, it is possible to prove an "approximation theorem." If the sectoral production functions F_i^j are close to being identical to each other (up to a scaling factor), in some sense we are in a situation where an aggregate production function "almost exists." Then it seems reasonable that the standard aggregate factor productivity calculation will be a good approximation to μ.

Suppose, then, there is a production function $f(K, L)$ homogeneous of degree one, such that

$$\|F_i^j(K, L) - f(K, L)\| < \epsilon \tag{11}$$

for some $\epsilon > 0$, all i, j, K, L. As usual, $\|\cdot\|$ denotes any legitimate distance norm. If ϵ is small, f is a good approximation to each of the sectoral production functions.

Let Y_α be the national product of economy α ($Y_\alpha = pD^\alpha$). Let Y_β be the national product of economy β measured in prices of α ($Y_\beta = pD^\beta$). Consider using f as an aggregate production function in the surrogate factor productivity comparison,

$$\hat{\mu} = \frac{Y_\alpha / f(\bar{K}_\alpha, \bar{L}_\alpha)}{Y_\beta / f(\bar{K}_\beta, \bar{L}_\beta)}. \tag{12}$$

The following result is not difficult to show (the proof is omitted).

$$\lim_{\epsilon \to 0} \hat{\mu}(\epsilon) = \mu. \tag{13}$$

Thus the aggregate approximation of comparative factor productivity (12) becomes the rigorous definition (6) in the limit as the aggregate production function f becomes a perfect approximation to each technique's production function F_i^j. If each technique's production function is close to Cobb-Douglas, with similar capital and labor elasticities, there is a rigorous justification for aggregating price weighted outputs arithmetically and inputs geometrically.

In summary, the economy can have a very complicated input-output structure, but so long as the substitution possibilities between capital and labor are similar for each technique, use of an aggregate production function and price weighted outputs will yield a serviceable approximation to a true measure of comparative factor productivity.

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