Technical Appendix for "Limits of Arbitrage: Theory and Evidence from the Mortgage-Backed Securities Market"

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Abstract

We present the continuous time version of our two period model. The economic conclusions are the same, though the two period model is much simpler to manipulate. Indeed, in general, the continuous time model requires the use of Malliavin calculus. This continuous time version is likely to be useful for practical implementations of the model.

1 Introduction and Preview of the Results

Gabaix, Krishnamurthy and Vigneron (2004, henceforth GKV) analyze the equilibrium price of risk in the MBS market. It uses a two period model, which gives the economic intuition. In the present Technical Appendix, we extend the analysis to continuous time settings. Economically, the conclusions are very similar. The machinery is much more involved, but lends itself better to practical implementation.

We start with the "quasi-static approximation" of the model — the approximation where we assume that interest rates and average coupon move

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slowly. This yield the following representation for the price of risk:

$$\lambda_t = \rho_t b\left(\overline{c}_t - r_t\right) \tag{1}$$

where $\overline{c}_t$ is the value-weighted average coupon outstanding in the MBS market, and r_t is the level of the interest rate, ρ_t the level of risk aversion, and b is a proportionality factor. This allows one to price any MBS. One takes a potentially sophisticated Wall Street model, which makes predictions for the cash-flow $dC\left(t\right)$ of an MBS security. For instance, this Wall Street model predicts a discounted present value of cash flows equal to $E\left[\int_0^\infty \exp\left(\int_0^t -r_s ds\right) dC(t)\right]$

We can enrich this model by pricing the prepayment risk, via the simple formula[1]:

$$\text{Price} = E\left[\int_0^\infty \exp\left(\int_0^t -r_s ds + \lambda_s dB_s - \frac{1}{2}\lambda_s^2 ds\right) dC(t)\right] \tag{2}$$

where $dC\left(t\right)$ is the cash-flow of the security, and dB_s is the common shock to prepayment risk, and the expectation is taken under the physical probability distribution. So, to price any security, one just simulates forward (1)-(2), together with an equation of motion for the average coupon outstanding $\overline{c}_t$. An example of such equation of motion is $d\overline{c}_t/dt = -\theta\left(r_t - \overline{c}_t\right)$, and a more sophisticated version would make the speed of adjustment θ depends on the incentive for issuance of new MBS.

The key result of Gabaix, Krishnamurthy and Vigneron (2004) is the expression (1) for the time-varying price of risk, and the empirical support for it. In the present paper, we show how to derive (1) in a continuous time framework. We show first how to do that in the "quasi-static" approximation. Then, we go on to the general case, and see that one needs to involve Malliavin calculus. We show how to calculate the corrections that result. This becomes analytically fairly untractable, but .is simple to do on a computer.

To sum up, while we derived (1) in a simplified context, it provides a first cut to value rather complex securities. The combination of our simple adjustment (1) and a complex model is indicated in (2). In this Technical Appendix, we also indicated how in principle more sophisticated adjustments could be made.

[1] To keep a positive λ_t in our model, we use the $\lambda_t dB_s$ convention in (2) rather than the more usual $-\lambda_t dB_s$.

2 The Continuous Time Model in the "Quasi-Static" Approximation

2.1 Notations

2.1.1 Cash-Flows and Prepayment

Working in continuous-time, we will call $dC(t)$ the cash-flow yielded by any MBS in dt units of time. The following definitions will be used throughout the paper.

- $A(t)$, the principal that would remain at date t if there were no prepayments. We have $A(t) = A(0)\frac{e^{c'T} - e^{c't}}{e^{c'T} - 1}$, where T is the maturity of the MBS[2], and c', the fixed interest paid by the mortgagor (the interest received by the owner of the MBS is $c < c'$; typically $c = c' - 50$ bp).
- $S^*(t)$ the pure cumulative prepayments, so that the remaining principal at time t is $a(t) = A(t)e^{-S^*(t)}$.
- $S(t)$ the cumulative prepayments, so that the remaining principal at time t is $a(t) = a(0)e^{-S(t)}$.
- The cash-flows $dC(t)$ or the securities are:
  - For an IO, $dC(t) = ca(t)\,dt$, the interest payment on the outstanding balance.
  - For a PO, $dC(t) = -da(t)$, how much of the principal has been paid down in dt units of time
  - For the collateral (also called pass-through): $dC(t) = ca(t)\,dt - da(t)$; the collateral cash-flow is sum of both IO and PO's.

There is an abuse of notation in the name of $S(t)$, because $S(t)$ includes the normal repayment of principal, in addition to the pure prepayments. But the advantage is that we have to carry only one object, $S(t)$, in the rest of the paper, instead of two. The name has the advantage of focusing the attention of the reader on the important aspect of $S(t)$, which is that it carries with it the stochasticity of prepayments.

[2] In amortizing mortgages, is a constant payment of principal and interest is scheduled to be paid each month so that no principal remains at T. Hence $A(T) = 0$, and $-dA(t) + i'A(t)dt = Cdt$, C a constant amount paid per unit of time. This yields the expression for $A(t)$ in the text.

In our definition $S(\tau)$ is a stochastic function. In current practice, people most often simplify their evaluations by considering a deterministic function, say $S_t^m(\tau)$, which is a complex function of interest rates and time, calibrated at date t (thus the t subscript) to best fit past prepayments observed. For simplicity, we assume here that this deterministic approximation of the true prepayment path is unbiased $e^{-S_t^m(\tau)} = E_t(e^{-S(\tau)})$ (corresponds to unbiased model for prices in a risk-neutral world).

2.1.2 The Option-Adjusted Spread (OAS)

In this paper, we will use two probability measures, P and $\hat{P}$. Both measures are understood to be risk-neutral with respect to interest-rate risk. The first one, P, corresponding to the expectation E, we call the physical probability (with respect to prepayment risk) to stress the fact that, given and interest rate path, its sample counterpart is the simple average of observed prepayment paths. The second one, $\hat{P}$, corresponding to $\hat{E}$, is the risk-neutral probability under which averaging has to be done with respect to an unobserved prepayment process whose drift is adjusted for risk. We will later make clear how this adjustment is made appealing to standard arbitrage-pricing arguments.

Defining the cumulative rate of interest by $R(\tau) = \int_0^\tau r_u du$, we have

Definition 1 *The OAS (option adjusted spread) of a security with cash-flow $dC(t)$, is the spread that we need to add to the discount curve in order to recover the market price:*

$$\text{Price} = E\left[\int_0^\infty e^{-R(t)-OAS\cdot t} dC(t)\right] \tag{3}$$

where E is the expectation under the physical probability for prepayment risk (but risk-neutral for interest rate risk).

If $d\widehat{B}_t$ is the Brownian increment under the risk-neutral measure, the gain process $G_t^k = D_t^k + P_t^k$ will have a drift equal to $r_t dt$ which means that it does not yield more than the riskless investment in expected value in the risk-neutral "world":

$$\frac{dG_t^k}{P_t^k} = r_t dt + \sigma_t^{G^k} d\widehat{B}_t$$

for some sensitivity σ_t^{Gk}. If we assume that the market is made of MBS, and the marginal investor holds the market portfolio, the instantaneous risk in his portfolio is:

$$dr_t^M = r_t dt + \sigma_M d\widehat{B}_t$$

with

$$\sigma_M = \frac{\sum_k P_t^k \sigma_t^{G,k}}{\sum_k P_t^k}$$

If the investor has no hedging demand in his portfolio (because his horizon is short, of we simply neglect the hedging demand terms), the price of risk is:

$$\lambda_t = \gamma \sigma_M \tag{4}$$

2.1.3 Simplifying Assumptions

The resolution of the market equilibrium we have described is not tractable when we consider the general case of a stochastic interest rate and MBS securities in the investment opportunity set. In the fixed point problem of Proposition 9, the price involves future prices of risks and risk premia, which makes the problem formidable.

We show in the next two sections that simplifying assumptions allow to use this concept of equilibrium to derive two testable implications for the MBS risk premia and the corresponding price of prepayment risk:

1. Constant interest rates, stochastic prepayment.
2. A discount rate that tends to infinity, which allows us to ignore the intertemporal hedging demand term.

2.2 Valuation Formulas for i.i.d. Prepayment Shocks: $S(t) = \phi \cdot t + n(t)$, $n(t)$ Prepayment Shock

Using relation (3), we now derive key valuation formulas, linking the OASs, OAS^{IO}, OAS^{PO} and OAS^{Col}. We will first present the theory in a simplified context, where the key relations can be derived in closed forms. Section 3 will deal with the general case. Still, we will get the main insights in this special case.

We assume that the interest rate r is constant, and, for the cumulated prepayments, take the expression $S(t) = \phi \cdot t + n(t)$, where ϕ is the constant mean prepayment rate, and $n(t)$ shocks, such that $n(0) = 0$ and $E[e^{-n(t)}] = 1$. This corresponds to the condition that the model be unbiased, i.e. $E[a(t)] = a(0)e^{-\phi t}$, when $a(t) = a(0)e^{-\phi t - n(t)}$. The reader should think of $n(t)$ as a jump process, or, as we will shortly assume, a functional of a low-dimensional Brownian motion. Finally, to simplify the notation, we normalized the amount outstanding at $t = 0$ as $a(0) = 1$.

For the IO, we have:

$$P^{IO} = i\hat{E}\int_0^\infty a(t)e^{-rt}dt = i\hat{E}\int_0^\infty e^{-rt-\phi t-n(t)}dt,$$

and in the other hand, the definition of the OAS of the IO, OAS^{IO}:

$$P^{IO} = iE\int_0^\infty a(t)e^{-rt-OAS^{IO}t}dt = iE\int_0^\infty e^{-rt-\phi t-OAS^{IO}t}dt = \frac{c}{r+\phi+OAS^{IO}}, \tag{5}$$

which leads to the relation, valid for all forms of $n(t)$:

$$\hat{E}\int_0^\infty e^{-rt-\phi t-n(t)}dt = \frac{1}{r+\phi+OAS^{IO}}, \tag{6}$$

We can use this relation to link the OAS of the PO and the collateral with the IO's. Indeed for the PO:

$$\begin{aligned} P^{PO} &= \hat{E}\int_0^\infty -da(t)e^{-rt} = a(0) - \hat{E}\int_0^\infty ra(t)e^{-rt}dt \\ &= 1 - r\hat{E}\int_0^\infty e^{-rt-\phi t-n(t)}dt = 1 - \frac{r}{r+\phi+OAS^{IO}}, \end{aligned} \tag{7}$$

by substitution of (6). By definition of OAS^{PO},

$$\begin{aligned} P^{PO} &= E\int_0^\infty -da(t)e^{-rt-\omega^{PO}t} = a(0) - E\int_0^\infty (r+OAS^{PO})a(t)e^{-rt-OAS^{PO}t}dt \\ &= 1 - \frac{r+OAS^{PO}}{r+\phi+OAS^{PO}}. \end{aligned} \tag{8}$$

We now can derive the link between the OAS of IO's and PO's:

Proposition 2

$$OAS^{PO} = -\frac{r}{\phi+OAS^{IO}}OAS^{IO} \tag{9}$$

The signs of OAS^{PO} and OAS^{IO} are opposite.

Proof. Indeed if $OAS^{IO}+\phi<0$, then $P^{PO}\notin[0,1]$ by (7), which is a contradiction. ■

Hence, we find the same intuition we had in GKV. This will also carries over in the model with stochastic interest rates. This result is important since it matches a robust empirical regularity, mentioned above. The typical values for r, ϕ, OAS^{IO} (see GKV 2004) also give the right order of magnitude for the ratio OAS^{PO}/OAS^{IO} (between one-third and one-fourth in absolute value).

For the collateral, likewise,

$$P^{Col} = P^{IO} + P^{PO} = 1 + \frac{c-r}{r+\phi+OAS^{IO}}. \tag{10}$$

We see that in the present case, the correspondence made above between premium environments ($P^{Col} > 1$, and $c > r$) and discount ones ($P^{Col} < 1$, and $c < r$) carries over. As the collateral's cash-flow is $dC = -da + iadt$, we get;

$$\begin{aligned} P^{Col} &= E\int_0^\infty (-da(t) + ca(t))e^{-rt-OAS^{Col}t}dt \\ &= 1 + E\int_0^\infty (c - r - OAS^{Col})a(t)e^{-rt-OAS^{Col}t}dt = 1 + \frac{c-r-OAS^{Col}}{r+\phi+OAS^{Col}}, \end{aligned}$$

so that OAS's of the collateral and the IO obey the following relationship:

Proposition 3 *The OAS of the collateral lies between those of the IO and PO, and*

$$OAS^{Col} = \frac{c-r}{c+\phi+OAS^{IO}}OAS^{IO}. \tag{11}$$

This result is very general and matches another empirical regularity. We find again our intuition of GKV. When $c > r$, OAS^{IO} has the sign of OAS^{Col}, i.e. positive, while the opposite is true when $c < r$. Relation (9) shows that the signs are opposite for the PO.

2.2.1 The Case of Brownian Motion: $S(t) = \phi t + \sigma B(t) + \sigma^2 t/2$

To obtain the OAS of the IO in closed form, one has to assume a particular form for the noisy part of prepayment. We take: $n(t) = \sigma B(t) + \sigma^2 t/2$, where $B(t)$ is a Brownian motion embodying the systematic risk of prepayment, and σ^2 is the variance of the shocks. The term $\sigma^2 t/2$ simply ensures $E\left[e^{-n(t)}\right] = 1$, as required to get an unbiased prepayment model[3].

In equation (6), one has to undertake a change of measure in order to compute the risk-neutral expectation of the discounted cash-flows. Using

[3]For clarity, we use the notation σ^k rather than GVK β^k for the loading on the aggregate prepayment shock.

standard arbitrage pricing arguments, there exists a constant λ – the *market price of prepayment risk* – such that, under the risk-neutral measure, one can[4] the original $B(t)$ with $\widehat{B}(t) + \lambda t = B(t)$, where $\widehat{B}(t)$ is the risk-neutral Brownian which will drive the new stochastic process for the cumulative prepayment function: $\hat{S}(t) = (\phi + \sigma\lambda)t + \sigma\hat{B}(t) + \sigma^2 t/2$.

Evaluation of the risk-neutral expectation gives:

$$\begin{aligned} \hat{E}\int_0^\infty e^{-rt-\phi t-n(t)}dt &= E\int_0^\infty e^{-rt-\phi t-\sigma(B(t)+\lambda t)-\sigma^2 t/2}dt \\ &= \frac{1}{r+\phi+\lambda\sigma}. \end{aligned}$$

Equating this last expression with the left hand side of (6) yields $OAS^{IO} = \sigma\lambda$. Suppose that the economy has N different IOs, indexed by k. We would have:

Proposition 4 *The OAS of the k-th IO depends on its systematic prepayment risk in the following way:*

$$OAS_k^{IO} = \sigma_k\lambda. \tag{12}$$

OAS of PO's and collateral take a non-linear form, as one sees by plugging (12) into (9) and (11). For the IO, in our theoretical framework, the OAS is literally the risk premium investors will ask to bear the prepayment risk orthogonal to interest rate risk (λ being the market price per unit of prepay risk). This formula is very intuitive and will carry over when interest rates are stochastic. If one abandons the simple form assumed for the noise $n(t)$ to include a more realistic process, such a closed-form formula is no longer available, although much of the insights gained with it will remain valid.

This proposition establishes the first empirically testable implications of the model. Equation (12) tells us that, at a given date, OAS's of all the traded IOs obey a proportional rule with the security's prepayment risk σ_k. A plot of $(\sigma_1, OAS_1^{IO}), ..., (\sigma_N, OAS_N^{IO})$ for N different IOs should lie along a line of slope λ. Although we derive it in an original way, a cousin of this relation has already been proposed by Cheyette (1994). The crucial step of the valuation model is the next one: the endogeneization of the market price of risk λ, using a market equilibrium.

[4] Our convention for the signs differs from the more usual one, which would use $\lambda' = -\lambda$. We chose this convention because it gives positive prices of risk in the more frequent premium environment. Of course, nothing in the analysis hinges on this choice of signs.

2.3 Market Equilibrium

The previous section showed how a simple framework could explain the signs and relative magnitudes of the OAS of the IO, PO, and collateral. The only ingredient required was some traditional arbitrage pricing theory, assuming a given price of risk λ. This subsection goes a step further and endogenizes the price of risk.

2.3.1 Preparation for the Equilibrium

We recall that the economy is made up of a representative agent (intuitively, she is a trader in the MBS market) who maximizes an objective function on her wealth (portfolio returns): $E_t\left[\int_t^{\infty} \delta e^{-\delta(\tau-t)} W_\tau^{1-\gamma}/(1-\gamma)d\tau\right]$, where W_τ is her wealth at instant τ. Hence her risk aversion is γ. She can invest in the riskless asset, or in MBS.

Her optimal portfolio leads, for the market as a whole, to:

$$\mu_c - r = \gamma\ \sigma_M \cdot \sigma_c^G$$

for an asset c (for instance a pass-through, an IO, a PO), with expected returns μ_c (under the physical probability) and a sensitivity to shocks σ_c, and denoting σ_M the sensitivity of market returns to prepayment shocks. More specifically if security c's price is P_c, its gain process dG_c (that includes capital appreciation and dividends) is: $dG_c = \mu_c P_c dt + \sigma_c^G P_c dB_t$. Its standard deviation is $\left\|\sigma_c^G\right\|$. The term $\sigma_M \cdot \sigma_c^G$ is the scalar product of two vectors, and represents the (instantaneous) covariance of returns of asset c with the returns of the market as a whole. We are going to endogenize the equilibrium values of σ_M and σ_c^G.

In our framework, the relation between the physical Brownian motion B_t and the risk-neutral one is $\hat{B}_t$ is: $dB_t = d\hat{B}_t + \lambda dt$. So the gain process for asset c can be written: $dG_c = \mu_c P_c dt + \sigma_c^G P_c(d\hat{B}_t + \lambda dt)$, and because under risk-neutral expectations, $\hat{E}[dG_c] = rP_c$, we can see that:

Proposition 5 *The market price of risk is, in equilibrium:*

$$\lambda = -\gamma\sigma_M, \tag{13}$$

where γ is the risk-aversion of the market, and σ_M the sensitivity of the market to shocks. Then, an asset with sensitivity σ_c^G has a (continuous-time) risk premium: $\mu_c - r = -\lambda\sigma_c^G$.

Suppose that the economy is made up of several assets, indexed by k, with a value P_k, and that their gain process has a sensitivity to prepayment σ_j^G. Then, the market portfolio has a total sensitivity:

$$\sigma_M = \frac{\sum_k P_k \sigma_k^G}{\sum_k P_k}. \tag{14}$$

Hence, to calculate the market equilibrium, we need an expression for the sensitivity of the gain process of a collateral to prepayment shocks. This is given in the following proposition:

Proposition 6 *Consider a collateral k, with coupon c_k, prepayment process $dS_t = (\phi_k - \sigma_k^2/2)dt + \sigma_k dB_t$, (so that the prepayment rate has a standard deviation $|\sigma_k|$), amount outstanding a_k, price P_k^{Col}, and gain process G_k. The sensitivity σ_k^G of the collateral to prepayment risk, defined by: $dG_k = rP_k^{Col}dt + \sigma_k^G P_k^{Col} d\hat{B}_t$, is:*

$$\sigma_k^G = -\frac{c_k - r}{r + \phi_k + OAS_k^{IO}} \sigma_k \tag{15}$$

Proof: The value P_k^{Col} of the collateral is given by equation (10), with a balance outstanding $a(t) = a(0)e^{-S(t)}$, which gives:

$$P_k^{Col}(t) = a_k(0)\left(1 + \frac{c_k - r}{r + \phi_k + OAS_k^{IO}}\right) e^{-(\phi_k + \sigma_k^2/2)t - \sigma_k B_t}.$$

Suppose there is a prepayment shock dB_t. This increases the prepayments of collateral k by $\sigma_k dB_t$. By the above expression, the owner of the pass-through k experiences a (positive or negative) capital gain of:

$$dP_k^{Col}(t) = -\phi_k P_k^{Col}(t)dt - \sigma_k P_k^{Col}(t)dB_t.$$

Following these shocks, she also receives both interest rates – $c_k a_k(t)dt$ – and partial repayment of the principal — equal to the amount $-da(t)$ in prepayment, i.e. she receives the dividend flow $c_k a_k(t)dt - da_k(t) = (c_k + \phi_k)a_k dt + \sigma_k a_k dB_t$. Hence, using equation (10), the total change in the gain process, $dG_k := c_k a_k(t)dt - da_k(t) + dP_k^{Col}(t)$, are risk-neutral shocks: $dG_k = rP_k^{Col} - dt + \sigma_k^G P_k^{Col}(dB_t - \lambda dt)$, with $\sigma_k^G = (a_k/P_k^{Col} - 1)\sigma_k$. Using equation (10) again, we get equation (15).□

We are ready to complete the market equilibrium. We need to identify what the market is. For the sake of clarity, we do it in three steps: in an economy where the only risky asset is collateral; in an economy where the risky assets are collaterals; and then in an economy in which the relevant market portfolios include non-MBS assets like stocks and bonds.

2.3.2 When the Only Risky Asset in a Pass-through

Suppose, for illustrative purposes, that the market portfolio is simply made up of the riskless asset and one risky asset, a pass-through with coupon c. Their values are respectively V^{Riskless} and V^{Col}. Applying formula (14), we get: $\sigma_M = V^{Col}\sigma^G/(V^{\text{Riskless}} + V^{Col})$. The expressions (13) and (15) then give:

$$\lambda = \gamma' \frac{c - r}{r + \phi + OAS^{IO}} \sigma,$$

with

$$\gamma' = \gamma \frac{V^{Col}}{V^{\text{Riskless}} + V^{Col}}.$$

Using expression (12) to get $OAS^{IO} = -\lambda\sigma$, and expression (10) to get the value of λ as the root of second-order polynomial equation. Hence it can be expressed as a closed, though messy, form. Rather than stating it explicitly, it seems most illuminating to take first order developments, and consider the case where $V^{\text{Riskless}} \ll V^{Col}$, and[5] $OAS^{IO} \ll \phi + r$. Then, we get the first-order development:

$$\lambda \simeq \gamma \frac{\sigma}{r + \phi}(c - r). \tag{16}$$

We get some features that we will find in more complicated versions of the equilibrium: λ is positive in a premium environment, i.e. when $c - r > 0$. The "behavioral" functions ϕ and σ — they depend only on prepayment behavior, and are exogenous in that sense — are both increasing in $c - r$, but the ratio $\sigma/(r + \phi)$ increases mildly with $c - r$, and tends to an asymptote for large $c - r$. The reason for this empirical behavior is intuitive: ϕ has a mean of 20%/year in sample, hence easily dominates c. Also, the uncertainty σ on ϕ is roughly proportional to ϕ itself (see GKV), hence $\sigma/\,(r + \phi)$ is roughly independent of c for large c, and is mildly increasing in c for $c - r$ near r.

The bottom line is that λ is an increasing function of $c - r$, and is approximately linear in $c - r$ when $c - r$ is large.

2.3.3 When the Risky Assets are All Pass-Throughs

Suppose, for illustrative purposes, that the market portfolio is made up of only the riskless asset and several types of collateral with index k. Their

[5] In term of primitive terms, $OAS^{IO} \ll \phi + r$ corresponds to $\gamma\sigma^2(c - r)/\phi \ll \phi + r$. Empirically, this is not a strong approximation.

value is respectively, V^{Riskless} and V_k^{Col}. Applying formula (14), we get:

$$\sigma_M = \frac{\sum_k V_k^{Col} \sigma_k^G}{V^{\text{Riskless}} + \sum_k V_k^{Col}}.$$

Using equations (13) and (15) we get (recall a_k is the outstanding balance of collateral k),

$$\lambda = \gamma' \frac{\sum_k a_k \sigma_k (c_k - r)/(\phi_k + r + OAS_k^{IO})}{V^{\text{MBS}}}, \tag{17}$$

with, $V^{\text{MBS}} = \sum_k V_k^{Col}$ being the total value of the MBS market, and:

$$\gamma' = \gamma \frac{V^{\text{MBS}}}{V^{\text{MBS}} + V^{\text{Riskless}}}.$$

Again, with equation (12) to get $OAS_k^{IO} = -\lambda\sigma_k$, and equation (10), we can get the value of λ. To gain some intuition for this, take first-order developments, and consider the case where $V^{\text{Riskless}} \ll V^{\text{MBS}}$, and $OAS_k^{IO} \ll \phi_k + r$. As argued in the case where there is only one type of collateral, the ratio σ_k/ϕ_k is approximately constant empirically, $\sigma_k/\phi_k \simeq \bar{\sigma}/\bar{\phi}$. Then, we get the first-order development:

$$\lambda \simeq \gamma \frac{\bar{\sigma}}{\bar{\phi}}.(\bar{c} - r), \tag{18}$$

where we introduce the average coupon $\bar{c}$, defined by

$$\bar{c} = \frac{\sum_k a_k c_k}{\sum_k a_k}, \tag{19}$$

i.e. $\bar{c}$ is the average coupon outstanding, the weights being given by the amount outstanding a_k of the securities.

We conclude that the results in the case where there are many pass-throughs in the market is much the same as when there is only one, i.e. as in the previous subsection.[6] One has just to consider the average coupon $\bar{c}$ and a measure of the average "noise-to-signal ratio" in prepayment, $\bar{\sigma}/\bar{\phi}$.

[6] Note that by taking developments of higher orders, we get more precise formulas for λ. For instance, as $OAS_k^{IO} = \lambda\sigma_k$ is quite a bit smaller than $s_k + r$, we can take

$$\lambda = \frac{\gamma}{V^{\text{MBS}}} \sum_k a_k \sigma_k \frac{i_k - r}{s_k + i_k} (1 + O(\frac{\lambda\sigma_k}{s_k + i_k})),$$

or, in next approximation:

3 The Theory in the General Case

3.1 OAS Formulas and Market Equilibrium

We chose to first present the theory in a very simplified context. In the present section, we present the more general formulation of our approach to MBS: we will drop the assumption of constant interest rate, and the special functional form we assumed for the prepayment function. Most insights gained in the simplified case carry over, and some new ones appear. The formulation adopted here will be most convenient for empirical work.

We will call $R(t,\tau) = \int_t^\tau r_u du$ the cumulative interest rate, and $S(t,\tau) = S(\tau) - S(t)$ the cumulative prepayment between t and τ. Again, we consider the original Brownian motion B (expressing the physical probability of prepayment shocks, and the risk-neutral probability for interest rate shocks), and its risk-neutral counterpart $\widehat{B}(t) = B(t) - \int_0^t \lambda(s)\, ds$, with $\lambda(s)$ the market price of prepayment shocks that occur at time s. Recall that to express the correspondence between the two, one introduces

$$Z(t) := \exp(\int_0^t \lambda(s) dB(s) - \frac{1}{2}\int_0^t \lambda(s)^2 ds)$$

and gets evaluations in the risk-neutral probability measure by $\widehat{E}_t[X] = E_t[XZ(T)/Z(t)]$ for X a $\mathcal{F}_T-$measurable variable, where $\mathcal{F}_T = \sigma(B(u), 0 \leq u \leq T)$.

Following the lines of Section 2, we get easily[7]:

$$\lambda \simeq \gamma \sum_k a_k \sigma_k \frac{i_k - r}{s_k + i_k}(1 - \frac{\lambda \sigma_k}{s_k + i_k}),$$

which leads to

$$\lambda \simeq \frac{\gamma \sum_k a_k \sigma_k \left(i_k - r\right) / \left(s_k + i_k\right)}{1 + \gamma \sum_k a_k \sigma_k^2 (i_k - r) / \left(s_k + i_k\right)^2}$$

second-order terms. It does not appear that we get much from using those higher-order formulas: if one wants greater sophistication, it is better in give up at the same time the assumptions of absence of burnout, infinitely long-lived securities, etc. that we made here for tractability, and use the general theory presented in section 3. For our illustrative purposes, the simplified first-order expression (18) is quite enough.

[7] We call ∞ the terminal date of the life of a security; this is just for ease of notation, because all our securities have a finite life, i.e. for all securities k, there is a date T_k at which $a_k(t) = 0$ for all $t > T_k$.

Proposition 7 *The values of the IO, PO, and Collateral are*

$$\begin{aligned} P^{IO}(t) &= a(t)\hat{E}_t \int_t^\infty c\, e^{-R(t,\tau)-S(t,\tau)} d\tau, \\ P^{PO}(t) &= a(t) - a(t)\hat{E}_t \int_t^\infty r_\tau e^{-R(t,\tau)-S(t,\tau)} d\tau, \\ P^{Col}(t) &= a(t) + a(t)\hat{E}_t \int_t^\infty (c - r_\tau) e^{-R(t,\tau)-S(t,\tau)} d\tau, \end{aligned} \tag{20}$$

or, expressed in terms of the OAS:

$$\begin{aligned} P^{IO}(t) &= a(t)E_t \int_t^\infty c\, e^{-R(t,\tau)-S(t,\tau)-\omega^{IO}(\tau-t)} d\tau, \\ P^{PO}(t) &= a(t) - a(t)E_t \int_t^\infty (r_\tau + \omega^{PO}) e^{-R(t,\tau)-S(t,\tau)-OAS^{PO}(\tau-t)} d\tau, \\ P^{Col}(t) &= a(t) + a(t)E_t \int_t^\infty (c - r_\tau - \omega^{Col}) e^{-R(t,\tau)-S(t,\tau)-OAS^{Col}(\tau-t)} d\tau. \end{aligned} \tag{21}$$

Proof. To derive these expressions, for instance, for the PO, we take the definition of its cash-flow, and integrate by parts:

$$P^{PO}(t) = \hat{E}_t \int_t^\infty -da(\tau) e^{-R(t,\tau)} d\tau = a(t) + \hat{E}_t \int_t^\infty r_\tau a(\tau) e^{-R(t,\tau)} d\tau.$$

The other relations are proved similarly. ■

The model of investor behavior is the same before. The investors solves a Merton problem, and, for a large class of utility functions, this leads to the same expression as in Proposition 13, which here becomes $\lambda(t) = -\gamma D_t G^{\mathrm{M}}(t)/V^{\mathrm{MBS}}(t)$, where $G^{\mathrm{M}}(t) := \sum_k G_k^{Col}(t)$ is the gain process for the market as a whole, and $V^{\mathrm{MBS}}(t) := \sum_k P_k^{Col}(t)$ is the total market value. Hence:

$$\lambda(t) = -\frac{\gamma}{V^{\mathrm{MBS}}(t)} \sum_k D_t G_k^{Col}(t). \tag{22}$$

$D_t G^{\mathrm{M}}(t)$ is the Malliavin derivative of $G^{\mathrm{M}}(t)$ (Appendix A gives a primer on Malliavin calculus), and can be simply understood as the impact of a shock dB_t to the prepayment process on the gain process of the holder of the market. As seen in the proof of Proposition 6, we have $dG_k^{Col}(t) = c_k a_k(t)dt - da_k(t) + dP_k^{Col}(t)$, so that $D_t G_k^{Col}(t) = -D_t a_k(t) + D_t P_k^{Col}(t)$. We can therefore define the equilibrium concept:

Definition 8 *An equilibrium is a set of values $\lambda(t)$, $t \geq 0$, such that, defining $\hat{P}$ as the risk neutral probability associated with $\lambda(t)$ (i.e., that makes*

$\hat{B}(t) = B(t) - \int_0^t \lambda(t)dt$ a Brownian motion), for all dates $t \geq 0$, relations (20) and (22) hold, with $D_t G_k^{Col}(t) := -D_t a_k(t) + D_t P_k^{Col}(t)$ and $P^{MBS}(t) := \sum_k P_k^{Col}(t)$.

We will not show the existence of the equilibrium or the Malliavin-differentiability of the equilibrium values of the variables. The reason is simple: The mathematical toolbox that would be required to rigorously derive our results is still largely to be developed[8]. We call for its elaboration. In the mean time, considering that those issues are orthogonal to the focus of this paper - a concrete understanding of the economics of the MBS market - we will take those results as given, and hope that, incidentally, our analysis will provide the motivation for proving these tools. Hence we make the following:

Assumption. *The equilibrium in definition 8 exists and is unique.*

Assuming the existence of the equilibrium, we now proceed to the exploration of its nature. The first step is to express how the values of securities change with prepayment shocks, i.e. to calculate the value of $D_t G_k^{Col}$.

Proposition 9 *The gain process $G_k^{Col}(t)$ for the collateral number k follows:*

$$dG_k^{Col}(t) = r_t P_k^{Col}(t)dt + D_t G_k^{Col}(t)d\hat{B}_t$$

where $D_t G_k^{Col}$ is its Malliavin derivative :

$$D_t G_k^{Col}(t) = -a_k(t)\hat{E}_t \int_t^\infty (c_k - r_\tau) e^{-R(t,\tau)-S_k(t,\tau)} \left(D_t S_k(\tau) - \int_t^\tau D_t \lambda_u d\widehat{B}_u \right) d\tau \tag{23}$$

Proof. By the generalized Clark-Ocone formula (30), we get $dG_k^{Col}(t) = xdt + D_t G_k^{Col}(t)d\hat{B}_t$, for some x. Because under the risk-neutral probability $\hat{P}$ the expected return on the security is r_t, we have $x = r_t P_k^{Col}(t)$. As $a_k(t) = a_k(0)e^{-S_k(t)}$, $D_t a_k(t) = -a_k(t) D_t S_k(t)$. From (20) and the generalized Clark-Ocone formula, and using the notation $C_\tau := (c - r_\tau)e^{-R(t,\tau)-S(t,\tau)}$,

[8] The situation appears considerably more complicated than the other market equilibrium with Malliavin calculus to be found in the literature is Serrat (2001), where the existence of the equilibrium could be reduced to the existence of the existence of two scalars (the Lagrange multipliers of the problem). Here we see no straightforward way reduce the continuous infinity of $\lambda(t)$.

and the fact that $S_k(t,\tau) := S_k(\tau) - S_k(t)$, we get:

$$\begin{aligned} D_t G_k^{Col}(t) &= -D_t a_k(t) + D_t P_k^{Col}(t) \\ &= -D_t a_k(t) + D_t a_k(t)\left(1 + \hat{E}_t \int_t^{\infty} C_\tau d\tau\right) + \\ &\quad a_k(t)\hat{E}_t \int_t^{\infty} C_\tau \left(D_t S_k(\tau) - D_t S_k(t) - \int_t^{\tau} D_t \lambda_u d\widehat{B}_u\right) d\tau \\ &= -a_k(t)\hat{E}_t \int_t^{\infty} C_\tau \left(D_t S_k(\tau) - \int_t^{\tau} D_t \lambda_u d\widehat{B}_u\right) d\tau \end{aligned}$$

which is the result to prove. ■

The interpretation of (23) is the following[9]. A prepayment shock has a direct effect on the value of the security via its impact on the prepayments of the security. For instance, suppose that at time t there has been a positive prepayment shock $dB_t > 0$. In the premium environments $c - r_t > 0$, this hurts the value of the principal, because the total impact will be a higher prepayment: $D_t S_k(\tau) > 0$. Hence, in those premia environments, it is likely that we have both $D_t G_k^{Col}(t) < 0$, and $-a_k(t)\hat{E}_t \int_t^{\infty}(c_k - r_\tau)e^{-R(t,\tau)-S_k(t,\tau)} D_t S_k(\tau) d\tau < 0$.

But there is also a second, indirect effect: A prepayment shock also affects the price of prepayment risk: these are the terms $D_t\lambda_u$, $u \geq t$. Suppose that the securities with the highest coupon $c_{k'}$ prepay disproportionately after a prepayment shock. Then, after a positive prepayment shock, the amount outstanding of those high-coupon securities will decrease disproportionately, so that the average coupon will decrease — which in turn leads the market price of risk $\lambda(u)$ to diminish, as in formula (22). Following that line of reasoning, it is likely that $D_t\lambda_u < 0$. Now, because S increases with B, we get $\hat{E}_t e^{-R(t,\tau)-S_k(t,\tau)} \int_t^{\tau} D_t\lambda_u d\widehat{B}_u > 0$, so that:

$$-a_k(t)\hat{E}_t \int_t^{\infty} (c_k - r_\tau) e^{-R(t,\tau)-S_k(t,\tau)} D_t S_k(\tau) d\tau < D_t G_k^{Col}(t) < 0,$$

$$0 < a_k(t)\hat{E}_t \int_t^{\infty} (c_k - r_\tau) e^{-R(t,\tau)-S_k(t,\tau)} \int_t^{\tau} D_t \lambda_u d\widehat{B}_u d\tau.$$

The second effect, the impact $D_t\lambda_u$ of current prepayments on the future value of the price of prepayment risk λ_u, somewhat dampens the first effect, which is that increased prepayments hurt the value of the collateral (in a premium environment).

[9]The link with the σ_k^G used in section 2 is that $D_t G_k^{Col}(t) = V_k^{Col}(t)\sigma_k^G(t)$.

The market price of risk $\lambda(t)$ depends on the securities outstanding at time t. Hence, its future values $\lambda(\tau)$ depend on the future of the amount outstanding of securities. This depends on which securities will be issued from now on. Hence, a complete theory of MBS must include a model of the issuance of future securities (this is also true of any financial model, of course). The question of the best such model is essentially empirical, and we encourage this question as a topic for future research. To close the model now, the following simple model is natural. First, take notations, and interpret k as the time at which security number k was issued, with an initial amount $a_k(k)$, and a coupon of c_k. Then the relation (22) becomes

$$\lambda(t) = -\frac{\gamma}{V^{\mathrm{MBS}}(t)} \int_{-\infty}^{t} D_t G_k^{Col}(t) dk.$$

To reflect the growing share of mortgages that are securitized, we could write that $a_k(k) = ae^{\gamma k}$, for some $\gamma \geq 0$, but a more appropriate function would take into account the state of the business cycle, seasonalities, and interest rates. The next level of refinement would take into account that part of the prepayments are refinanced. If, indeed, in a fraction ϕ of prepayments, people repay their old mortgage to subscribe to a new one, then a pure prepayment[10] of $dS_k^*(t)$ of security k will increase the issues of securities at time t by: $\phi a_k(t) dS_k^*(t)$, so that the model of the issuance of MBS would become:

$$a_t(t) = ae^{\gamma t} + \phi \int_{-\infty}^{t} a_k(t) \frac{dS_k^*(t)}{dt} dk.$$

Then there would be a positive correlation between the creation of new securities and prepayment shocks, which would refine the predictions on the future λ in interesting ways.

3.2 First-order Expansion of the Expressions

The above expression for $\lambda(t)$ might appear formidable because the current $\lambda(t)$ is a function of the future $\lambda(u)$, as seen in (23) and (22). However, the following remark will help both the theorist and the empirical researcher: at least when the risk premia are small, the value of $\lambda(t)$ does not depend, to the first order, on the future values of $\lambda(u)$. This simplifies considerably both the actual implementation, and the theoretical work.

To be more precise, say that one can decompose the cumulative prepayment $S(t)$ into $S_k(t) = S_k^0(t) + \sigma S_k^1(t) + o(\sigma)$, where $S_k^0(t)$ is a deterministic

[10] See Definition 1 for the notion of pure prepayment.

function of interest rates, burnout etc., σ is a scalar, and $S_k^1(t)$ is the first order noise around the deterministic part $S_k^0(t)$. Hence σ is a measure of the uncertainty around $S_k(t)$, and the limit $\sigma \to 0$, which we shall consider here, correspond to the small uncertainty in $S_k(t)$.

We need a technical assumption to ensure the existence of a useful norm. We will make the hypothesis that the nominal interest rate is always bounded below by a[11] positive number $\delta/2$: $\forall t, r_t \geq \delta/2 > 0$. Then the $L^2([0,+\infty))$ norm we define is: $||u|| := E\left[\int_0^{+\infty} e^{-\delta t} u_t^2 dt\right]^{1/2}$, and we define the associated norm on $\mathcal{D}_{2,1}$ as in Appendix A: $||F||_{2,1} = E[||F||^2 + ||DF||^2]^{1/2}$. This gives the norm that allows us to talk about the developments of the variables powers of σ, the variability of prepayments. It is fairly intuitive that, in a well-behaved equilibrium, the equilibrium values of $\lambda(t)$ will depend smoothly on σ. We will not prove it here[12], but will just assume it. Again, we call for the elaboration of general theorems that proving these smoothness properties for equilibria of the type we consider here. We state:

Proposition 10 *Suppose that S_k and λ are differentiable at $\sigma = 0$, with $S_k(t) = S_k^0(t) + \sigma S_k^1(t) + o(\sigma)$. Then we have, in the limit of small uncertainty in prepayment ($\sigma \to 0$)*

$$D_t G_k^{Col}(t) = -\sigma a_k(t) E_t \int_t^{\infty} (c_k - r_\tau) e^{-R(t,\tau) - S_k(t,\tau)} D_t S_k^1(\tau) d\tau + o(\sigma)$$

When $D_t S_k^1(\tau) = \sigma_k$, we have: Hence, via formula (22), we get a first-order development of $\lambda(t)$:

$$\lambda(t) = \gamma \frac{\sum_k a_k(t) \left(P_k^{Col} - 1\right) Q_k^{Col} \sigma_k}{\sum_k a_k(t) P_k^{Col} Q_k^{Col}} + o\left(\sigma\right). \tag{24}$$

Proof. See Appendix B. ■

This suggests the following way of approximating λ. Assuming that $S_k(t)$ can be developed in integer powers of σ, doing the same analysis as above, we get with (24) an approximation of λ_1 of λ to $O(\sigma^2)$ terms. We

[11] This technical assumption, which is not very demanding (say that $r_t \geq 1$ basis point), could be itself weakened: the nominal interest rate could become null, or even negative (which is possible if carrying cash if costly or dangerous) - but only for limited periods of time, and then would have to go back above a normal, positive floor.

[12] Again, the existence, uniqueness and smoothness of the type of equilibrium we see here should amount to a sizable and interesting mathematical agenda.

take this first-order expansion λ_1 of λ, plug it into (23), and using (22), we get a second-order expansion – up to $O(\sigma^3)$ terms – λ_2 of[13] λ.

The advantage of expression (24) is that we do not need to know the future $\lambda(u)$ to know, to a first approximation, the current $\lambda(t)$. This is useful both for empirical work, and to clarify the theoretical analysis. A side payoff of this development is that we can provide a justification for the simplified model we used in section 2. More precisely, we can prove:

Proposition 11 *Suppose that the interest rates follows a process* $dr_t = \varepsilon(\mu_r dt + \sigma_r dW_t)$ *for some bounded adapted processes* μ_r, σ_r*, a Brownian motion* W_t*, and a scalar* ε*, and, as in section 2 of the paper* $dS_k(t) = (\phi(r_t) + \sigma_k(r_t)^2/2)dt + \sigma_k(r_t)dB_t$*, with* $\sigma_k(r_t) = \sigma v_k(r_t)$*,* $v_k(r_t)$ *being some bounded adapted process. Then, under the assumptions of Proposition 10, and in the limit where prepayment uncertainty is small (*σ *is in a neighborhood of 0), and the interest rates move slowly (*ε *is a neighborhood of 0), the approach of section 2 is correct (the expressions found in it are those of the general theory of this section), up to lower order terms in* ε*,* σ*. In particular,*

$$\lambda(t) = \frac{\gamma}{V^{MBS}(t)} \sum_k a_k(t) \frac{\sigma_k}{r_t + \phi(r_t)} (c_k - r_t) + o(\sigma) + o(\varepsilon) \tag{25}$$

Proof. The proof is the same as for Proposition 10, but only simpler. ■

The conclusion is that the "quasi-static" approach of section 2 is correct, up a first order. As was intuitive, this vindicates the thought experiments we considered, when we assumed constant interest rates, derived an expression for $\lambda(t)$, and saw how it changes with the interest rate. This approximation corresponded to slow movements in the interest rate[14].

4 Conclusion

We have shown that the first order expansion of the continuous time model yields the a formula for the price of risk λ_t that is the same as the two period model of Gabaix, Krishnamurthy and Vigneron 2004. Our continuous time

[13] It is plausible that an iteration of this "Picard" procedure would make the estimates λ_n of λ converge to their true value, but no check of this will be attempted here.

[14] We could add here a whole analysis trying to analyze quantitatively the quality of this approximation (which, by the way, is likely to be good up to $O(\varepsilon^2)$ terms, given the the nominal interest rate is relatively close to a random walk). Back of the envelope calculations suggest their quality is quite good indeed, and the good fit revealed by the empirical analysis confirms this.

formulation is likely to be of some use in practical pricing of MBS. When we venture into the further terms in the expansion, interesting economic effects arise. For instance, an increase in prepayment lowers the value of future average coupon outstanding, hence lower the future value of the price of risk. This shows up as corrective terms in the expressions of section 3. In practice, though the first order formulation (1)-(2) is the one that is likely to be most useful.

5 Appendix A: Elements of Malliavin Calculus

In this appendix we present the notions of Malliavin calculus useful for this paper, following Detemple and Zapatero (1991) and Nualart (1991). Ocksendal (1996) provides a pedagogical introduction. Malliavin (1997) and Nualart (1995) offer in-depth treatments. Other papers using Malliavin calculus include Detemple and Zapatero (1991), Fournié et al. (1999), Ocone and Karaztas (1991) and Serrat (2001).

Definition Take the d-dimensional Wiener space $(B, \mathcal{B}(W_0), P)$, where W_0 is the space of continuous functions $w : [0,T] \to \mathbb{R}^d$ such that $w(0) = 0$, $\mathcal{B}(W_0)$ is the associated Borel σ-field, and P is the Wiener measure on $(W_0, \mathcal{B}(W_0))$. A measurable function on that space is called a Brownian functional.

Let $\mathcal{S}$ be the class of smooth Brownian functionals, i.e. random variables of the form

$$F = f\left(\int_0^T h_1(u)dB_u, ..., \int_0^T h_n(u)dB_u\right), \tag{26}$$

where $h_1, ..., h_n \in L^2(0,T)$, $f \in C^\infty(\mathbb{R})$, for some $T \in \mathbb{R}$.

The Malliavin derivative of such a process is the stochastic process $(D_tF, 0 \leq t \leq T)$ defined by:

$$D_tF = \sum_{c=1}^{n} \frac{\partial f}{\partial x_c}\left(\int_0^T h_1(u)dB_u, ..., \int_0^T h_n(u)dB_u\right) h_c(t). \tag{27}$$

The intuitive meaning of the derivative is useful to keep in mind: D_tF can be interpreted as the change in the F caused by a change of the Brownian motion B at time t. For example, $D_tW_s = 1_{t\leq s}$, and $D_t(\int_0^T \mu_u dt + \sigma_u dB_u) = \sigma_t$ if the μ_u and σ_u are deterministic (the Malliavin derivative of deterministic functions is 0).

Let $||\cdot||$ denote the $L^2(0,T)$ norm. Introduce the norm $||F||_{2,1} = E[||F||^2 + ||DF||^2]^{1/2}$ on $\mathcal{S}$, and we denote $\mathcal{D}_{2,1}$ the Sobolev space which is the closure of $\mathcal{S}$ under $||\cdot||_{2,1}$. We can extend in a natural way the definition of the Malliavin derivative on $\mathcal{D}_{2,1}$.

Chain rule The chain rule of Malliavin calculus is formally an immediate consequence of the definition. Let $F = (F^1, ..., F^m)$, $F^n \in \mathcal{D}_{2,1}$, and consider $\phi : \mathbb{R}^m \to \mathbb{R}$, continuously differentiable with bounded partial derivatives. Then $\phi(F) \in \mathcal{D}_{2,1}$, and:

$$D_t\phi(F) = \sum_{n=1}^{m} \frac{\partial \phi}{\partial F^n} D_t F^n. \tag{28}$$

As an example, this gives $D_t W_s^2 = 2W_s 1_{t \leq s}$. Likewise, $D_t(\int_0^T \mu_u dt + \sigma_u dB_u) = \sigma_t + \int_t^T D_t\mu_u du + D_t\sigma_u dB_u$ when coefficients μ_u and σ_u are stochastic.

Clark-Ocone formula and generalized Clark-Ocone formula Call the expectation of a variable X at time t: $E_t[X] := E[X|\mathcal{F}_t]$,where $\mathcal{F}_t = \sigma(B(u), 0 \leq u \leq t)$. The Clark-Ocone formula states[15]:

$$F = E[F] + \int_0^T E_s[D_s F] dB_s. \tag{29}$$

Hence the Clark-Ocone formula allows us to evaluate as Ito processes expressions of the type $V(t) := E_t[F]$, which is obviously quite useful in finance. Indeed, for this $V(t)$, we get $V(t) = E[F] + \int_0^t E_s[D_s F] dB_s$, and $dV(t) = E_t[D_t F] dB_t$.

The Generalized Clark-Ocone formula (Ocone and Karatzas, 1991) extends this to the case representation under an equivalent (Girsanov) probability measure. From the original Brownian motion B, define a new (e.g., in the applications, risk-neutral) one $\widehat{B}$ by $\widehat{B}(t) = B(t) - \int_0^t \lambda(s) ds$. With $Z(t) := \exp\left(\int_0^t \lambda(s) dB(s) - \frac{1}{2}\int_0^t |\lambda(s)|^2 ds\right)$, we get the new probability measure by $\widehat{E}[X] = E[XZ(T)]$ for X in $\mathcal{F}_T$. The generalized Clark-Ocone formula then reads:

$$F = \widehat{E}[F] + \int_0^T \widehat{E}_s \left[D_s F + F \int_s^T D_s \lambda(u) d\widehat{B}(u) \right] d\widehat{B}(s). \tag{30}$$

As a simple exercise in Malliavin calculus, we advise the reader to derive (at least formally) this formula from the original Clark-Ocone formula (29) (use the chain rule). Malliavin calculus has other properties, notably the integration by part formula used extensively in Fournié et al. (1999), but we will not need those here.

[15] An elementary proof of the Clark-Ocone formula works the following way. First, it is easy to prove (29) in the case where $f(y_1, ..., y_n) = \exp(\sum_{i=1}^n \nu_i y_i)$, for some ν_i. Then the linearity of the Clark-Ocone formula and density arguments allow to conclude.

6 Appendix B: Longer Derivations

6.1 Derivation of Proposition 10

Because $S_k(t)$ and $\lambda(t)$ are differentiable at $\sigma = 0$, and that for $\sigma = 0$, we have $\lambda = 0$, we can express:

$$\begin{aligned} S_k(t) &= S_k^0(t) + \sigma S_k^1(t) + \sigma S_k^R(t) \\ \lambda(t) &= \sigma \lambda^1(t) + \sigma \lambda^R(t) \end{aligned}$$

where the remainders $||S_k^R||$ and $||\lambda^R||$ tend to 0 when $\sigma \to 0$, and the norm is the norm $||\cdot||_{2,1}$ defined in the text above Proposition 10.

Here $S_k^0(t)$ is a deterministic function of interest rates, burnout etc., σ is a scalar, and $S_k^1(t,\sigma)$ is the noise around the deterministic part $S_k^0(t)$. Hence σ is a measure of the uncertainty around $S_k(t)$, and the limit $\sigma \to 0$, which we shall consider here, correspond to the small uncertainty in $S_k(t)$.

Expressing (23) in the physical probability gives:

$$\begin{aligned} D_t G_k^{Col}(t) &= -a_k(t) E_t \int_t^{T_k} (c_k - r_\tau) e^{-R(t,\tau) - S_k(t,\tau)} e^{-\int_t^\tau \lambda_u dB_u - \lambda_u^2/2du} \qquad (31) \\ & \left(D_t S_k(\tau) - \int_t^\tau D_t \lambda_u (dB_u - \lambda_u du) \right) d\tau \end{aligned}$$

where T_k is the date at which security k expires, i.e. $a_k(T_k) = 0$. We can write: $D_t G_k^{Col}(t) = -a_k(t)(A + C)$, with

$$\begin{aligned} A &= E_t \int_t^{T_k} (c_k - r_\tau) e^{-R(t,\tau) - S_k(t,\tau)} e^{-\int_t^\tau \lambda_u dB_u - \lambda_u^2/2du} D_t S_k(\tau) d\tau \\ &= \sigma E_t \int_t^{T_k} (c_k - r_\tau) e^{-R(t,\tau) - S_k(t,\tau)} e^{-\int_t^\tau \lambda_u dB_u - \lambda_u^2/2du} (D_t S_k^1(\tau) + D_t S_k^R(t)) d\tau \\ &= \sigma E_t \int_t^{T_k} (c_k - r_\tau) e^{-R(t,\tau) - S_k^0(t,\tau)} D S_k^1(t) d\tau + o(\sigma) \end{aligned}$$

and

$$\begin{aligned}
C &= E_t \int_t^{T_k} (c_k - r_\tau) e^{-R(t,\tau) - S_k(t,\tau)} e^{-\int_t^\tau \lambda_u dB_u - \lambda_u^2/2du} \left(\int_t^\tau D_t \lambda_u (dB_u - \lambda_u du) \right) d\tau \\
&= E_t \int_t^{T_k} (c_k - r_\tau) e^{-R(t,\tau) - S_k(t,\tau)} e^{-\int_t^\tau \lambda_u dB_u - \lambda_u^2/2du} \left(\int_t^\tau D_t \lambda_u dB_u \right) d\tau + O(\sigma^2) \\
&= E_t \int_t^{T_k} (c_k - r_\tau) e^{-R(t,\tau) - S_k^0(t,\tau)} \left(\int_t^\tau D_t \lambda_u dB_u \right) d\tau + O(\sigma^2) \\
&= E_t \int_t^{T_k} (c_k - r_\tau) e^{-R(t,\tau) - S_k^0(t,\tau)} E_t \left[\int_t^\tau D_t \lambda_u dB_u | r_v, v \le \tau \right] d\tau + O\left(\sigma^2\right) \\
&= O\left(\sigma^2\right)
\end{aligned}$$

as $E_t \left[\int_t^\tau D_t \lambda_u dB_u | r_v, v \le \tau \right] = 0$. Thus we get:

$$D_t G_k^{Col}(t) = -\sigma a_k(t) E_t \int_t^\infty (c_k - r_\tau) e^{-R(t,\tau) - S_k^0(t,\tau)} D_t S_k(\tau) d\tau + o(\sigma)$$

which was the expression to prove.□

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