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The Optimal Development of Resource Pools

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INTRODUCTION

At each instant of time a depletable resource can be drawn from any one of a number of pools. The cost of removing an extra unit from a pool depends on how much has already been taken out of it. What policy supplies a fixed flow of the resource at minimum present discounted cost?

It is easy to characterize the optimal rule in a classical environment where every pool has nondecreasing extraction costs. At any time simply draw the required amount from the source with lowest marginal cost. But what happens if, as in the real world, incremental extraction costs decline in some initial range? In the general case, a marginalist policy of exploiting the least cost source would be suboptimal.

The present paper shows that a natural generalization of the marginalist rule is optimal when resource pools have *arbitrary* extraction costs. A key solution concept turns out to be the "equivalent stationary cost" of exploiting the resource. Properties of an optimal policy are derived and interpreted.

THE MODEL

The present paper is primarily an exercise in trying to characterize, at a high level of abstraction, the basic form of an optimal policy for exploiting natural resources under very general cost conditions, including decreasing costs. This goal can be most conveniently served by treating the underlying prototype problem in a discrete formulation. (A continuous version yields completely analogous results, but the mathematical analysis is intrinsically more technical. Actually, the form of an optimal policy in the continuous case is most easily proved by first treating the corresponding discrete or integer mode and then carefully taking the limit as the step size approaches zero.)

A "generalized resource" is extracted from "generalized resource pools," indexed by i. The system is in state $\{m_i\}$, where each m_i is a non-negative integer, if amount m_i of the resource has already been extracted from pool i. A function

$$F_i(m)$$

gives the cost of extracting one more unit of the resource from pool i when m units have already been taken from that source. If no more than amount m can be gotten from pool i, we symbolically write $F_i(m) = \infty$.

Let v_i be an indicator variable that takes on value unity if pool i is being exploited, and zero, otherwise. A transition can be made from state

$$\{m_i\}$$

to state

$$\{m_i + v_i\}$$

where

$$v_i = 0 \text{ or } 1,$$
$$\sum_i v_i = 1,$$

at a cost of

$$\sum_i v_i F_i(m_i).$$

There is a constant demand for the resource which must be satisfied. Without loss of generality, time is measured so that demand equals one. Let α be the appropriate one-period discount factor. The basic problem is to determine control variables $\{v_i(t)\}$ which will

$$\text{minimize} \sum_{t=0}^{\infty} \alpha^t \sum_i v_i(t) F_i(m_i(t)) \tag{1}$$

subject to

$$m_i(t+1) = m_i(t) + v_i(t), \tag{2}$$
$$\sum_i v_i(t) = 1, \tag{3}$$
$$v_i(t) = 0 \text{ or } 1. \tag{4}$$

(Implicitly we are assuming that (1)–(4) is well formulated. In fact it is not difficult to specify sufficient conditions for guaranteeing the existence of a solution.)

There are many ways in which the actual economics of extraction is more complicated than the simple model presented here. (Just to mention a few: Uncertainty may be present; demands and discount factors could vary with time; extraction costs might depend on current outflow as well as on how much has been cumulatively taken from a pool. Equation (4) could be replaced by the seemingly weaker "mixing" condition $0 \leqslant v_i(t) \leqslant 1$, but only at the expense of making the analysis more intricate, and without otherwise altering results. Note that when time units are sufficiently small any fractional extraction policy can be approximated to an arbitrary degree of accuracy by a discrete policy of the form (4) (which rapidly switches back and forth among the pools being "mixed").) Nevertheless, the above formulation might be defended as a rough approximation to the general issue of finding the best pattern for developing a resource from alternative sources or pools. The fact that it is possible to sharply characterize an optimal solution makes problem (1)–(4) a natural preliminary to any more general analysis. And it may even be a reasonable description of some situations.

Depending on the context, the various pools might be mines, pits, wells, deposits, fields, regions, or even resource-based technologies. Actually, with a judicious interpretation of "resource pool," formulation (1)–(4) is sufficiently general to include as special cases many of the standard deterministic operations–research models for such problems as equipment selection and replacement, inventory and production scheduling, or capacity expansion. (In such situations there are typically several *classes* of pools, each of which contains an infinite number of identical members. For equipment problems a "resource pool" is a certain piece of equipment and the "amount extracted" is the length of time it has been in service. With scheduling, inventory, and expansion problems, a pool is a certain strategy-schedule (like ordering inventory or building capacity) which goes up to some expiration or regeneration point (after which costs are interpreted as being infinite); the "amount extracted" is the length of time the given strategy-schedule has been carried out.)

Nondecreasing Costs: A Special Case

Consider now the classical assumption of constant or increasing marginal cost. In symbols,

$$F_i(m) \leqslant F_i(m') \tag{5}$$

for all i, m, m' such that

$$m \leqslant m'.$$

When restriction (5) is imposed, it is easy to characterize an optimal solution to problem (1)–(4). Always extract the next unit from the cheapest source currently available. At time t, fix

$$v_{i^*}(t) = 1$$

for any integer i^* satisfying

$$F_{i^*}(m_{i^*}(t)) = \min_i F_i(m_i(t)). \tag{6}$$

That such a policy is optimal follows directly from applying the standard marginalist argument to a situation with convex cumulative cost functions and positive time discounting.

It is important to realize that the marginalist selection rule (6) does *not* give correct answers in a nonconvex environment where (5) fails to hold. If extraction costs can decline, an optimal policy may require exploiting a source having high initial but low eventual costs. There is no way that a completely myopic algorithm that looks only at incremental costs will pick out this kind of policy. The correct solution concept, if there is one for a problem seemingly as complex as (1)–(4), would have to somehow look ahead at the entire pattern of future costs.

Now we can pose the following basic question. Is there any simple abstraction or generalization of the marginalist principle which works when *no structure whatsoever* is placed on the cost coefficients $\{F_i(m)\}$? The answer turns out to be yes.

The algorithm which is developed in this paper for dealing with an arbitrary cost structure may be of some interest. Situations with a range of decreasing marginal cost are frequently the rule rather than the exception. Substantial set-up costs in overhead, research, development, or the like must usually be incurred just to open up a resource pool. Then there is the pervasive feature of learning by doing or cumulative technological progress, which is likely to lower costs over at least the beginning stages of exploiting a new source. It is true that costs of extracting from a given pool will eventually rise if and when resource deposits start to give out. (Even then, fresh investments in secondary and tertiary recovery methods may cut subsequent operating costs, at least temporarily.) But the evidence suggests that incremental costs are not universally nondecreasing over their entire range.

Solving the General Case

If amount m has already been drawn from pool i, the equivalent stationary cost of taking the next n units in a row from it is the weighted average

$$\Psi_i^n(m) = \sum_{j=0}^{n-1} F_i(m+j)\, \alpha^j \Big/ \sum_{j=0}^{n-1} \alpha^j. \tag{7}$$

Let $\hat{n}$ be the time horizon which minimizes (7),

$$\Psi_i^{\hat{n}}(m) = \min_{n=1,2,\ldots} \Psi_i^n(m). \tag{8}$$

We permit the case $\hat{n} = \infty$. (Strictly speaking, the operation *inf* should replace *min* in (8). Likewise for formulae (6) and (11) if the number of resource pools is infinite.) This could occur, for example, when costs are everywhere nonincreasing.

The *implicit cost* of pool i (when m units have been extracted from it) is defined to be its smallest equivalent stationary cost

$$\Phi_i(m) \equiv \Psi_i^{\hat{n}}(m). \tag{9}$$

The following decision rule completely characterizes an optimal policy in the general case. (That is, it constitutes a necessary and sufficient condition for an optimum.)

Always extract the next resource unit from the pool with lowest implicit cost. Symbolically, in an optimal policy, at each t

$$v_{i^*}(t) = 1 \tag{10}$$

for any integer i^* satisfying

$$\Phi_{i^*}(m_{i^*}(t)) = \min_i \Phi_i(m_i(t)). \tag{11}$$

Converting arbitrary cost streams to stationary equivalents for the purpose of finding the cheapest alternative is an old economist's trick. The optimality of decision rule (10), (11) can in a sense be interpreted as justifying this heuristic procedure under certain conditions.

Note that the implicit cost of a pool reduces to its marginal cost for the special case of nondecreasing costs. This is because when (5) holds, the horizon $\hat{n} = 1$ is always a solution of (8); and then (9) becomes

$$\Phi_i(m) = F_i(m).$$

So the decision rule (11) is indeed a generalization of the marginalist principle (6).

It is not necessary to recalculate implicit costs in each time period. If a pool is not used, its implicit cost does not change. If it is used, its implicit cost must have been lower, or at least no higher, than the implicit cost of any other pool. But then it will remain the lowest implicit cost pool (and hence, will continue to be exploited) for at least a number of periods equal to the equivalent stationary cost minimizing horizon $\hat{n}$ of (8).

This last conclusion can be derived as follows. Split the time interval $[0, \hat{n})$ into two subintervals $[0, h)$ and $[h, \hat{n})$ for h any integer between 1 and $\hat{n} - 1$. The equivalent stationary cost of extracting over the interval $[0, \hat{n})$ is a weighted average of the equivalent stationary costs over its component subintervals,

$$\Psi_i^{\hat{n}}(m) = \left(\frac{1-\alpha^h}{1-\alpha^{\hat{n}}}\right) \Psi_i^h(m) + \left(1 - \left[\frac{1-\alpha^h}{1-\alpha^{\hat{n}}}\right]\right) \Psi_i^{\hat{n}-h}(m+h). \quad (12)$$

We know from (8) that

$$\Psi_i^{\hat{n}}(m) \leqslant \Psi_i^h(m).$$

It follows that

$$\Psi_i^{\hat{n}-h}(m+h) \leqslant \Psi_i^{\hat{n}}(m).$$

Using (9), the above expression becomes

$$\Psi_i^{\hat{n}-h}(m+h) \leqslant \Phi_i(m). \quad (13)$$

But

$$\Phi_i(m+h) \leqslant \Psi_i^{\hat{n}-h}(m+h).$$

Thus,

$$\Phi_i(m+h) \leqslant \Phi_i(m)$$

for $h = 1, 2, \ldots, \hat{n} - 1$, the result to be proved.

To illustrate the typical form of an optimal policy for exploiting depletable natural resources, consider the following simple example. Suppose there are but three resource pools. Each pool has an initial range of decreasing costs, followed by a final section of increasing costs. Let the pools be ordered so that the first has lower implicit cost than the second, which in its turn is lower than the third. The optimal strategy will be to initially exploit the pool with lowest implicit cost, pool number one. This will be done until the marginal cost of extracting one more unit (in the increasing cost range) becomes greater than the implicit cost of the second pool. At that time pool two will start being exploited, and it will be the exclusive source until its marginal cost in the increasing cost range becomes greater than the marginal cost of pool one. Then pool one or two will alternately be exploited, depending on which is currently the cheaper source at the margin (in a continuous formulation, both would simultaneously be tapped). That is, if pool one has lower incremental cost than two, it will be exploited until its incremental cost exceeds two's, then two will be exploited until its incremental cost exceeds one's, etc. This rising marginal cost phase ends when the implicit cost of the third pool is found to be lower than the incremental costs of the first two pools. Then the third pool alone is tapped, up to the point where its incremental cost in the increasing cost range becomes greater than the incremental cost of the first or the second pool. From then on all three pools are alternately exploited, that pool being tapped at any given time which has lowest marginal cost.

Note that in a situation where all pools are the same and there is an unlimited collection of them, the optimal policy will be cyclic or recursive. The same conclusion holds if there are several classes of pools, each class containing an infinite number of identical pools (because in an optimal strategy only pools from one class will be tapped). This is why so many of the standard operations research models with renewable options (for example, the simple deterministic models of equipment selection and replacement, inventory and production scheduling, or capacity expansion) end up having a repetitive solution which may be universally characterized as follows. At each decision node, choose the strategy element with lowest equivalent stationary cost.

That such an elementary decision rule as (10), (11) is optimal depends crucially on the simplifying assumptions of the model. There does not seem to be available a sharp characterization of an optimal solution when, for example, demand varies with time or the discount rate is not constant. (About all that might be said of a general character in such cases is that a limiting argument could be used to show the results presented here are valid as an approximation when the stipulated preconditions are close to being met.)

The Basic Monotonicity Property

The correctness of decision rule (10), (11) derives ultimately from an important monotonicity property.

Pick any time t. Suppose that pool i is being exploited throughout the interval $[t, t + k]$ but not in period $t + k$, for k some positive integer. (If $k = \infty$, the monotonicity property is empty. But an important corollary, to be explained later, still holds.) Then pick any time $t' > t + k$. Suppose that pool i' is exploited throughout the interval $[t' - k', t')$ but not in period $t' - k' - 1$, for some positive integer k'. (Naturally, $t + k \leqslant t' - k'$.)

The basic monotonicity property is that with an optimal policy the equivalent stationary cost of exploiting pool i over the interval $[t, t + k)$ must be no greater than the equivelent stationary cost of exploiting i' over $[t' - k', t')$. In symbols,

$$\Psi_i^k(m_i(t)) \leqslant \Psi_{i'}^{k'}(m_{i'}(t' - k')). \tag{14}$$

When $t + k = t' - k'$, this fundamental lemma can be proved by a simple variational argument that reverses the order of pool exploitation. Suppose the original exploitation pattern looks like this:

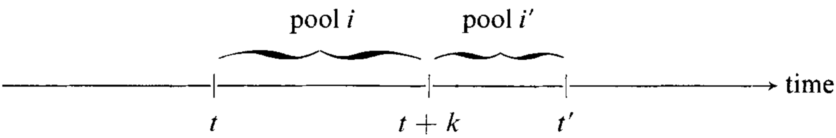

FIGURE 1

If the equivalent stationary cost of exploiting pool i over the interval $[t, t + k)$ is greater than pool i' over $[t' - k', t')$, it will be cheaper to delay incurring the higher cost by changing the exploitation pattern to:

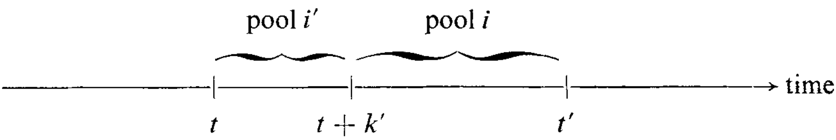

FIGURE 2

More formally, when $t + k = t' - k'$, the present discounted cost over the interval $[t, t')$ of the allegedly optimal policy which exploits pool i in the subinterval $[t, t + k)$ and pool i' in the subinterval $[t' - k', t') = [t + k, t')$ is

$$\alpha^t(1 - \alpha^k)/(1 - \alpha)\, \Psi_i^k(m_i(t)) + \alpha^{t+k}(1 - \alpha^{k'})/(1 - \alpha)\, \Psi_{i'}^{k'}(m_{i'}(t + k)). \tag{15}$$

By comparison, the present discounted cost of the feasible alternative policy which would reverse the order of exploitation of pools i and i' in the interval $[t, t')$, but would otherwise leave the system in the same state at times t and t', is

$$\alpha^t(1 - \alpha^{k'})/(1 - \alpha)\, \Psi_{i'}^{k'}(m_{i'}(t + k)) + \alpha^{t+k'}(1 - \alpha^k)/(1 - \alpha)\, \Psi_i^k(m_i(t)). \tag{16}$$

Because the original policy was optimal, the value of expression (15) must not be higher than (16). The resulting inequality reduces to

$$\Psi_i^k(m_i(t)) \leqslant \Psi_{i'}^{k'}(m_{i'}(t + k)), \tag{17}$$

which is equivalent to (14) for $t + k = t' - k'$.

Suppose now that $t + k < t' - k'$. Divide the time interval $[t + k, t' - k')$ into r contiguous subintervals of the form $[t_{j-1}, t_j)$ with $t_j > t_{j-1}$. The divisions are made so that during each subinterval one distinct pool is being exploited. Let pool $i_j (i_j \neq i_{j-1}$ and $i_j \neq i_{j+1})$ be tapped in the time interval $[t_{j-1}, t_j)$ for $j = 1, 2,..., r$ where $t_0 \equiv t + k$, $t_r \equiv t' - k'$, $i_0 \equiv i$, $i_{r+1} \equiv i'$. Repeatedly applying the appropriate form of result (17) at each of the $r + 1$ transition times $\{t_j\}_{j=0}^{r}$ (where exploitation of one pool ends and of another begins) yields a string of inequalities which can be collapsed into (14).

The basic idea is illustrated in Fig. 3 for the case $r = 2$. From what has previously been shown, the equivalent stationary cost over

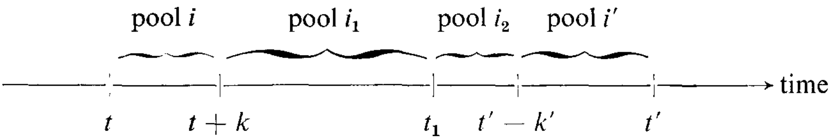

FIGURE 3

the interval $[t, t + k)$ is $\leqslant$ over $[t + k, t_1)$, which is $\leqslant$ over $[t_1, t' - k')$, which is $\leqslant$ over $[t' - k', t')$. This concludes our proof of the basic monotonicity property.

The following corollary to the basic monotonicity property is important.

Let i' be any pool and k' any positive integer. Suppose there is a time T such that pool i' is never tapped after T. Let the equivalent stationary cost of (hypothetically) exploiting i' over the time interval $[T, T + k')$ be

$$\mu \equiv \Psi_{i'}^{k'}(m_{i'}(T)). \tag{18}$$

Suppose pool i is exploited throughout the interval $[T, T + k)$ but not in period $T + k$, for some positive integer k. (We also allow the possibility $k = \infty$. In this case it will be evident that the proof of the corollary still goes through, although there is no opportunity or need to apply the basic monotonicity property. If $k' = \infty$, the statement and proof of the corollary are likewise unaffected.) Let the equivalent stationary cost of exploiting i over the time interval $[T, T + k)$ be

$$\lambda \equiv \Psi_i^k(m_i(T)). \tag{19}$$

In an optimal policy, it must be that

$$\lambda \leqslant \mu. \tag{20}$$

The proof of corollary (20) is as follows. Let the equivalent stationary cost at time T of the optimal policy over the entire interval $[T, \infty)$, during which there is no extraction from i', be β. In other words, if the alledgedly optimal policy costs $f(\theta)$ in period θ,

$$\beta \equiv (1 - \alpha) \sum_{\theta=T}^{\infty} f(\theta)\, \alpha^{\theta-T}. \tag{21}$$

By the basic monotonicity property, λ must be no greater than the equivalent stationary cost over any subinterval of $[T, \infty)$ during which one distinct pool is being exploited. But there is an easy generalization of (12) which says that the equivalent stationary cost of extracting over any interval is a weighted average of the equivalent stationary costs of extracting over each of its component subintervals. It follows that

$$\lambda \leqslant \beta. \tag{22}$$

Consider an alternative policy which exploits pool i' in the interval $[T, T + k')$ while postponing to a starting date of $T + k'$ the allegedly optimal policy which had previously begun at T. In other words, what was policy over $[T, \infty)$ now becomes policy over $[T + k', \infty)$ and pool i' is exploited in the interval $[T, T + k')$. Before time T, the original and alternative strategies coincide.

The cost over $[T, \infty)$ of the alternative variation discounted to time T is

$$\frac{1 - \alpha^{k'}}{1 - \alpha}\, \mu + \frac{\alpha^{k'}}{1 - \alpha}\, \beta.$$

This must be no less than the corresponding discounted cost of the original optimal policy

$$\beta/(1 - \alpha).$$

It follows that

$$\beta \leqslant \mu. \tag{23}$$

Combining (22) and (23) yields the desired corollary (20).

Proof of Optimality

Our proof of the main result is by contradiction. Suppose that the proposed decision rule (10), (11) is nonoptimal. If some other policy is optimal, there must be a time t and a pool $j(\neq i^*)$ such that

$$\Phi_j(m_j(t)) > \Phi_{i^*}(m_{i^*}(t)), \tag{24}$$

and yet,

$$v_j(t) = 1, \tag{25}$$

instead of $v_{i^*}(t) = 1$.

In this allegedly optimal policy, let $t + k$ be the first period after t that some pool other than j is exploited. (The case $k = \infty$ is allowed, which means that pool j alone is exploited after time t.) The equivalent stationary cost of exploiting pool j over the interval $[t, t + k)$ is

$$\gamma = \sum_{h=0}^{k-1} F_j(m_j(t + h))\, \alpha^h \Big/ \sum_{h=0}^{k-1} \alpha^h. \tag{26}$$

From the definition of $\Phi_j(m_j(t))$,

$$\Phi_j(m_j(t)) \leqslant \gamma,$$

which can be combined with (24) to yield

$$\Phi_{i^*}(m_{i^*}(t)) < \gamma. \tag{27}$$

Let n^* be the value of $\hat{n}$ which satisfies (8) for $i = i^*, m = m_{i^*}(t)$. If $n^* = \infty$, a direct argument based on (27) and the monotonicity property show that exploiting only pool i^* from time t on has a present discounted cost lower than the proposed policy. In what follows, the case $n^* < \infty$ is treated.

Although pool j is being exploited at time t instead of pool i^* it may be that i^* will be exploited again at some future dates. Suppose, as one case, that at least $m_{i^*}(t) + n^*$ units are eventually extracted from pool i in the allegedly optimal policy with (24), (25) holding. Let t' be the first time when

$$m_{i^*}(t') = m_{i^*}(t) + n^*.$$

Let k' be the positive integer such that pool i^* is exploited throughout the interval $[t' - k', t')$, but not at time $t' - k' - 1$.

For the values $i = i^*$, $m = m_{i^*}(t)$, $\hat{n} = n^*$, $h = n^* - k'$, expression (13) becomes

$$\Psi_{i^*}^{k'}(m_{i^*}(t) + n^* - k') \leqslant \Phi_{i^*}(m_{i^*}(t)). \tag{28}$$

Combining (28) with (27) yields

$$\Psi_{i^*}^{k'}(m_{i^*}(t) + n^* - k') < \gamma. \tag{29}$$

Now inequality (29) is a direct contradiction with the basic monotonicity property (14). A time segment $[t, t + k)$ with *higher* equivalent stationary cost γ is preceding a segment $[t' - k', t')$ with *lower* equivalent stationary cost $\Psi_{i^*}^{k'}(m_{i^*}(t) + n^* - k')$.

A contradiction (now with the corollary) is also implied if after time t a total of *less* than n^* additional units are to be taken from pool i^* in an allegedly optimal policy. Let T be the time right after the last unit has been taken from pool i^* if this occurs after t, and t otherwise. In the present context let k' be defined as the positive integer satisfying

$$m_{i^*}(T) = \lim_{s \to \infty} m_{i^*}(s) = m_{i^*}(t) + n^* - k'. \tag{30}$$

For k' so defined, (29) continues to hold. Using (30), expression (29) becomes

$$\Psi_{i^*}^{k'}(m_{i^*}(T)) < \gamma. \tag{31}$$

Employing definition (18) for $i' \equiv i^*$, (20) becomes

$$\Psi_{i^*}^{k'}(m_{i^*}(T)) \geqslant \lambda. \tag{32}$$

By the basic monotonicity property and the definition of T,

$$\gamma \leqslant \lambda. \tag{33}$$

Inequalities (32) and (33) can be combined to yield

$$\Psi_{i^*}^{k'}(m_{i^*}(T)) \geqslant \gamma$$

which is a direct contradiction with (31).

This concludes our proof of the form of an optimal policy. (Strictly speaking, we have proved the necessity of (10), (11). That rule specifies a unique selection of i^* for each t (except in the case of ties, for which it is easy to show that how the tie is broken makes no difference to the value of the objective function). Thus, provided an optimum exists, sufficiency of (10), (11) has also been demonstrated.)

Scarcity Price of the Resource

By "efficiency," "shadow," or "scarcity" price of the resource at time t is meant its marginal value at that time, defined in terms of the overall objective (1), discounted to t. (Note that efficiency prices will not necessarily have the "decentralization" property of inducing and supporting optimal decisions unless we are operating in the convex environment of nondecreasing extraction costs.) It turns out that the efficiency price of the generalized resource can be represented by a simple expression with a direct economic interpretation.

The shadow price of the resource at time t is

$$p_t = (1 - \alpha) \sum_{s=t}^{\infty} F^*(s)\, \alpha^{s-t}, \tag{34}$$

where

$$F^*(s) \equiv F_{i^*(s)}(m_{i^*}(s)) \tag{35}$$

is the extraction cost in period s of an optimal policy.

The above result has a most interesting interpretation. The current shadow price is an exponentially weighted (by the discount rate) average of all future costs. At any time the scarcity price of an exhaustible resource is the equivalent stationary cost of exploitation henceforth. In other words, the efficiency price is exactly equal to that hypothetical constant extraction cost per unit which would yield the same present discounted cost as an optimal policy.

That the shadow price must have the form (34) is easily demonstrated. Suppose that an extra unit of the resource is given as a free gift in period t (recall that the resource could have been measured in arbitrarily small units to begin with). Consider the following proposed policy, which is now feasible. Up to time t, follow exactly the same extraction pattern as the original optimal solution (10), (11). During the time interval $[t, t + 1)$. take advantage of the free gift and extract nothing. Throughout the interval $[s, s + 1)$ for $s \geqslant t + 1$, extract from the pool $i^*(s - 1)$ which in the original optimal solution had been exploited during the previous interval $[s - 1, s)$.

The cost of such a policy discounted to period t is

$$C_t' = \sum_{s=0}^{t-1} F^*(s)\, \alpha^{s-t} + \sum_{t+1}^{\infty} F^*(s - 1)\, \alpha^{s-t}.$$

The cost of the original optimal policy discounted to period t is

$$C_t = \sum_{s=0}^{\infty} F^*(s)\, \alpha^{s-t}.$$

The difference is

$$C_t - C_t' = (1 - \alpha) \sum_{s=t}^{\infty} F^*(s)\, \alpha^{s-t}.$$

Thus, the gain achieved by the gift of an extra unit of the resource in period t is at least p_t, defined by (34). (It would be exactly p_t if the gift came as an unforseen "surprise" at t, in which case the proposed policy would be optimal. If the date of the gift is known beforehand, a planner may be able to cut present discounted costs by altering the proposed extraction schedule.) Using a similar argument, the loss incurred by taking away one unit of the resource in period $t - 1$ is no greater than p_t. The efficiency price of the resource at time t, defined as its undiscounted marginal value, is therefore given by expression (34). (A few mathematical fine points are being glossed over in favor of a cleaner exposition. For example, left- and right-hand-side marginal products need not be equal to each other; the proposed shadow price is merely "trapped between" them. Also, it is implicitly being assumed that time periods are "sufficiently small" to justify the variational arguments being employed.)

As a consequence of (34), the efficiency price of a depletable resource can have a complicated and interesting time profile. It may temporarily decline, corresponding to the development of a source having a significant range of decreasing costs (the decline will be permanent if the decreasing cost stage lasts forever). In the long run, when different pools are exploited the shadow price trends upward because the resource is becoming more expensive as the cheaper sources are exhausted; this increase will of course be monotonic in the classical case where each pool has nondecreasing costs. For operations-research type problems with renewable options, the shadow price exhibits a wave-like recursive pattern.