# Linearity-Generating Processes: A Modelling Tool Yielding Closed Forms for Asset Prices

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**Abstract**

This methodological paper presents a class of stochastic processes with potentially appealing properties for theoretical and empirical work in finance and macroeconomics, the "linearity-generating" class. Its key property is that it yields simple exact closed-form expressions for stocks and bonds, with an arbitrary number of factors. It operates in discrete and continuous time. Controlling for the covariance with the stochastic discount factor, the distribution of many disturbances does not affect stock or bond prices, which simplifies the modeller's task. The paper presents a series of illustrative examples, including stocks with stochastic risk premia or stochastic dividend growth rates, macroeconomic environments with changing trend growth rates, and yield curve analysis.

KEYWORDS: Modified Gordon growth model, stochastic discount factor, growth rate risk, interest rate processes, yield curve, bond premia, equity premium, factor models.

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# 1 Introduction

This methodological paper defines and analyzes a class of stochastic processes that has a number of potentially attractive properties for economics and finance, the "linearity-generating" (LG) processes. The LG class generates closed-form solutions for the prices of stocks and bonds. It is simple and flexible, applies to an arbitrary number of factors with a rich correlation structure, and works in discrete or continuous time. These features make it an easy-to-use tool for pure and applied financial modelling.

The main advantage of the LG class is that it generates, with little effort, tractable multifactor stock and bond models, in a way that incorporates stochastic growth rates of dividends, and a stochastic equity premium. Stock and bond prices are linear in the factors – hence the name "linearity-generating" processes.

A few moment conditions have to be verified for a process to be in the LG class (see Eq. 8-9). Given only those moments, one can price stocks and bonds (i.e., finite maturity claims on dividends). Higher order moments (e.g., the distribution of the noise of the factors) do not matter. In many applications, controlling for the covariance with the stochastic discount factor, the variance of processes can be changed almost arbitrarily and the prices will not change. The fact that a few moments are enough to derive prices makes modelling easier.

Linearity-generating processes are meant to be a practical tool for several areas in economics. They are likely to be useful in: macroeconomics, with models with stochastic trend growth rate or probability of disaster; asset pricing, with models with stochastic equity premium, interest rate, or earnings growth rate.

Several literatures motivate the need for a tool such as LG processes. Many recent studies investigate the importance of long-term risk for asset pricing and macroeconomics, e.g., Bansal and Yaron (2004), Croce, Lettau and Ludvigson (2006), Gabaix and Laibson (2002), Hansen, Heaton and Li (2008), Hansen and Scheinkman (2009), Julliard and Parker (2004). The LG processes offer a way to model long-term risk, while keeping a closed form for stock prices. In addition, there is debate about the existence and mechanism of the time-varying expected stock market returns, e.g., Campbell and Shiller (1988), Cochrane (2008) and many others. Because of the lack of closed forms, the literature relies on simulations and approximations. The LG processes offer closed forms for stocks with time-varying equity premium, which is useful for thinking about those issues.

The motivation for the LG class is inspired by the broad applicability and empirical success of the affine class identified by Duffie and Kan (1996), and further developed by Dai and Singleton (2000) and Duffie, Pan and Singleton (2000), which includes the Vasicek (1977) and the Cox, Ingersoll, Ross (1985) processes as special cases. Much theoretical and empirical work is done with the affine class. Some of this could be done with the LG class. Section 5.3 develops the link between the LG class and the affine class. The two classes give the same quantitative answers to a first order. The main advantage of the LG class is for stocks. The LG class gives a simple closed-form expression for stocks, whereas the affine class needs to express stocks as an infinite sum. Hence, while the affine class can be expected to be remain for long the central model for options and bonds, one can think that the LG class may be a auxiliary technique for bonds, but will be particularly useful for stocks.

Closed forms for stocks, or perpetuities, are not available with the current popular processes, such as the affine models of Ornstein-Uhlenbeck / Vasicek (1977) and Cox, Ingersoll, Ross (1985), or models in the affine class (Duffie and Kan 1996). Those models simply generate an infinite sum of terms. Several papers have derived closed forms for stocks. Bakshi and Chen (1996) derive a closed form, which is an exponential-affine function of a square root process. Mamayski (2002) derives another closed form, though in a non-stationary setting. Cochrane, Longstaff and Pedro Santa (2008) contains nice closed form solutions. We confirm results from Mele (2003, 2007), who obtains general results (particularly with one factor) for having bond and stock prices that are convex, concave, or linear in the factors. LG processes satisfy Mele's conditions for linearity. Mele, however, did not derive the closed forms for stocks and bonds in the linear case.

Linear expressions in asset prices are in Bhattacharya (1978), Buraschi and Jiltsov (2007), Veronesi (2000), Menzly, Santos and Veronesi (2004)[1], Santos and Veronesi (2006).[2] Their process turns out to belong to the LG class (see Example 9). Indeed, we show that *any* process yielding linear expressions for fine-maturity claims has to belong to the LG class. In view of those earlier findings, the present paper does two things. First, it defines and analyzes the unified class that underlines disparate results of the literature (as Duffie and Kan (1996) did for affine processes that unified pockets of tractability exemplified by Vasicek

[1] It is indeed the Menzly, Santos and Veronesi (2004) paper that alerted me to the possibility of a class with closed forms for stocks.

[2] After the initial submission of this paper, Philip Dybvig told me about never-typed notes with Jonathan Ingersoll that contained other linear expressions. However, they do not seem to have obtained the general LG structure.

(1977) and Cox, Ingersoll and Ross (1985)). Second, it proposes what appears to be some novel processes, such as those using the "linearity-generating twist".

Finally, we contribute to the vast literature on interest rate processes, by presenting a new, flexible process. The main advantage is probably that, because the LG processes are so easy to analyze, they lend themselves easily to economic analysis. Potential disadvantages are discussed later in the paper. Using the LG class, Gabaix (2009) develops a model of stocks and bonds, and Farhi and Gabaix (2009) a model of exchange rates and the forward premium puzzle.

This paper follows a productive literature that (proudly) reverse-engineers processes for preferences and payoffs, e.g., Campbell and Cochrane (1999), Cox, Ingersoll, Ross (1985), Liu (2007), Ljungqvist and Uhlig (2000), Pastor and Veronesi (2005), Ross (1978), Sims (1990), Veronesi (2000), and, particularly, Menzly, Santos and Veronesi (2004). Indeed, the two LG moment conditions of Definition 2 give a recipe to "reverse-engineer" processes to ensure tractability.

Section 2 is a gentle introduction to LG processes. Section 3 contains the basic results of the paper. Section 4 presents a variety of examples illustrating LG processes. Section 5 presents some additional results, notably on the range of admissible conditions, and on the projection of non-LG processes onto LG processes. Section 6 concludes.

## 2 A Simple Introduction to Linearity-Generating Processes

To motivate LG processes, this section presents a very simple, almost trivial example – the Gordon formula in discrete time.[3] We want to calculate the price $P_t = E_t\left[\sum_{s=1}^{\infty} D_{t+s}/\left(1+r\right)^s\right]$ of a stock with dividend growth:

$$\frac{D_{t+1}}{D_t} = \left(1+g_*\right)\left(1+x_t\right) + \varepsilon_{t+1}, \tag{1}$$

[3] This example is so simple that it would not be surprising if it had already been done elsewhere, even though I did not find it in the previous literature. For instance, a referee pointed out that the process leads to an ARIMA in $D_t$, which has been developed e.g. in Hansen and Sargent (1991). Still, process (5) seems new. In any case, it seems quite certain that the class of LG processes (including the general structure with several factors, stocks bonds and continuous time), as a class, is identified and analyzed in the present paper for the first time.

where $r > 0$ is the riskless rate (later, we will add risk premia), $g_* \in (-1, r)$ is the trend growth rate, and $x_t$ is a deviation of the growth rate from trend, which may be autocorrelated, and $\varepsilon_{t+1}$ has mean 0. This is a prototypical example of a stock with stochastic trend growth. Even in this example, the usual processes for $x_t$ typically yield infinite sums of exponential terms, so they are more cumbersome particularly for paper-and-pencil theory.

Let us reverse engineer the process for $x_t$, and see if the price-dividend ratio can have the form:

$$\frac{P_t}{D_t} = A + Bx_t \tag{2}$$

for some constants $A$ and $B$. The no-arbitrage equation for the stock is $P_t = \frac{1}{1+r} E_t \left[ D_{t+1} + P_{t+1} \right]$ i.e.

$$\frac{P_t}{D_t} = \frac{1}{1+r} E_t \left[ \frac{D_{t+1}}{D_t} \left( 1 + \frac{P_{t+1}}{D_{t+1}} \right) \right]. \tag{3}$$

Plugging in (1) and (2), and assuming that $E\left[\varepsilon_{t+1}\right] = E_t\left[\varepsilon_{t+1} x_{t+1}\right] = 0$, the no-arbitrage equation reads:

$$A + Bx_t = \frac{1}{1+r} E_t \left[ (1+g_*)(1+x_t)(1 + A + Bx_{t+1}) \right], \text{ i.e.}$$

$$A + Bx_t = \frac{1+g_*}{1+r} \left( (1+x_t)(1+A) + (1+x_t) E_t\left[x_{t+1}\right] B \right). \tag{4}$$

If $x_t$ is an autoregressive process of order 1 (AR(1)), i.e. $E_t\left[x_{t+1}\right] = \rho x_t$, then (4) cannot hold: we have linear terms on the left-hand side, and a non-linear term $(1+x_t) E_t\left[x_{t+1}\right]$ on the right-hand side. However, (4) can hold if we postulate that $x_t$ follows the following "twisted" AR(1), with $|\rho| < 1$:

$$\text{Linearity-generating twist: } E_t\left[x_{t+1}\right] = \frac{\rho x_t}{1+x_t}, \tag{5}$$

If $x_t$ is close to 0, then to the first order, $E_t\left[x_{t+1}\right] \sim \rho x_t$, so that $x_{t+1}$ behaves approximately like an AR(1). It is a twisted AR(1), because of the term $1+x_t$ in the denominator. However, in many applications, $x_t$ will be within a few percentage points from 0, so materially, the twist is small (more on this later). In any case, process (5) is meant to be a stand-alone modelling proposal, rather than the approximation of another process. If (5) holds, then (4) reads:

$$A + Bx_t = \frac{1+g_*}{1+r} \left( (1+x_t)(1+A) + \rho x_t B \right),$$

which features only linear terms, and admits a solution. Indeed, identifying the constant and $x_t$ terms, we obtain $A = \frac{1+g_*}{1+r}(1+A)$, i.e. $A = (1+g_*)/(r-g_*)$, and $B = \frac{1+g_*}{1+r}[1+A+\rho B]$, i.e. $B = A/\left(1-\rho\frac{1+g_*}{1+r}\right)$. Finally, plugging those values of $A$ and $B$ back in (2) gives:

$$\frac{P_t}{D_t} = \frac{1+g_*}{r-g_*}\left(1+\frac{x_t}{1-\rho\frac{1+g_*}{1+r}}\right). \tag{6}$$

We conclude that (6) solves (3). It is actually easy to show that the stock price satisfies (6): by induction on $T$, one shows that for all $T \geq 0$, $E_t[D_{t+T}] = (1+g_*)^T\left(1+\frac{1-\rho^T}{1-\rho}x_t\right)D_t$, and direct calculation yields (6). We note that (6) is a generalization of the Gordon growth formula with constant growth rate (which is simply $\frac{P_t}{D_t} = \frac{1+g_*}{r-g_*}$) to the case of time-varying growth rate. [4] We gather our result in Example 1.

**Example 1** *(Simple stock example with LG stochastic trend growth rate). Consider a stock with dividend growth rate $x_t$, with $D_{t+1}/D_t = (1+g_*)(1+x_t)+\varepsilon_{t+1}$, where $\varepsilon_{t+1}$ has mean 0 and is uncorrelated with $x_{t+1}$, with the linearity-generating "twist" for the growth rate, $E_t[x_{t+1}] = \rho x_t/(1+x_t)$, and price $P_t = E_t\left[\sum_{s=1}^{\infty} D_{t+s}/(1+r)^s\right]$. Suppose that, with probability 1, $\forall t, x_t > -1$. Then, the price-dividend ratio, $P_t/D_t$ is given by (6). The rest of the paper develops this systematically.*

This example illustrates several general traits of LG processes. Eq. 5 imposes just one moment condition. Higher order moments do not matter for the price. For instance, we could have a complicated nonlinear function for the variance of the growth rate, but it would not affect the stock price. Likewise, the distribution of the noise does not matter, so that one can have jumps and the like, without changing the price. This may be useful in many cases, though there might be drawback to that, as discussed in section 5.3. In LG models with risk premia, the covariance of variables with the stochastic discount factor will of course matter for the price, but most other moments (i.e., those that do not explicit figure in the LG moments (8)-(9) below) will not matter.

We need restrictions on the domain of $x_t$. Mostly obviously, one needs $x_t > -1$. Actually, the stronger condition $x_t > \rho - 1$ is needed (see section 5.1). In particular, the variance of

[4] The economic interpretation of (6) is straightforward. A stock with an unually high current growth rate $x_t$ should have a high price. The effect is larger the growth rate is more persistent (high $\rho$), and the future is discounted less (low $r$ or high $g_*$).

$x_t$ has to go to 0 near that boundary.

With the affine models of Duffie and Kan (1996), we might model: $D_{t+1}/D_t = e^{g_* + x_t}$, $x_{t+1} = \rho x_t + \varepsilon_{t+1}$. That would lead to $E_t\left[D_{t+T}/D_t\right] = e^{a(T)+b(T)x_t}$, for some functions $a(T)$, $b(T)$, and finally: $\frac{P_t}{D_t} = \sum_{T \geq 1} e^{a(T)+b(T)x_t}$. We get an infinite sum over maturities, rather than the compact expression (6). Hence, LG processes are particularly tractable for stocks.

The twisted process (5) is similar to an AR(1), $E_t x_{t+1} = \rho x_t$, up to second order terms. Hence, the behavior is likely to be close to an AR(1). To illustrate this, the online appendix to this paper reports the simulation of the above example, with and without the twisted terms. The values for the growth rates are quite close, and hard to distinguish visually. Likewise, the associated price-dividend ratios are quite close. Of course, even if they had been quite different, this would not have been an important drawback for LG processes. We do not want to say that the true model is an AR(1), that a LG process approximates. It could as well be that the true model is a LG process, than an AR(1) model approximates. Or rather, as a model is just idealization of a complex economic reality, the respective advantage of LG vs. affine models depends on the specific task at hand. The modeler should be able to pick whichever modelling idealization is most expedient, and LG processes offer one such choice.

Finally, from the regular Gordon formula, the reader might expect the stock price at time $t$ to be a convex function of $x_t$. It is true that it is a convex function of the future growth rates $(x_{t+s})_{s \geq 1}$. However, the LG twist (5) makes future growth rates be a concave function of the initial growth rates $x_t$. That is how the LG stock price, as it compounds a concave and a convex relation, might a priori be concave or convex function of $x_t$ (see Mele 2003, 2007). With the LG twist, it is exactly a linear function of the initial growth rate.

We now start our systematic treatment of LG processes.

# 3 Basic Theory

We fix a probability space $\left(\Omega^P, \mathcal{F}, P\right)$ and an information filtration $\mathcal{F}_t$ satisfying the usual technical conditions (see, for example, Karatzas and Shreve 1991). A process $(x_t)_{t \geq 0}$ is $L^1$ if it is integrable and (i) in discrete time, $\forall T > 0$, $E_0\left[\sum_{t=0}^{T} |x_t|\right] < \infty$, or (ii) in continuous time, $\forall T > 0$, $E_0\left[\int_0^T |x_t|\, dt\right] < \infty$.

We want to price an asset with dividend process $D_t$, given a discount factor process $M_t$.[5] The price at time $t$ of a claim yielding a stochastic dividend $D_{t+T}$ at maturity $T > 0$ is $P_t = E\left[\sum_{T=1}^{\infty} M_{t+T}D_{t+T}\right]/M_t$ if time is discrete, $P_t = E_t\left[\int_0^{\infty} M_{t+T}D_{t+T}dT\right]/M_t$ if time is continuous. For instance, the price at $t$ of a ("zero coupon") bond yielding 1 in $T$ periods is: $Z_t(T) = E_t[M_{t+T}]/M_t$. Throughout the paper, the number of factors $n$ is a positive integer.

## 3.1 Linearity-Generating Processes in Discrete Time

We start with the following definition.

**Definition 1** *(Abstract version of LG processes). A LG process is quadruplet* $\left(\Omega, \nu, (Y_t)_{t\geq 0}, (M_tD_t)_{t\geq 0}\right)$ *with* $\Omega$ *a* $(n+1)\times(n+1)$ *matrix (called the generator of the process),* $\nu = (1, 0, \ldots, 0)' \in \mathbb{R}^{n+1}$*, a* $L^1$ *state vector process* $(Y_t)_{t=0,1,\ldots}$ *with values in* $\mathbb{R}^{n+1}$*, and a process* $M_tD_t$ *with non-zero values such that for all* $t \in \mathbb{N}$, $M_tD_t = \nu' Y_t$ *and*

$$E_t[Y_{t+1}] = \Omega Y_t. \tag{7}$$

Hence, the (dividend-augmented) stochastic discount factor of a LG process is simply the first component of an autoregressive process, $Y_t$. The tractability of LG processes comes from the tractability of autoregressive processes.

Definition 1 is a bit abstract. In practice, it is often easier to apply the following (and, we will see, equivalent) definition. We consider a state vector $X_t \in \mathbb{R}^n$ which can generally be thought of as stationary, while $M_tD_t$ generally has a trend, and is not stationary. The definition of the LG process is the following.

**Definition 2** *(Concrete version of LG processes). The process* $M_tD_t(1, X_t')_{t=0,1,2,\ldots}$*, with* $M_tD_t \in \mathbb{R}\smallsetminus\{0\}$ *and* $X_t \in \mathbb{R}^n$*, is a linearity-generating process if it is* $L^1$ *and there are*

[5] The simplest example is $M_t = (1+r)^{-t}$. If a consumer with utility $\sum_t \delta^t U(C_t)$ prices assets, then $M_t = \delta^t U'(C_t)$. Also, some authors call $M_{t+1}/M_t$ the "stochastic discount factor". In the present article, there is no confusion.

*constants $\alpha \in \mathbb{R}, \gamma, \delta \in \mathbb{R}^n, \Gamma \in \mathbb{R}^{n\times n}$, such that the following relations hold at all $t \in \mathbb{N}$:*

$$E_t\left[\frac{M_{t+1}D_{t+1}}{M_tD_t}\right] = \alpha + \delta' X_t, \tag{8}$$

$$E_t\left[\frac{M_{t+1}D_{t+1}}{M_tD_t}X_{t+1}\right] = \gamma + \Gamma X_t. \tag{9}$$

To interpret (8), consider first the case of bonds, $D_t \equiv 1$; Eq. 8 says that the properly-defined interest rate is linear in the factors. When $M_t = (1+r)^{-t}$ with general $D_t$, (8) says that expected dividend growth is linear in the factors. In general, (8) means that the expected value of the (dividend augmented) stochastic discount factor growth is linear in the factors.[6]

Condition (9) means that $X_t$ follows a "twisted" AR(1). It behaves in some sense like $E_t[X_{t+1}] = \gamma + \Gamma X_t$, but it is twisted by the $\frac{M_{t+1}D_{t+1}}{M_tD_t}$ term.

What kinds of models are compatible with Definition 2? As the examples below show, it is not difficult to write toy economic models satisfying conditions (8)-(9), e.g. in Lucas (1978) and Campbell-Cochrane (1999) economies with exogenous consumption, dividend or marginal utility processes, or models with learning. Farhi and Gabaix (2009) and Gabaix (2009) present fully worked-out general-equilibrium macroeconomic models satisfying the LG conditions.[7] Indeed, conditions (8)-(9) give a prescription to "reverse-engineer" macro or micro fundamentals, so as to make the model tractable: The modeler has to make sure that the endowment, technology etc. is such that (8)-(9) hold.[8]

To see the link between the two definitions, consider the elements of Definition 2, and define

$$\Omega \equiv \begin{pmatrix} \alpha & \delta' \\ \gamma & \Gamma \end{pmatrix}, \tag{10}$$

$$Y_t \equiv \begin{pmatrix} M_tD_t \\ M_tD_tX_t \end{pmatrix} = (M_tD_t, M_tD_tX_{1t}, \cdots, M_tD_tX_{nt})', \tag{11}$$

[6]Hence, if the interest rate is constant, $y_t = D_t(1, X_t)$ is an autoregressive process under the risk-neutral probability induced by $M_t$.

[7]So cash-flows and discount factors may be intertwined (as they typically are in general equilibirum), or not with LG models. The present work and Gabaix (2009) contains examples of both situations.

[8]In addition, models that do not directly fit into the conditions of Definition 2, could be approximated by projecting linearly in (8)-(9), as we will discuss later.

so that the vector $Y_t$ stacks all the information relevant to the prices of the claims derived below.[9] Conditions (8)-(9) can be written (7). Hence, Definition 2 implies Definition 1. Conversely, Definition 1 implies Definition 2 by defining $\alpha, \gamma, \delta$ and $\Gamma$ as in (10).[10] The two definitions are equivalent.

The basic pricing properties are the following two theorems.

**Theorem 1** *(Bond prices, discrete time). The price-dividend ratio of a zero-coupon equity or bond of maturity $T$, $Z_t(T) = E_t[M_{t+T}D_{t+T}]/(M_t D_t)$, is*

$$Z_t(T) = \begin{pmatrix} 1 & 0_n \end{pmatrix} \Omega^T \begin{pmatrix} 1 \\ X_t \end{pmatrix}. \tag{12}$$

*When $\gamma = 0$ (i.e., when the process $X_t$ is centered around 0) this simplifies to*

$$Z_t(T) = \alpha^T + \delta' (\alpha I_n - \Gamma)^{-1} \left(\alpha^T I_n - \Gamma^T\right) X_t. \tag{13}$$

*In the above expressions, $I_n$ is the identity matrix of dimension $n$, and $0_n$ is the row vector with $n$ zeros.*

For instance, when $D_t \equiv 1$, the above theorem can price bonds, with $n$ factors, in closed form. Theorem 1 highlights that when $\gamma = 0$, a simplification arises. The case $\gamma = 0$ means the state variables are re-centered around 0, which is easy to do in practice, as the examples below will illustrate.

The second main result is the most useful property of LG processes: the existence of a closed-form formula for stock prices.

**Theorem 2** *(Stock prices, discrete time). Suppose that $\Omega$'s eigenvalues have modulus less than 1 (finiteness of the price). Then, the price-dividend ratio of the stock, $P_t/D_t$ =*

[9] Other assets, e.g. options, require of course to know more moments.

[10] In addition, Definition 2 can be used with a different $\nu$, which is sometimes useful. But it is equivalent to the definition with $\nu = (1, 0, ..., 0)'$ by reordering the basis for the vector of $Y_t$.

$E_t\left[\sum_{s=t+1}^{\infty} M_s D_s\right]/(M_t D_t)$, *is*

$$P_t/D_t = \frac{1}{1-\alpha-\delta'(I_n-\Gamma)^{-1}\gamma}\left(\alpha+\delta'(I_n-\Gamma)^{-1}(X_t+\gamma)\right) \tag{14}$$

$$= \begin{pmatrix} 1 & 0_n \end{pmatrix}\Omega\left(I_{n+1}-\Omega\right)^{-1}\begin{pmatrix} 1 \\ X_t \end{pmatrix}. \tag{15}$$

Theorem 2 allows to generate stock prices with an arbitrary number of factors, including time-varying growth rate and time-varying risk premia. Formula (14) may look a bit complicated a first, but in Example 1 and the examples below, it typically gives simple expressions. Indeed, to make formulas concrete, consider the case where $\Gamma$ is a diagonal matrix: $\Gamma = Diag(\Gamma_1, \ldots, \Gamma_n)$. Then, $\frac{\alpha^T I_n - \Gamma^T}{\alpha I_n - \Gamma} = Diag\left(\left(\alpha^T - \Gamma_i^T\right)/(\alpha-\Gamma_i)\right)$,[11] so that (13) and (14) read:

$$Z_t(T) = \alpha^T + \sum_{i=1}^{n}\frac{\alpha^T-\Gamma_i^T}{\alpha-\Gamma_i}\delta_i X_{it} \text{ if } \gamma = 0, \tag{16}$$

$$P_t/D_t = \frac{\alpha+\sum_{i=1}^{n}\frac{\delta_i(X_i+\gamma_i)}{1-\Gamma_i}}{1-\alpha-\sum_{i=1}^{n}\frac{\delta_i\gamma_i}{1-\Gamma_i}}. \tag{17}$$

In applications, it is useful to have the price of a claim yielding not just $D_t$, but any linear functional $D_t X_t$. For instance, in a bond model, a futures price has this form. The following two propositions show how to do that. The proofs are exactly identical to those of the previous two theorems.

**Proposition 1** *(Value of a single-maturity claim yielding $D_{t+T} f' X_{t+T}$). Given the LG process $M_t D_t(1, X_t')$, the price of a claim yielding a dividend $d_{t+T} = D_{t=T}\sum_{i=1}^{n} f_i X_{it+T} = D_{t+T}(f' X_{t+T})$, $P_t = E_t[M_{t+T} d_{t+T}]/M_t$, is $P_t = \begin{pmatrix} 0 \\ f \end{pmatrix}' \Omega^T \begin{pmatrix} 1 \\ X_t \end{pmatrix} D_t$, and*

$$P_t = f'\Gamma^T X_t D_t \text{ when } \gamma = 0. \tag{18}$$

[11] If $A$ is a matrix, and $f(x) = \sum_{n=0}^{\infty} f_n x^n$ is an analytic function of a real variable $x$, then $f(A) \equiv \sum_{n=0}^{\infty} f_n A^n$. If $A = Diag(a_1, .., a_n)$, $f(A) = Diag(f(a_1), ..., f(a_n))$.

**Proposition 2** *(Value of an asset yielding $D_t f' X_t$ at each period). Under the conditions of Theorem 2, the price of a claim yielding a dividend $d_t = D_t \sum_{i=1}^{n} f_i X_{it} = D_t f' X_t$, $P_t = E_t \left[ \sum_{s=t+1}^{\infty} M_s D_s \right] / M_t$ satisfies: $P_t / D_t = \begin{pmatrix} 0 \\ f \end{pmatrix}' \Omega \left( I_{n+1} - \Omega \right)^{-1} \begin{pmatrix} 1 \\ X_t \end{pmatrix}$, i.e.*

$$\frac{P_t}{D_t} = \frac{f'(I-\Gamma)^{-1}\gamma}{1-\alpha-\delta'(I-\Gamma)^{-1}\gamma} + f' \left( \Gamma + \frac{1}{1-\alpha}\gamma\delta' \right) \left( I - \Gamma - \frac{1}{1-\alpha}\gamma\delta' \right)^{-1} X_t. \tag{19}$$

For instance, when $\Gamma = Diag\left(\Gamma_1, \ldots, \Gamma_n\right)$ and $\gamma = 0$, (18) reads $P_t / D_t = \sum_{i=1}^{n} f_i \Gamma_i^T X_{it}$, and (19) reads $P_t / D_t = \frac{1}{1-\alpha} \sum_{i=1}^{n} f_i \frac{\Gamma_i}{1-\Gamma_i} X_{it}$.

We next turn to the continuous time version of what we have seen so far. The reader may wish to skip directly to the examples of section 4.

## 3.2 Linearity-Generating Processes in Continuous Time

The following notation is useful when using LG processes. For $x_t, \mu_t$ processes in a vector space $V$, we say $E_t\left[dx_t\right] = \mu_t dt$, or $E_t\left[dx_t\right]/dt = \mu_t$, to signify that there exists a martingale $N_t$ with values in $V$ such that: $x_t = x_0 + \int_0^t \mu_s ds + N_t$.

The definition in continuous time is analogous to the definition in discrete time.

**Definition 3** *(Abstract version of LG processes, continuous time). A LG process is quadruplet $\left(\omega, \nu, \left(Y_t\right)_{t\geq 0}, \left(M_t D_t\right)_{t\geq 0}\right)$ with $\omega$ a $(n+1)\times(n+1)$ matrix (called the generator of the process), $\nu = (1, 0, \ldots, 0)' \in \mathbb{R}^{n+1}$, a $L^1$ state vector process $\left(Y_t\right)_{t\geq 0}$ with values in $\mathbb{R}^{n+1}$, and a process $\left(M_t D_t\right)_{t\geq 0}$ with non-zero values, such that for all $t \in \mathbb{R}_+$, $M_t D_t = \nu' Y_t$ and*

$$E_t\left[dY_t\right] = -\omega Y_t dt. \tag{20}$$

In the "concrete" version of the definition, the vector of factors is $X_t$.

**Definition 4** *(Concrete version of LG processes, continuous time). The process $M_t D_t \left(1, X_t'\right)_{t\in\mathbb{R}_+}$, with $M_t D_t \in \mathbb{R} \setminus \{0\}$ and $X_t \in \mathbb{R}^n$, is a linearity-generating process if it is $L^1$ and there are* *constants* $a \in \mathbb{R}, b, \beta \in \mathbb{R}^n, \Phi \in \mathbb{R}^{n \times n}$, *such that the following relations hold at all* $t \in \mathbb{R}_+$,

$$E_t \left[ d \left( M_t D_t \right) \right] = - \left( a + \beta' X_t \right) M_t D_t dt \tag{21}$$

$$E_t \left[ d \left( M_t D_t X_t \right) \right] = - \left( b + \Phi X_t \right) M_t D_t dt. \tag{22}$$

The interpretation is exactly the same as for Definition 2. Eq. 21 means that the expected growth rate of $M_t D_t$ is linear in the factors. Eq. 22 means that $X_t$ follows a twisted AR(1). For instance, in the case $D_t \equiv 1$ and $dM_t/M_t = -\left(a + \beta' X_t\right) dt$, Eq. 22 gives

$$dX_t + \langle dX_t, dM_t/M_t \rangle = -bdt - \left( \Phi - aI_n \right) X_t dt + \left( \beta' X_t \right) X_t dt + dN_t, \tag{23}$$

where $N_t \in \mathbb{R}^n$ is a martingale, and $\langle dX_t, dM_t/M_t \rangle$ is the usual bracket, $cov\left(dX_t, dM_t/M_t\right)$. Hence, the process contains an AR(1) term, $-b - \left(\Phi - aI_n\right) X_t$, plus a "twist" quadratic term, $\left(\beta' X_t\right) X_t$. It is a "twisted" AR(1). In many applications, $X_t$ represents a small deviation from trend, and the quadratic term $\left(\beta' X_t\right) X_t$ is small. The term $\langle dX_t, dM_t/M_t \rangle$ indicates that it could be absent in the physical probability, but present under the risk-neutral measure.

So $E_t\left[dN_t\right] = 0$, but its components $dN_{it}$, $dN_{jt}$ *can* be correlated. The simplest type of martingale is $dN_t = \sigma\left(X_t\right) dB_t$, for $B_t$ a Brownian motion, but richer structures, e.g. with jumps, are allowed. As in the one-factor process, the volatility of $dN_t$ must go to zero in some limit regions for the process to be well-defined. We defer this more technical issue until section 5.1.

As in discrete time, the two definitions, abstract and concrete, are equivalent, with the generator

$$\omega = \begin{pmatrix} a & \beta' \\ b & \Phi \end{pmatrix}, \tag{24}$$

and with $Y_t$ defined as in (11). Conditions (21)-(22) are then equivalent to (20). The above process leads to a LG discrete-time process with time increments $\Delta t$, with a generator $\Omega = e^{-\omega \Delta t}$.

The next theorem prices claims of finite maturity.

**Theorem 3** *(Bond prices, continuous time). The price-dividend ratio of a claim on a divi-*

*dend of maturity $T$, $Z_t(T) = E_t[M_{t+T}D_{t+T}]/(M_tD_t)$, is*

$$Z_t(T) = \begin{pmatrix} 1 & 0_n \end{pmatrix} \exp(-\omega T) \begin{pmatrix} 1 \\ X_t \end{pmatrix}. \tag{25}$$

*When $b = 0$ (i.e., when the process is centered around 0) this simplifies to*

$$Z_t(T) = e^{-aT} + \beta' (\Phi - aI_n)^{-1} \left(e^{-\Phi T} - e^{-aT} I_n\right) X_t. \tag{26}$$

As an example, bond prices come from $D_t = 1$. In many applications, $b = 0$, which can generically be obtained by re-centering the variables. The next theorem gives the stock price.

**Theorem 4** *(Stock prices, continuous time). Suppose that $\omega$'s eigenvalues have positive real part (finite stock price). Then, the price-dividend ratio of the stock, $P_t/D_t = E_t\left[\int_t^\infty M_s D_s ds\right]/(M_tD_t)$, is*

$$P_t/D_t = \begin{pmatrix} 1 & 0_n \end{pmatrix} \omega^{-1} \begin{pmatrix} 1 \\ X_t \end{pmatrix} = \frac{1 - \beta'\Phi^{-1}X_t}{a - \beta'\Phi^{-1}b}. \tag{27}$$

To make things more concrete, consider the case where $\Phi = Diag(\Phi_1, \ldots, \Phi_n)$. Then, $e^{-\Phi T} = Diag\left(e^{-\Phi_i T}\right)$, and then (26) and 27) read:

$$\begin{aligned} Z_t(T) &= e^{-aT} + \sum_{i=1}^n \frac{e^{-\Phi_i T} - e^{-aT}}{\Phi_i - a} \beta_i X_{it} \text{ if } b = 0 & (28) \\ P_t/D_t &= \frac{1 - \sum_{i=1}^n \frac{\beta_i X_{it}}{\Phi_i}}{a - \sum_{i=1}^n \frac{\beta_i b_i}{\Phi_i}}. & (29) \end{aligned}$$

Finally, the following propositions show that one can price claims that have dividend a linear function of $D_tX_t$.

**Proposition 3** *(Value of a single-maturity claim yielding $D_{t+T}f'X_{t+T}$). Given the LG process $M_tD_t(1, X_t)$, the price of a claim yielding a dividend $d_{t+T} = D_{t+T}(f'X_{t+T})$, $P_t = E_t[M_{t+T}d_{t+T}]/M_t$, is $P_t = \begin{pmatrix} 0 \\ f \end{pmatrix}' \exp(-\omega T) \begin{pmatrix} 1 \\ X_t \end{pmatrix} D_t$, and $P_t = f'e^{-\Phi T}D_tX_t$ when $b = 0$.*

**Proposition 4** *(Value of an asset yielding $D_t f' X_t$ at each period). Under the conditions of Theorem 4, the price of a claim yielding a dividend $d_t = D_t f' X_t$, $P_t = E_t \left[ \int_t^\infty M_s d_s ds \right] / M_t$, satisfies:* $P_t/D_t = \begin{pmatrix} 0 \\ f \end{pmatrix}' \omega^{-1} \begin{pmatrix} 1 \\ X_t \end{pmatrix} = -\frac{1}{\alpha - \beta' \Phi^{-1} b} f' \Phi^{-1} b + f' \left( \Phi - \frac{1}{\alpha} b \beta' \right)^{-1} X_t.$

# 4 Some Examples

We present some examples illustrating the use of LG processes. Some derivations are in the Appendix.

## 4.1 Examples: Stocks

**Example 2** *Gordon growth formula with time-varying dividend growth.*

In this example, we apply the general mechanics of LG processes to our introductory example. The discount factor is $M_t = (1+r)^{-t}$. We calculate the two LG moments (8)-(9):

$$\begin{aligned} E_t \left[ \frac{M_{t+1} D_{t+1}}{M_t D_t} \right] &= \frac{1+g_*}{1+r} (1+x_t) \\ E_t \left[ \frac{M_{t+1} D_{t+1}}{M_t D_t} x_{t+1} \right] &= E_t \left[ \frac{M_{t+1} D_{t+1}}{M_t D_t} \right] E_t \left[ x_{t+1} \right] = \frac{1+g_*}{1+r} (1+x_t) \frac{\rho x_t}{1+x_t} = \frac{1+g_*}{1+r} \rho x_t. \end{aligned}$$

In the above equation, the $1+x_t$ terms cancel out, because of the $1+x_t$ term in the denominator of (5). We designed the process so that the LG equation (9) holds.

Therefore $M_t D_t (1, x_t)$ is LG, with generator $\Omega = \begin{pmatrix} \frac{1+g_*}{1+r} & \frac{1+g_*}{1+r} \\ 0 & \frac{1+g_*}{1+r} \rho \end{pmatrix} = \begin{pmatrix} \alpha & \delta' \\ \gamma & \Gamma \end{pmatrix}$. Hence, we apply Theorem 2, with a dimension $n = 1$, $\gamma = 0$, $\alpha = \delta = \frac{1+g_*}{1+r}$, $\Gamma = \alpha \rho$. We obtain $P_t/D_t = \frac{1}{1-\alpha} \left( \alpha + \delta' \left( I_n - \Gamma \right)^{-1} X_t \right)$, i.e. $P_t/D_t = \frac{\alpha}{1-\alpha} \left( 1 + \frac{1}{1-\alpha\rho} x_t \right)$, i.e. (6). Hence we see how Example 1 comes from the general structure of LG processes.

We use this example to illustrate LG processes in continuous time. Suppose $M_T = e^{-rT}$, $D_T = D_0 \exp \left( \int_0^T g_t dt \right)$, $g_t = g_* + x_t$, where $x_t$ follows what is, formally at least, the continuous time limit of (5) when $\rho = 1 - \phi \Delta t$ and $\Delta t \to 0$:

$$dx_t = \left( -\phi x_t - x_t^2 \right) dt + \sigma \left( x_t \right) dW_t. \tag{30}$$

where $W_t$ is a standard Brownian motion. In the equation above, the coefficient on $x_t^2$ has to be $-1$. The Appendix calculates the LG moments, and Theorem 4 yields:[12]

$$P_t/D_t = \frac{1}{r-g_*}\left(1+\frac{x_t}{r-g_*+\phi}\right). \tag{31}$$

We see that (31) is the limit of (6) when $\rho = 1-\phi\Delta t$ and $\Delta t \to 0$. Section 5.1 will present the condition $x_t \geq -\phi$ for the process to be well defined.

**Example 3** *Stock price with stochastic equity premium.*

Consider a discount factor and dividend process:

$$\frac{M_{t+1}}{M_t} = \frac{1}{1+r}\left(1+\varepsilon_{t+1}^M\right), \qquad \frac{D_{t+1}}{D_t} = (1+g_*)\left(1+\varepsilon_{t+1}^D\right),$$

and the risk premium $\pi_t = -cov_t\left(\varepsilon_{t+1}^M, \varepsilon_{t+1}^D\right)$ is modelled to follow a twisted AR(1):

$$\pi_{t+1} = \pi_* + \frac{1-\pi_*}{1-\pi_t}\rho_\pi\left(\pi_t - \pi_*\right) + \varepsilon_{t+1}^\pi, \tag{32}$$

where $E_t\left[\varepsilon_{t+1}^x\right] = 0$ for $x = M, D, \pi$ and $E_t\left[\frac{M_{t+1}D_{t+1}}{M_t D_t}\varepsilon_{t+1}^\pi\right] = 0$. The term $\frac{1-\pi_*}{1-\pi_t}$ will be close to 1 in many applications, as $\pi_t$ is close to $\pi_*$. So, we have $\pi_{t+1} = \pi_* + \rho_\pi\left(\pi_t - \pi_*\right) + \varepsilon_{t+1}^\pi$ up to second-order terms, which is the simple AR(1). However, the LG twist allows for computation of many interesting quantities, such as the stock price. Defining $\alpha = (1+g_*)(1-\pi_*)/(1+r)$, simple calculations in the Appendix show that

$$P_t/D_t = \frac{\alpha}{1-\alpha}\left(1-\frac{\pi_t-\pi_*}{(1-\pi_*)(1-\alpha\rho_\pi)}\right). \tag{33}$$

When $\pi_t = \pi_*$, this is simply the traditional Gordon growth formula, $P_t/D_t = \alpha/(1-\alpha)$. The economics of this generalized formula is intuitive: a temporarily high level of the equity

[12] The Fisher-Wright process contains a quadratic term, but it does not highlight the tractability coming from the unit coefficient on the quadratic term in a LG context. In addition, it is more special than the LG class, because it imposes a specific functional form on the variance. Cochrane, Longstaff, and Santa-Clara (2008) apply the Fisher-Wright process. Mele (2003, 2007) identifies a condition for the process to be linear in the factor, but does not derive stocks and bond prices such as (31). Other papers introduce different quadratic terms in stochastic process, for instance Ahn et al. (2002), and Constantidines (1992) but they do not take the form of this paper.

premium leads to a low price, particularly this level is persistent. Note that the PD ratio is independent of the variance of the dividend shock $\varepsilon_{t+1}^D$ per se, but it is depends on the covariance with the $cov\left(\varepsilon_{t+1}^M, \varepsilon_{t+1}^D\right) = -\pi_t$.

**Example 4** *Stock price with stochastic growth rate and stochastic equity premium.*

**Discrete time.** Consider a discount factor and dividend process, $\frac{M_{t+1}}{M_t} = \frac{1}{1+r}\left(1+\varepsilon_{t+1}^M\right)$, $\frac{D_{t+1}}{D_t} = 1 + g_t + \varepsilon_{t+1}^D$, where $g_t$ is the stochastic trend growth rate of the dividend, and $\pi_t = -cov_t\left(\varepsilon_{t+1}^M, \varepsilon_{t+1}^D\right)$ is a risk premium.[13] We postulate:

$$\begin{aligned} g_{t+1} &= g_* + \frac{1+g_*-\pi_*}{1+g_t-\pi_t}\rho_g\left(g_t-g_*\right)+\varepsilon_{t+1}^g \\ \pi_{t+1} &= \pi_* + \frac{1+g_*-\pi_*}{1+g_t-\pi_t}\rho_\pi\left(\pi_t-\pi_*\right)+\varepsilon_{t+1}^\pi, \end{aligned}$$

where $E_t\left[\varepsilon_{t+1}^x\right] = 0$ $(x = M, D, g, \pi)$ and $E_t\left[\frac{M_{t+1}D_{t+1}}{M_tD_t}\varepsilon_{t+1}^x\right] = 0$ $(x = g, \pi)$. So, the growth rate and the risk premium hover around their trend, $g_*$ and $\pi_*$. The term $\frac{1+g_*-\pi_*}{1+g_t-\pi_t}$ will be close to 1 in many applications. So, we have $g_{t+1} = g_* + \rho_g\left(g_t - g_*\right) + \varepsilon_{t+1}^g$ up to second-order terms, an AR(1). However, the LG twist allows for computation of many interesting quantities, such as the stock price. Defining $\alpha = \left(1+g_*-\pi_*\right)/\left(1+r\right)$, the Gordon discount factor, Theorem 2 yields:

$$P_t/D_t = \frac{1}{1-\alpha}\left(\alpha + \frac{1}{1+r}\frac{g_t-g_*}{1-\alpha\rho_g} - \frac{1}{1+r}\frac{\pi_t-\pi_*}{1-\alpha\rho_\pi}\right). \tag{34}$$

In the limit of small time intervals, with $\rho_g = 1-\phi_g$, $\rho_\pi = 1-\phi_\pi$, with $r$ and $\phi_g, \phi_\pi$ small ($\phi_g$ is the speed of mean-reversion of $g$ to its trend), we obtain:

$$P_t/D_t = \frac{1}{R}\left(1+\frac{g_t-g_*}{R+\phi_g} - \frac{\pi_t-\pi_*}{R+\phi_\pi}\right), \qquad R \equiv r + \pi_* - g_* \tag{35}$$

Equation (35) nests the three main sources of variations of stock prices in a simple and natural way. Stock prices can increase because the level of dividends increases (that's the $D_t$ terms), because the expected future dividend growth rate increases (the $g_t - g_*$ term), or

[13] The risk premium is on the innovations to dividends. One could also have a risk premium on the innovation to the expected dividend growth rate (as in Bansal and Yaron 2004), an exercise that we leave to the reader.

because the equity premium decreases (the $\pi_t - \pi_*$ term). The two growth or discount factors ($g_t$ and $\pi_t$) enter linearly, weighted by their duration (e.g., $1/(R+\phi_\pi)$), which depends on the speed of mean-reversion of each process (parametrized by $\phi_\pi, \phi_g$), and the effective discount rate, $R$. The volatility terms do not enter in (34), and the price does not change if one changes the correlation between the instantaneous innovation in $g_t$ and $\pi_t$.

**Continuous time.** The analogue is $dM_t/M_t = -rdt - \frac{\pi_t}{\sigma} dW_t$ and $dD_t/D_t = g_t dt + \sigma dW_t$. We assume that $\pi_t$ (which will be the risk premium on the whole stock) and $g_t$ follow the following LG processes, best expressed in terms of their deviation from trend, $\widehat{\pi}_t = \pi_t - \pi_*, \widehat{g}_t = g_t - g_*$:

$$\begin{aligned} d\widehat{g}_t &= -\phi_g \widehat{g}_t dt + (\widehat{\pi}_t - \widehat{g}_t)\,\widehat{g}_t dt + \sigma_g(\widehat{g}_t, \widehat{\pi}_t) \cdot dB_t \\ d\widehat{\pi}_t &= -\phi_\pi \widehat{\pi}_t dt + (\widehat{\pi}_t - \widehat{g}_t)\,\widehat{\pi}_t dt + \sigma_\pi(\widehat{g}_t, \widehat{\pi}_t) \cdot dB_t, \end{aligned}$$

where the $B_t$ is a multidimensional Brownian process independent of $W_t$, and $\sigma_g$ and $\sigma_\pi$ are vector-valued functions. The term $(\widehat{\pi}_t - \widehat{g}_t)\,\widehat{g}_t dt$ is a (in practice often small) LG twist term. The stock price $P_t = E_t\left[\int_t^\infty M_s D_s ds\right]/M_t$ is exactly given by (35).

**Example 5** *Dividend growth rate as a sum of mean-reverting processes (e.g., a slow and a fast process).*

Suppose $M_T = e^{-rT}$, $D_T = D_0 \exp\left(\int_0^T g_t dt\right)$, with $g_t = g_* + \sum_{i=1}^n x_{it}$ and $E_t[dx_{it}]/dt = -\phi_i x_{it} + (g_* - g_t)\,x_{it}$. The growth rate $g_t$ is a steady state value $g_*$, plus the sum of mean-reverting processes $x_{it}$. Each $x_{it}$ mean-reverts with speed $\phi_i$, and also has second-order LG perturbation $(g_* - g_t)\,x_{it}$. The price-dividend ratio is

$$P_t/D_t = \frac{1}{r - g_*}\left(1 + \sum_{i=1}^n \frac{x_{it}}{r - g_* + \phi_i}\right). \tag{36}$$

Each component $x_{it}$ perturbs the baseline Gordon expression $1/(r-g_*)$. The perturbation is $x_{it}$, times the duration of $x_i$, discounted at rate $r - g_*$, which is the term $1/(r - g_* + \phi_i)$. Terms that mean-revert more slowly have a higher impact on the price. Finally, Theorem 3 yields $E_t[D_{t+T}] = e^{g_* T}\left(1 + \sum_{i=1}^n \frac{1-e^{-\phi_i T}}{\phi_i} x_{it}\right) D_t$. Hence the model could be extended (see Gabaix 2009) to different their risk premia for the frequency – which may be a good way to represent asset prices (see e.g. Hansen, Heaton and Li (2008)).

## 4.2 Examples: Bonds

We next turn to LG models of bonds.

**Example 6** *A one-factor bond model, with an always positive nominal rate.*

**Discrete time.** The following example merely illustrates LG processes. It has just one factor, whereas multifactor models (presented in the next example) are necessary to capture the yield curve. The stochastic discount factor is $\frac{M_{t+1}}{M_t} = \frac{1}{1+r_*}(1 - \widehat{r}_t)$ where the constant $r_* > 0$ can be interpreted as a central tendency for interest rates, and $\widehat{r}_t$ is a deviation of the interest from trend. The short term rate is $r_t = 1/E_t\left[\frac{M_{t+1}}{M_t}\right] - 1 \simeq r_* + \widehat{r}_t$ if the $r$'s are small. We postulate the LG-twisted process:

$$\widehat{r}_{t+1} = \frac{(1-\phi)\,\widehat{r}_t + \sigma(\widehat{r}_t)\,\eta_{t+1}}{1 - \widehat{r}_t}, \tag{37}$$

where $E_t\left[\eta_{t+1}\right] = 0$. Using Theorem 1 with $D_t \equiv 1$, the bond price is:

$$Z_t(T) = \frac{1}{(1+r_*)^T}\left(1 - \frac{1-(1-\phi)^T}{\phi}\widehat{r}_t\right). \tag{38}$$

We can ensure that interest rates remain positive, i.e. $\widehat{r}_t > -r_*$: this way, we have a discrete-time one factor model with always positive rate. To ensure that the process is well-defined, we need to ensure $\widehat{r}_t < \phi$. This is ensured if $\eta_{t+1}$ has support in a bounded interval $\left[\underline{\eta}, \overline{\eta}\right]$, and $\sigma(\widehat{r}_t)$ is enough near boundaries. For the interest rate to remain in $(\underline{r}, \overline{r}) = (-r_*, \phi)$, the condition is that $\sigma(\widehat{r})$ is between $\left[\underline{r} - (1-\phi+\underline{r})\,\widehat{r}\right] / \left|\underline{\eta}\right|$and $\left[\overline{r} - (1-\phi+\overline{r})\,\widehat{r}\right]/\overline{\eta}$.

**Continuous time.** Suppose $M_t = \exp\left(-\int_0^t r_s ds\right)$, with $r_t = r_* + \widehat{r}_t$, with $d\widehat{r}_t = -(\phi - \widehat{r}_t)\,\widehat{r}_t dt + dN_t$, where $\phi > 0$, $\widehat{r}_t \leq \phi$, and $N_t$ is a martingale, which could include a diffusive part and a jump part, a rich stochastic volatility structure. The bond price is:

$$Z_t(T) = e^{-r_* T}\left(1 - \frac{1-e^{-\phi T}}{\phi}\widehat{r}_t\right). \tag{39}$$

The independence of bond prices from volatility greatly simplifies the analysis. In particular, $dN_t$ could have jumps, which model a decision by the central bank, or fat-tailed innovations of other kinds (Gabaix et al. 2006). One does not need to specify the volatility process to obtain the prices of bonds: only the drift part is necessary. This leaves a high margin of flexibility to calibrate volatility, for instance on interest rate derivatives, a topic we do not pursue here.

How can we ensure that the interest rate always remains positive? That is very easy (with $r_* > 0$). For instance, we could have $dN_t = \sigma(r_t)\, dW_t$, with $\sigma(r) \sim k' r^{\kappa'}$, $\kappa' > 1/2$ for $r$ in a right neighborhood of 0, and $k' > 0$, so that the local drift at $r_t = 0$ is positive. By the usual Feller conditions on natural boundaries, the process admits a strong solution, and $r_t \geq 0$ always (Cheridito and Gabaix (2008) spells out the technical conditions). And, the bond price (39) is not changed by this assumption about the volatility process. One can indeed change the lower bound for the process (if it is less than $r_*$) without changing the bond price.

Section 5.1 will detail the conditions for the existence of the process. The interest rate needs to remains below some upper bound $\overline{r} \in (r_*, r_* + \phi]$, so as to not explode. One way is to assume that $\sigma(r) \sim k(\overline{r} - r)^{\kappa}$, for $r$ in a left neighborhood of $\overline{r}$, $\kappa > 1/2$ and $k > 0$. Given the drift is negative around $\overline{r}$, that will ensure that $\overline{r}$ is a natural boundary, and $\{\forall t, r_t \leq \overline{r}\}$ almost surely, as detailed in Cheridito and Gabaix (2008). We next turn to a multifactor bond model.

**Example 7** *A multifactor bond model with bond risk premia.*

**Discrete time.** The stochastic discount factor evolves as $\frac{M_{t+1}}{M_t} = \frac{1}{1+r_*}\left(1 - \sum_{j=1}^n r_{jt}\right)$ where the constant $r_*$ is the central value of the interest rate, and the $r_{jt}$ are factors centered around 0. The short term rate is $r_t = 1/E_t\left[\frac{M_{t+1}}{M_t}\right] - 1 \simeq r_* + \sum r_{it}$ if the $r$'s are small. Each factor $r_{it}$ is postulated to evolve as:

$$r_{i,t+1} = \frac{\rho_i r_{i,t}}{1 - \sum_{j=1}^n r_{jt}} + \eta_{i,t+1}, \tag{40}$$

where $E_t\left[\eta_{i,t+1}\right] = 0$, but the $\eta_{i,t+1}$ can have any correlation structure. This is a LG process. The bond price is:

$$Z_t(T) = \frac{1}{(1+r_*)^T}\left(1 - \sum_{i=1}^n \frac{1-\rho_i^T}{1-\rho_i} r_{it}\right). \tag{41}$$

This expression is quite simple, and accommodates a wide variety of specifications for the factors, Eq. 40, and work with both nominal and real interest rates or stochastic discount factors. Furthermore, it accommodates bonds with risk premia. Just take a stochastic discount factor: $\frac{M_{t+1}}{M_t} = \frac{1}{1+r_*}\left(1 - \sum_{j=1}^n r_{jt}\right) - \varepsilon_{t+1}$, where $E_t \varepsilon_{t+1} = 0$, but otherwise $\varepsilon_{t+1}$ is unspecified, and can be heteroskedastic, and postulate: $r_{i,t+1} = \frac{\rho_i r_{i,t}}{1-\sum r_{jt}} + \eta_{i,t+1} + \frac{E_t\left[\varepsilon_{t+1}\eta_{i,t+1}\right]}{E_t[M_{t+1}/M_t]}$, which means that $r_{it}$ follows the process (40) under the risk-neutral measure. Then, Eq. 41 holds. The risk premium on the $T$ maturity bond is:

$$\text{Risk premium} = \frac{cov\left(\varepsilon_{t+1}, Z_{t+1}(T-1)\right)}{Z_t(T)} = \frac{(1+r_*)\sum_i \frac{1-\rho_i^{T-1}}{1-\rho_i} cov\left(\varepsilon_{t+1}, \eta_{i,t+1}\right)}{1-\sum \frac{1-\rho_i^T}{1-\rho_i} r_{it}} \tag{42}$$

Hence the linear bond models from the LG class could complement the search for tractable bond models with economic microfoundations (see Buraschi and Jiltsov (2007) for a recent example).

**Continuous time.** Suppose $dM_t/M_t = -r_t dt + dN_t$, where $N_t$ is a martingale, and $r_t = r_* + \sum_{i=1}^n r_{it}$, with:

$$E_t\left[dr_{it}\right] + \langle dr_{it}, dM_t/M_t\rangle = \left[-\phi_i r_{it} + (r_t - r_*) r_{it}\right] dt,$$

Then $M_t(1, r_{1t}, \ldots, r_{nt})$ is LG, and the bond price is

$$Z_t(T) = e^{-r_* T}\left(1 - \sum_{i=1}^n \frac{1-e^{-\phi_i T}}{\phi_i} r_{it}\right). \tag{43}$$

This LG framework gives the (see Proposition 6) extension of previous models with that have linear bond prices,

## 4.3 Other Examples of Potential Methodological Interest

We next conclude with a few examples that may be of methodological interest.

**Example 8** *Lucas economy where stocks, bonds, and a continuum of moments can be calculated.*

We consider a Lucas economy with: $\frac{dC_t}{C_t} = (g_* + \widehat{g}_t)\, dt + dN_t^C$, $var\left(dN_t^C\right) = \sigma^2 dt$, $d\widehat{g}_t = -\phi\widehat{g}_t dt + dN_t^g$, $\left\langle d\widehat{g}_t, \frac{dC_t}{C_t}\right\rangle = -\widehat{g}_t(\widehat{g}_t - A)\, dt$ with $A > 0$, $\left(N_t^g, N_t^C\right)$ is a martingale, and $g_t \leq A$. Then:

$$\forall\alpha \leq 0, \forall T \geq 0,\ E_t\left[C_{t+T}^\alpha\right] = C_t^\alpha e^{\alpha\left(g_* + (\alpha-1)\frac{\sigma^2}{2}\right)T}\left(1 + \frac{1-e^{-(\phi-\alpha A)T}}{\phi - \alpha A}\alpha\widehat{g}_t\right). \tag{44}$$

The first part of the right-hand side is the traditional term. The novel term is the $\widehat{g}_t$ term. This way, if the agent has marginal utility $M_t = e^{-\rho t} C_t^{-\gamma}$, one can calculate bond and stock prices. The price $E_t [M_{t+T}] / M_t$ of a zero coupon bond with maturity $T$ is

$$Z_t(T) = e^{-\rho T - \gamma\left(g_* - (\gamma+1)\frac{\sigma^2}{2}\right)T} \left(1 - \frac{1 - e^{-(\phi+\gamma A)T}}{\phi + \gamma A} \gamma \widehat{g}_t\right).$$

The price of a leveraged claim on consumption yielding dividends $C_t^\lambda$ is (when $R > 0$):

$$P_t = \frac{C_t^\lambda}{R}\left(1 + \frac{(\lambda - \gamma)\widehat{g}_t}{R + \phi + (\gamma - \lambda) A}\right), \qquad R \equiv \rho + (\gamma - \lambda)\left(g_* + (\lambda - \gamma - 1)\frac{\sigma^2}{2}\right).$$

**Example 9** *The aggregate model of Menzly, Santos and Veronesi (2004), and the Bhattacharya (1978) mean-reverting process, belong to the linearity-generating class.*

The following point is simple and formal. Menzly, Santos and Veronesi (MSV, 2004) rely on an Ornstein-Uhlenbeck process. The inverse of their consumption-surplus ratio, $y_t$, follows: $E_t [dy_t] = k(\overline{y} - y_t) dt$. The price-consumption ratio in their economy is $V_t = y_t^{-1} E_t \left[\int_0^\infty e^{-\rho s} y_{t+s} ds\right]$. In terms of the LG process, the state variable is $y_t$, and $M_t = e^{-\rho t}$. We have $E_t [dM_t/dt] / M_t = -\rho dt$, and $E_t [d(M_t y_t)/dt] / M_t = -\rho y_t + k(\overline{y} - y_t)$. So $M_t(1, y_t)$ is a LG process with generator $\omega = \begin{pmatrix} \rho & 0 \\ -k\overline{y} & \rho + k \end{pmatrix}$. The MSV pricing equation 17 comes directly from Proposition 4 of the present article (with $y_t$ as the payoff), $y_t V_t = (k\overline{y} + \rho y_t) / [\rho(\rho + k)]$. Hence, in retrospect, the MSV (2004) process is tractable because it belongs to the LG class. The same remark holds about Veronesi (2000) and Santos and Veronesi (2006) (see also Proposition 6). In a more elementary context, Bhattacharya (1978) models the dividend $y_t$ as an Ornstein-Uhlenbeck process, yielding the same closed form solution for the price. Also, Ljungqvist and Uhlig (2000) postulate an AR(1) process for the unit labor cost, which allows them to solve their model in closed form. These remarks allow for extensions of those models. For instance, they immediately suggest a way to formulate MSV (2004) in discrete time.

# 5 Discussion

This section presents additional results and remarks on LG processes.

## 5.1 Conditions to Keep the Process Well-Defined

This paper requires that the process be defined for $t \geq 0$, and in particular that $M_t D_t > 0$, which ensures that prices are positive. This section provides simple sufficient conditions to ensure that. We start in discrete time, with Example 1. Write $\widehat{g}_{t+1} = \frac{\rho \widehat{g}_t + \sigma(\widehat{g}_t)\eta_{t+1}}{1+\widehat{g}_t}$ with $E_t\left[\eta_{t+1}\right] = 0$. First, take the deterministic case, $\sigma\left(\widehat{g}_t\right) \equiv 0$. Function $\widehat{g} \mapsto \rho \widehat{g} / \left(1+\widehat{g}\right)$ has two fixed points, an attractive one $\widehat{g} = 0$, and a repelling one, $\widehat{g} = \rho - 1$. To ensure that the process is stable, we require that $\widehat{g}_t$ stays on the right side of the repelling fixed point, i.e.

$$\widehat{g}_t > \underline{g} \equiv \rho - 1. \tag{45}$$

This implies that the volatility $\sigma\left(\widehat{g}\right)$ of the noise should go to 0 fast enough near the boundary $\underline{g}$. Indeed, it is easy to show that if (i) there is an $m > 0$ such that for all $t$, $\eta_{t+1} > -m$, (ii) $0 \leq \sigma\left(\widehat{g}\right) \leq \frac{g - \underline{g}}{m}$ and (iii) $\widehat{g}_0 > \underline{g}$, then for all $t \geq 0$, $\widehat{g}_t > \underline{g}$, and the process is well-defined at all times.

We next present the generalization of this idea to several factors. Consider the discrete-time case where the generator (10) has $\gamma = 0$, $\Gamma = Diag\left(\Gamma_1, \ldots, \Gamma_n\right)$ with $\alpha > 0$ and $\alpha > \max_i \Gamma_i$, and consider the process $Y_{t+1} = \Omega Y_t + \nu' Y_t u_{t+1}$, where $\nu' = (1, 0, \ldots, 0)$ and $E_t\left[u_{t+1}\right] = 0$. The $n-$factor version of criterion (45) is the following:

**Proposition 5** *(Condition to ensure a well-behaved process, with positive stochastic discount factor, discrete time). Define the following condition:*

$$\textit{Condition } \mathcal{C}_t \textit{ (discrete time): } 1 + \sum_{i=1}^{n} \frac{\min\left(\delta_i X_{it}, 0\right)}{\alpha - \Gamma_i} > 0. \tag{46}$$

*Suppose that $M_0 D_0 > 0$, and that at $t = 0$, condition $\mathcal{C}_0$ is satisfied. Suppose also that the noise $u_{t+1}$ is bounded and goes to 0 fast enough near the boundary of (46). Then, for all times $t \geq 0$, $M_t D_t > 0$, and condition $\mathcal{C}_t$ holds.*

The first part of the proposition implies that, if the noise is small enough, then all prices derived above will be positive. The second part means that if condition $\mathcal{C}_t$ is satisfied at time $t = 0$, then it will be satisfied for all future times $t$, i.e. it is "self-perpetuating." It means that $\delta_i X_{it}$ terms should not be too negative: growth rates should not be too low, and interest rate and risk premia terms should not be too high. This makes sense, because otherwise prices could threaten to be negative.

To illustrate the condition, consider first Example 2. There, $\alpha = \delta = (1 + g_*) / (1 + r)$, $\Gamma_1 = \rho\alpha$, and the condition reads: $1 + \min(\widehat{g}_t, 0) / (1 - \rho) > 0$, i.e. $\widehat{g}_t > \rho - 1$: this is equation (45). Next, for Example 4, the condition reads: $1 + \min(g_t - g_*, 0) / \left(1 - \rho_g\right) - \max(\pi_t - \pi_*, 0) / (1 - \rho_\pi) > 0$. We see that condition (46) is quite easy to verify in practice. Before concluding, we express it in continuous time.

*Continuous Time.* In the case where $b = 0$, $\Phi = Diag(\Phi_1, \ldots, \Phi_n)$, and $a < \min_i \Phi_i$, the condition is:

$$\text{Condition } \mathcal{C}_t \text{ (continuous time): } 1 - \sum_{i=1}^{n} \frac{\max(\beta_i X_{it}, 0)}{\Phi_i - a} > 0. \tag{47}$$

For instance, for the simple growth model of Example 2, we have $X_t = x_t$, $a = r - g_*$, $\beta = -1$, $\Phi = a + \phi$, so the condition is: $1 - \max(-x_t, 0) / \phi > 0$, i.e. $x_t > -\phi$. This is the continuous time limit of (45). Likewise, in the multi-factor model of Example 7, $a = r_*$, $\beta_i = 1, \Phi_i = r_* + \phi_i$, so Condition $\mathcal{C}_t$ is: $1 - \sum_{i=1}^{n} \max(r_{it}, 0) / \phi_i > 0$.

We conclude that we have simple sufficient conditions to ensure that LG processes are well-defined, and prices are positive. Cheridito and Gabaix (2008) provide more general conditions.

## 5.2 Approximating Processes with LG Processes

We will show how to use LG processes to calculate approximate expressions for stock and bond prices, in more general models. Consider a state vector $x_t$ (for simplicity, we use a unidimensional notation, but it will be clear that everything goes through with multidimensional vectors), which follows a time-homogenous diffusion: $dx_t = \mu(x_t)\, dt + \sigma(x_t)\, dW_t$, and a "bond" (really, a finite-maturity asset) with price: $Z_T(x) = E_{x_0 = x}\left[e^{\int_0^T a(x_s) ds} q(x_T)\right]$ and a stock with price $V(x) = \int_0^\infty Z_T(x)\, dT$. For instance, in a basic stock model, we have $a(x) = -R + x$ and $q(x) = 1$.

**Heuristic Motivation: Infinite-Dimensional LG Processes.** We proceed heuristically and informally first. The basic idea is to form the infinite-dimensional state vector $Y_t = e^{\int_0^t a(x_s) ds}(1, x_t, x_t^2, \ldots)'$, and study its LG moments. Denoting the $i$-th coordinate of $Y_t$ by $Y_{it}$ we have:

$$E[dY_{it}] / dt = e^{\int_0^t a(x_s) ds}\left(a(x_t)\, x_t^i + \mathcal{A} x_t^i\right) = e^{\int_0^t a(x_s) ds}\left(\mathcal{B} x_t^i\right) = -\sum_{j \geq 0} \omega_{ij} Y_{jt},$$

where we define the operators $\mathcal{A} = \mu(x)\partial_x + \frac{\sigma^2(x)}{2}\partial_{xx}$, and $\mathcal{B} = \mathcal{A} + a(x)$, we define $\omega$ by:

$$\mathcal{B}x^i = -\sum_{j=0}^{\infty} \omega_{ij} x^j. \tag{48}$$

We assume that $a(x)$, $\mu(x)$ and $\sigma^2(x)$ are analytic, so the the expression converges. So $Y_t$ satisfies $E_t[dY_t] = -\omega Y_t dt$ with the infinite matrix $\omega$. Hence, $Y_t$ is an infinite-dimensional LG process. We have projected (at least informally) our regular process onto a LG space.

Hence $E_0[Y_t] = e^{-\omega t} Y_0$. Decomposing $q(x) = \sum_n \overline{q}_n x^n$, we have

$$Z_T(x) = E_0\left[e^{\int_0^T a(x_s)ds} q(x_T)\right] = E_0\left[\sum_n \overline{q}_n Y_{nT}\right] = E_0[\overline{q}' Y_T] = \overline{q}' E_0[Y_T] = \overline{q}' e^{-\omega T} Y_0$$

For instance, for a simple bond, we have $q(x) = 1$, hence $\overline{q}' = (1, 0, 0, \ldots)$. This expression is the "infinite-dimensional" LG version of the bond price. Likewise, the stock price is, formally $V(x) = \int_0^\infty Z_T(x)\, dT = \overline{q}' \omega^{-1} Y_0$.

To obtain a finite-dimensional expression, we simply truncate. Formally, this leads to the following definition.

**Definition 5** *The LG projection of the bond and stock values up to $m$ terms are, respectively:*

$$Z_T^{[m]}(x) = \overline{q}^{[m]\prime} e^{-\omega^{[m]} T} \left(1, x, x^2, \ldots, x^m\right)' \tag{49}$$

$$V^{[m]}(x) = \overline{q}^{[m]\prime} \left(\omega^{[m]}\right)^{-1} \left(1, x, x^2, \ldots, x^m\right)', \tag{50}$$

*where $\cdot^{[m]}$ is the truncation operator on the $m+1$ first coordinates (e.g., $(y_1, y_2, \ldots)^{[m]} = (y_1, \ldots, y_{m+1})$ and $\omega^{[m]} = (\omega_{ij})_{i,j \le m+1}$).*

Before stating a result on the convergence, we illustrate an example, an Ornstein-Uhlenbeck process: $dx_t = -\phi x_t dt + \sigma dW_t$, where $x_t$ is the growth rate of the dividend $g_t - g_*$, or minus a risk premium, $x_t = \pi_* - \pi_t$. The price-dividend ratio is $V(x) = E_{x_0=x}\left[\int_0^\infty e^{-RT + \int_0^T x_s ds} dT\right]$

and is stated in (52) below (changing $x$ into $-x$). Its generator is:

$$\omega = \begin{pmatrix} R & -1 & 0 & 0 & \cdots \\ 0 & R+\phi & -1 & 0 & \cdots \\ -\sigma^2 & 0 & R+2\phi & -1 & \cdots \\ 0 & -3\sigma^2 & 0 & R+3\phi & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \tag{51}$$

(the subdiagonal term is $-(i-1)(i-2)\sigma^2/2$). The LG projection of order 1 is:

$$V^{[1]}(x) = (1,0)\begin{pmatrix} R & -1 \\ 0 & R+\phi \end{pmatrix}^{-1}\begin{pmatrix} 1 \\ x \end{pmatrix} = \frac{1}{R}\left(1+\frac{x}{R+\phi}\right).$$

We now state a rigorous result.

**Theorem 5** *(Convergence of the LG projection to the exact value). Suppose (i) the bond price is analytic in $x$: the bond price $Z(x,T)$ (which satisfies $\partial_T Z(x,T) = \mathcal{B}Z(x,T)$, $Z(x,0) = q(x)$ and growth condition $\sup_{x,t\le T}|Z(x,t)| \le Ke^{cx^2}$ for some $K, c < T/2$) can be represented as $Z(x,T) = \sum_{n\ge 0} a_n(T)\, x^n$ with $\sup_{T,n}|a_n(T)|\, Q^n < \infty$ for all $Q$; and (ii) coefficients $\omega_{ij}$ defined by (48) satisfy $\omega_{ij} \le 0$ for $i \ne j$. Then, we have:*

$$\lim_{m\to\infty} Z_T^{[m]}(x,T) = Z(x,T).$$

*Suppose further that (iii) (analytic stock price) for all $Q$, $\sup_n \left|\int_0^\infty a_n(T)\, dT\right| Q^n < \infty$. Then,*

$$\lim_{m\to\infty} V^{[m]}(x) = V(x).$$

*In both cases, the convergence is uniform in $x$ on any compact subset.*

Assumption (i) that the bond price is analytic (entire, to be precise) is easy to satisfy. For instance, affine-yield models and LG models satisfy it. Assumption (ii) though restrictive, is satisfied in natural cases, such as the basic Ornstein-Uhlenbeck case (51). We conjecture that convergence holds under much broader assumptions. Finally, Theorem 5 also holds for a multidimensional $x_t$.[14]

[14] The only cost is one of greater notational complexity. Call $x = (x_1, ..., x_k)$, and use the multi-indices

The online appendix studies numerical examples, and proposes related decompositions (e.g., using Hermite polynomials) that increase the accuracy. In the Ornstein-Uhlenbeck case, even with one factor the accuracy is quite good. We obtain closed-form approximations to asset prices quite easily. Theorem 5 tells us the process will converge, but perhaps even more importantly, for some examples, the LG linear projection is quite good with just a few terms, which is useful for paper-and-pencil. In conclusion, LG processes might be as a way to linearize non-LG models. This is analogous to the Campbell-Shiller (1988) loglinearization, a comparison we proceed to detail further.

## 5.3 Comparison with Other Modelling Approaches

To handle processes with time-varying risk premia, interest rate or growth rate, the two most used techniques are the affine-yield models (AY, Duffie and Kan 1996; Dai and Singleton 2000; Duffie, Pan and Singleton 2000; Duffee 2002; Cheridito, Filipovic and Kimmel 2007), and the Campbell-Schiller (CS, 1998) decomposition. Denoting the state vector by $X_t$, AY models have zero-coupon bond prices of the form $Z_t^{\mathrm{AY}}(T) = e^{a_T + b_T X_t}$, where $a_T$ and $b_T$ follow difference equations that typically are solved numerically. The Campbell-Shiller decomposition expression log linearizes the stock price and stock returns, and postulate an AR(1) structure for disturbances such as the equity premium or the expected dividend growth rate. It leads to expressions such as as $(P_t/D_t)^{\mathrm{CS}} \simeq e^{A+BX_t}$, where $B$ is linked to the approximate vector of factors.

For instance, suppose we have a stock with an equity premium varying as: $\pi_* + x_t$, where $x_t$ being an AR(1) or a twisted AR(1), we obtain the decompositions (taking the continuous time limit, which is simpler)

$$\text{AY: } \frac{P_t}{D_t} = \int_0^\infty \exp\left[-RT - \frac{1-e^{-\phi T}}{\phi} x_t + \frac{\sigma^2}{2\phi^3}\left(\phi T + 2e^{-\phi T} - \frac{e^{-2\phi T}+3}{2}\right)\right] dT \qquad (52)$$

$$\text{CS: } \frac{P_t}{D_t} \simeq \frac{1}{R} e^{-x_t/(R+\phi)}, \qquad \text{LG: } \frac{P_t}{D_t} = \frac{1}{R}\left(1 - \frac{x_t}{R+\phi}\right)$$

We can see that the three expressions are the same up second order terms in $x_t$ and $\sigma$. This

---

$\alpha = (\alpha_1, ..., \alpha_k)$, $\alpha_j$ nonnegative integers. For instance, we decompose $\phi$ as a sum of terms $x^\alpha = x_1^{\alpha_1} ... x_k^{\alpha_k}$, $\phi = \sum_\alpha \phi_\alpha x^\alpha$. The same condition holds with $\sup_{T,\alpha} \sup |a_\alpha(T)| Q^{|\alpha|} < \infty$. It is straightforward to see that the proofs go through.

is general: all models are the same up to first order approximation.

The relative advantages of LG, AY and CS models may be provisionally assessed as follows.

*Ease of handling different distributions*: Distributions are tightly constrained for AY models (roughly, the noise follows a Gaussian diffusion process augmented by jumps), while to use LG and CS, there is no need to specify the noise: they allow for jumps and non-Gaussian behavior, and a free type of heteroskedascity. However, with LG, one sometimes needs to take care of boundary conditions (e.g., in some empirical analysis).

*Stocks*: The LG model yields simple closed forms for stocks (and stock-like assets such as exchange rates, see Farhi and Gabaix (2009)). AY models are more cumbersome: a stock price can only be expressed $P_t^{\mathrm{AY}}/D_t = \int_0^\infty Z_t^{\mathrm{AY}}(T)\,dT$. Those are infinite sums of exponentials, which is a great progress over stochastic sums, but are still a bit complicated, especially for paper-and-pencil work. CS is tractable, but approximate. CS is very versatile, as one can always make an log-linear approximation. LG models can also linearize non-LG models.

*Bonds*: AY models have been very useful (perhaps as a result, CS has been less used for bonds). AY give a closed form (up to a numerically solved difference equation), LG give a fully solved closed form. With AY modelling, the yields and log returns of zero coupon bonds (but not of coupon bonds) are linear in the factors, which is very convenient. With LG modelling, prices are linear, but the yields and returns are not (however, closed forms for yields and returns obtain easily). Economically, LG bond prices are independent of volatility (controlling for the covariances, see Eq. 43), LG processes naturally generate "unspanned volatility," which some authors propose is a relevant feature of the data (Collin-Dufresne and Goldstein (2002), Andersen and Benzoni (2007)), although there is no unanimity about the importance of this feature (Joslin (2007)). By contrast, affine models typically impose a tight link between bond prices and volatility. On the other hand, a potential drawback of pricing bonds with the LG process, is that, in the simplest version at least, bonds have no mechanically-induced convexity in the LG framework.[15] However, this may not be such a problem, as Joslin (2007) estimates that bond convexity plays a small role in bond prices. All in all, for bonds, affine models will continue to be tremendously useful, but LG models

[15] Convexity effects can arise in LG models. The online appendix contains an illlustrative example where the stock price is a convex function of the expected dividend growth rate.

may complement them, particularly in theoretical research.

*Options*: AY have proven very useful there, while the analysis of LG models is just beginning (Carr, Gabaix and Wu 2009). So, for now, using the AY class for options is the most efficient thing to do.

In sum, no technology dominates. This is as in a toolbox, where many tools are useful, though none dominates. It seems that LG processes are especially convenient for stocks, affine-yield models for bonds, and the Campbell-Shiller linearization for approximations.

### 5.4 LG Processes are the Only Ones that Yield Linearity

We show that, in a certain sense, if bond prices are linear in the factors, then they come from an LG process. Hence all models generating bond prices linear in factors (including the papers mentioned in the introduction) turn out to belong to the LG class.

**Proposition 6** *(LG processes are the only processes generating linear bond prices). Suppose that there are coefficients* $(\alpha_T, \beta_T,)_{T\geq 0}$*, with* $\{(\alpha_T, \beta_T), T = 1, 2..\}$ *spanning* $\mathbb{R}^{n+1}$*, such that* $\forall t, T \geq 0$*,* $E_t[M_{t+T}/M_t] = \alpha_T + \beta_T' X_t$*, Then,* $M_t(1, X_t)$ *is a LG process, i.e. there is a matrix* $\Omega$*, such that* $Y_t = M_t(1, X_t)'$ *follows* $E_t[Y_{t+1}] = \Omega Y_t$*.*

## 6 Concluding Remarks

Linearity-generating processes are quite tractable, as they yield closed forms for stocks and bonds, and prices that are linear in factors. They are likely to be useful in several parts of economics, when trend growth rates, or risk premia, are time-varying. The results of this paper suggest the following research directions.

Most importantly, LG processes allow the construction of paper and pencil tractable general equilibrium models, with closed forms for stocks and bonds. Indeed they suggest a way to "reverse engineer" the processes for endowments and technology, so that the model is tractable. Farhi and Gabaix (2009) and Gabaix (2009) present such models.

The flexibility and closed forms of LG processes may be useful for empirical work, as some recent papers illustrate. Binsbergen and Koijen (2009) estimate a LG stock model with time-varying equity premium and dividend growth rate. Bekker and Bouwman (2009) estimate a three-factor LG term structure model. Carr, Gabaix and Wu (2009) show how to calculate options in closed form in a LG term structure model, and perform an empirical analysis of bonds and fixed-income options.

Third, LG processes suggest a new way to linearize models. Given a model, one could do a Taylor expansion expressing moments $E_t\left[M_{t+1}D_{t+1}/M_tD_t\right]$ and $E_t\left[M_{t+1}D_{t+1}X_{t+1}/M_tD_t\right]$ as a linear function of the factors, thereby making Eq. 8-9 hold to a first order approximation. The projected model is then in the LG class, and its asset prices are approximations of the prices of the initial problem. Hence the LG class offers a way to derive linear approximations of the asset prices of more complicated models.

Fourth, LG processes can be enriched by a decision variable, and offer a way to do multifactor, closed-form dynamic programming. Ongoing research explores this issue.

In conclusion, LG processes may be a useful addition to the economist's toolbox.

# Appendix. Additional Derivations

The following standard results are often useful. We take $n > 0$, $a \in \mathbb{R}$, $b, c \in \mathbb{R}^n$, $d \in \mathbb{R}^{n\times n}$. If real numbers $a$ and $a - b'd^{-1}c$ are non-zero, and matrices $d$ and $d - \frac{1}{a}cb'$ are invertible:

$$\begin{pmatrix} a & b' \\ c & d \end{pmatrix}^{-1} = \begin{pmatrix} D & -Db'd^{-1} \\ -Dd^{-1}c & (d - \frac{1}{a}cb')^{-1} \end{pmatrix}, \qquad D \equiv 1/\left(a - b'd^{-1}c\right) \tag{53}$$

If $aI_n - d$ is invertible:

$$\forall T \in \mathbb{N}, \begin{pmatrix} a & b' \\ 0_n & d \end{pmatrix}^T = \begin{pmatrix} a^T & b'\left(aI_n - d\right)^{-1}\left(a^T I_n - d^T\right) \\ 0_n & d^T \end{pmatrix}. \tag{54}$$

$$\forall T \in \mathbb{R}, \exp\left[\begin{pmatrix} a & b' \\ 0_n & d \end{pmatrix} T\right] = \begin{pmatrix} e^{aT} & b'\left(aI_n - d\right)^{-1}\left(e^{aT} I_n - e^{dT}\right) \\ 0_n & e^{dT} \end{pmatrix}. \tag{55}$$

## 6.1 Derivation of Theorems and Propositions

**Proof of Theorem 1** Recall (7), $E_t[Y_{t+1}] = \Omega Y_t$. Iterating on $T$, it implies that for all $T \geq 0$, $E_t[Y_{t+T}] = \Omega^T Y_t$. Given $M_{t+T}D_{t+T} = \nu' Y_{t+T}$,

$$
\begin{aligned}
Z_t(T) &= (M_t D_t)^{-1} E_t[M_{t+T}D_{t+T}] = (M_t D_t)^{-1} E_t[\nu' Y_{t+T}] = (M_t D_t)^{-1} \nu' E_t[Y_{t+T}] \\
&= (M_t D_t)^{-1} \nu' \Omega^T Y_t = \nu' \Omega^T \left((M_t D_t)^{-1} Y_t\right) = \nu' \Omega^T \begin{pmatrix} 1 \\ X_t \end{pmatrix} = \begin{pmatrix} 1 & 0_n \end{pmatrix} \Omega^T \begin{pmatrix} 1 \\ X_t \end{pmatrix},
\end{aligned}
$$

i.e. (12). The formula for $\gamma = 0$ uses (54).

**Proof of Theorem 2** We use (12), which gives the perpetuity price:

$$
P_t/D_t = \sum_{T=1}^{\infty} Z_t(T) = \nu' \left(\sum_{T=1}^{\infty} \Omega^T\right) \begin{pmatrix} 1 \\ X_t \end{pmatrix} = \nu' \Omega \left(I_{n+1} - \Omega\right)^{-1} \begin{pmatrix} 1 \\ X_t \end{pmatrix}.
$$

$\sum_{T=1}^{\infty} \Omega^T$ is summable because all eigenvalues of $\Omega$ have a modulus less than 1. We use (53) to calculate $(I_n - \Omega)^{-1}$, and conclude.

**Proof of Theorem 3** Recall the definition of $\omega$ in (24), and $E_t[dY_t] = -\omega Y_t dt$. This implies: $\forall T \geq 0, E_t[Y_{t+T}] = e^{-\omega T} Y_t$. Given $M_{t+T}D_{t+T} = \nu' Y_{t+T}$,

$$
\begin{aligned}
Z_t(T) &= (M_t D_t)^{-1} E_t[M_{t+T}D_{t+T}] = (M_t D_t)^{-1} E_t[\nu' Y_{t+T}] = (M_t D_t)^{-1} \nu' E_t[Y_{t+T}] \\
&= (M_t D_t)^{-1} \nu' e^{-\omega T} Y_t = \nu' e^{-\omega T} \left((M_t D_t)^{-1} Y_t\right) = \nu' e^{-\omega T} \begin{pmatrix} 1 \\ X_t \end{pmatrix} = \begin{pmatrix} 1 & 0_n \end{pmatrix} e^{-\omega T} \begin{pmatrix} 1 \\ X_t \end{pmatrix}.
\end{aligned}
$$

i.e. Eq. 25. The formula for $b = 0$ uses (55).

**Proof of Theorem 4** We use (25). The perpetuity price is:

$$
P_t/D_t = \int_0^{\infty} Z_t(T)\, dT = \nu' \left(\int_0^{\infty} e^{-\omega T} dT\right) \begin{pmatrix} 1 \\ X_t \end{pmatrix} = \nu' \omega^{-1} \begin{pmatrix} 1 \\ X_t \end{pmatrix}.
$$

We use (53) to calculate $\omega^{-1}$, and conclude.

**Proof of Proposition 5** Write $Y_t = (Y_{0t}, \ldots, Y_{nt})$, with $Y_{0t} = M_t D_t$, and define $H_t = Y_{0t} + \sum_{i=1}^{n} \frac{\min(\delta_i Y_{it}, 0)}{\alpha - \Gamma_i}$. Start with the case of where there is no noise, i.e. $\forall t, Y_{t+1} = \Omega Y_t$. Given $\Omega$, this means $Y_{0,t+1} = \alpha Y_{0t} + \sum_i \delta_i Y_{it}$, and for $i \geq 1$, $Y_{i,t+1} = \Gamma_i Y_{i,t}$. So:

$$
\begin{aligned}
H_{t+1} &= \alpha Y_{0t} + \sum_i \delta_i Y_{it} + \sum_i \frac{\min\left(\delta_i \Gamma_i Y_{it}, 0\right)}{\alpha - \Gamma_i} \\
&= \alpha Y_{0t} + \sum_{i \text{ s.t. } \delta_i Y_{it} > 0} \delta_i Y_{it} + \sum_{i \text{ s.t. } \delta_i Y_{it} \leq 0} \left(1 + \frac{\Gamma_i}{\alpha - \Gamma_i}\right) \delta_i Y_{it} \\
&\geq \alpha Y_{0t} + \sum_{i \text{ s.t. } \delta_i Y_{it} \leq 0} \frac{\alpha}{\alpha - \Gamma_i} \delta_i Y_{it} = \alpha H_t.
\end{aligned}
$$

Hence, $H_{t+1} \geq \alpha H_t$. Hence, if $H_0 > 0$, then $\forall t \geq 0$, $H_t > 0$, and so that $M_t D_t = Y_{0t} \geq H_t > 0$.

In the case with noise, say that $Y_{t+1} = \Omega Y_t + u_{t+1}$, for some mean 0 noise $u_{t+1}$, and suppose that $u_{t+1}$ is bounded. By continuity, if $H_t > 0$, $H_{t+1}$ is positive with probability 1, if $u_{t+1}$ is small enough. And again, $M_t D_t = Y_{0t} \geq H_t > 0$.

**Proof of Theorem 5** *Step 1.* We put the basic objects in place. We define $L = -\omega'$, a matrix in $\mathbb{R}^{\mathbb{N} \times \mathbb{N}}$. We observe that for $\sum_j a_j x^j$ an entire series,

$$
\mathcal{B} \sum_j a_j x^j = \sum_j a_j \mathcal{B} x^j = \sum_j a_j \sum_i \left(-\omega_{ji}\right) x^i = \sum_i \left(\sum_j L_{ij} a_j\right) x^i,
$$

so $\partial_t Z(x,t) = \mathcal{B} Z(x,t)$ writes: $\dot{a}_i(t) = L_{ij} a_j(t)$, using the dot for time derivatives. Defining $A(t) = (a_0(t), a_1(t), \ldots)' \in \mathbb{R}^{\mathbb{N}}$, $\dot{A}(t) = LA(t)$, with $A(0) = \overline{q} = (q_0, q_1, \ldots)'$. Likewise, calling $L^m = (L_{ij})_{0 \leq i,j \leq m}$ the restricted matrix (we omit the brackets in the proof), $\dot{A}^m(t) = L^m A^m(t)$, and $A^m(0) = q^m = (q_0, \ldots, q_m)'$. Finally, we use the notation $a \preceq b$ if $a_i \leq b_i$ for all components $i$. Without loss of generality, we will assume that $q \succeq 0$. (If need be, we can decompose $q$ as the difference of two functions with nonnegative coefficients $q_n^+$ and $q_n^-$ respectively, and prove the result for each function).

*Step 2.* We will use the following well-known fact:

**Lemma 1** *Consider a solution of a Cauchy problem for a linear ordinary differential equation $\dot{Y}(t) = PY(t) + b(t)$ with $Y(0) = y$, where $Y(t) \in \mathbb{R}^N$, $P \in \mathbb{R}^{N \times N}$, $y \in \mathbb{R}^N$, $b(t) \in \mathbb{R}^N$,* *with initial vector $y \succeq 0$, $P$ such that $P_{ij} \geq 0$ for all $i \neq j$, and $b(t) \succeq 0$ is continuous. Then, $Y(t) \succeq 0$ for all $t \geq 0$.*

**Proof.** Write $P = D + J$, with $D$ diagonal, $J$ has zero-diagonal elements, and non-negative non-diagonal elements. Define $X(t) = e^{-Dt} Y(t)$. Then,

$$\dot{X}(t) = e^{-Dt}\dot{Y}(t) - De^{-Dt}Y(t) = e^{-Dt}(DY(t) + JY(t)) - De^{-Dt}Y(t) + e^{-Dt}b(t) = J_t X(t) + B_t$$

with $J_t \equiv e^{-Dt} J e^{Dt} \succeq 0$ and $B_t \equiv e^{-Dt} b_t \succeq 0$. Consider the Picard iterates $X_n(t)$ of the ODE, defined by $X_0(t) = y$ and $X_{n+1}(t) = y + \int_0^t (J_\tau X_n(\tau) + B_\tau)\, d\tau$. By induction on $n$, $X_n(t) \succeq 0$. Because $X(t) = \lim_{n\to\infty} X_n(t)$, we have $X(t) \succeq 0$, which implies $Y(t) = e^{Dt} X(t) \succeq 0$. A similar argument would prove the Lemma for infinite $N$. ■

We will show that for all $t \geq 0$,

$$0 \preceq A^m(t) \preceq A^n(t) \preceq A(t) \text{ for } m < n. \tag{56}$$

Because $\dot{A}^m(t) = L^m A^m(t)$, Lemma 1 proves that $A^m(t) \succeq 0$. Next, suppose $m < n$. Denote $\hat{A}^m = (A^{m\prime}\ 0'_{n-m})'$, and likewise extend $L^m$ by adding $n-m$ zero rows and columns after the last ones to get $\hat{L}^m$. Observe $\frac{d}{dt}(A^n - \hat{A}^m) = L^n A^n - \hat{L}^m \hat{A}^m = L^n(A^n - \hat{A}^m) + (L^n - \hat{L}^m)\hat{A}^m$. Decompose $\hat{L}^m = \hat{D}^m + \hat{J}^m$, where $\hat{D}^m$ contains the diagonal terms, and $\hat{J}^m$ the non-diagonal terms, and note that $(L^n - \hat{L}^m)\hat{A}^m = (D^n - \hat{D}^m + J^n - \hat{J}^m)\hat{A}^m = (J^n - \hat{J}^m)\hat{A}^m$ because $\hat{A}^m$ contains zeros in the last $n-m$ positions. Since $(J^n - \hat{J}^m)\hat{A}^m \succeq 0$, Lemma 1 establishes that $A^n - \hat{A}^m \succeq 0$. A similar argument shows $A - A^m \succeq 0$.

*Step 3.* By (56), for component $i$ and each $t$, $A_i^m(t)$ is weakly increasing in $m$ and $\leq A_i(t)$. So, define $A_i^\infty(t) = \lim_{m\to\infty} A_i^m(t)$. We have $0 \leq A_i^\infty(t) \leq A_i(t)$, so the $Z^\infty(x,t) = \sum_i A_i^\infty(t)\, x^i$ is an entire function of $x$. We next check that it satisfies the basic PDE. Because $\dot{A}^m(t) = L^m A(t)$, we have: $A_i^m(t) = A_i(0) + \int_0^t \sum_{j\leq m} L_{ij} A_j^m(s)\, ds$. Considering positive and negative terms separately, the monotone convergence theorem implies, $\lim_{m\to\infty} \int_0^t \sum_{j\leq m, j\neq i} L_{ij} A_j^m(s)\, ds = \int_0^t \sum_{j, j\neq i} L_{ij} A_j^\infty(s)\, ds$, and $\lim_{m\to\infty} \int_0^t L_{ii} A_i^m(s)\, ds = \int_0^t L_{ii} A_i^\infty(s)\, ds$. So $A_i^\infty(t) = A_i(0) + \int_0^t \sum_j L_{ij} A_j^\infty(s)\, ds$, which implies that $Z^\infty(x,t) = q(x) + \int_0^t \mathcal{B} Z^\infty(x,s)\, ds$. Also, $|Z^\infty(x,t)| \leq Z^\infty(|x|,t) \leq Z(|x|,t) \leq Ke^{cx^2}$. So, by the uniqueness part of the Feynman-Kac theorem (Karatzas and Shreve 1991, Theorem 4.4.2), $Z^\infty(x,t) = Z(x,t)$. This concludes the "bond" part of the Theorem. The uniform convergence comes from the fact the coefficients $0 \leq A_n^m \leq a_n$ for all $n, m$. We next turn to stocks.

*Step 4: Stocks.* Call $v_i = \int_0^\infty A_i(T)\,dT$, assumed to be finite, and define $v^m \in \mathbb{R}^m$ by $v_i^m = \int_0^\infty A_i^m(T)\,dT \le v_i$. By monotone convergence, $\lim_{m\to\infty} v_i^m = v_i$. Also, as $\lim_{T\to\infty} A_i(T) = 0$ and $0 \le A_i^m(T) \le A_i(T)$, we have $\lim_{T\to\infty} A_i^m(T) = 0$. Also:

$$L^m v^m = L^m \int_0^\infty A^m(T)\,dT = \int_0^\infty L^m A^m(T)\,dT = \int_0^\infty \dot{A}^m(T)\,dT = [A^m(T)]_0^\infty = -A^m(0) = -q^m,$$

so $v^m$ is a solution of $L^m v = -q^m$. Furthermore, $V^m(x) = \sum_{n\le m} v_n^m x^n$ satisfies $V^m(x) = \int_0^\infty \sum_{n\le m} A_n^m(t)\,x^n dt$. By dominated convergence, $V^m(x) \to \int_0^\infty \sum_n A_n(t)\,x^n dt = V(x)$.

**Proof of Proposition 6** Call $Y_t = M_t(1, X_t)'$, $\gamma_T = (\alpha_T, \beta_T)'$, so that $E_t[M_{t+T}] = \gamma_T' Y_t$. That implies $\gamma_{T+1}' Y_t = E_t[M_{t+T+1}] = E_t[E_{t+1}[M_{t+T+1}]] = E_t[\gamma_T Y_{t+1}]$, hence: $\gamma_{T+1}' Y_t = E_t[\gamma_T Y_{t+1}]$. Call $e_k \in \mathbb{R}^{n+1}$, the vector with $k-$th coordinate equal to 1, and other coordinates equal to 0. As $\{\gamma_T, T = 1, 2, ..\}$ spans $\mathbb{R}^{n+1}$, there are reals $\lambda_{kT}$ (with at most $n+1$ non-zero values) such that: $e_k = \sum_T \lambda_{kT}\gamma_T$. Define $\Omega = \sum_{k,T} e_k \lambda_{kT} \gamma_{T+1}'$. Given $I_{n+1} = \sum_{k=1}^{n+1} e_k e_k'$, we have:

$$\begin{aligned} E_t[Y_{t+1}] &= (\sum_k e_k e_k') E_t[Y_{t+1}] = (\sum_k e_k (\sum_T \lambda_{kT} \gamma_T')) E_t[Y_{t+1}] \\ &= \sum_{k,T} e_k \lambda_{kT} E_t[\gamma_T' Y_{t+1}] = \sum_{k,T} e_k \lambda_{kT} \gamma_{T+1}' Y_t = \Omega Y_t. \end{aligned}$$

## 6.2 Derivations of Examples

**Example 2** $E_t[d(M_t D_t)/(M_t D_t)] = (-r + g_* + x_t)\,dt$ and

$$E_t\left[\frac{d(M_t x_t)}{M_t}\right] = x_t E_t\left[\frac{dM_t}{M_t}\right] + E_t[dx_t] = x_t(-r + g_* + x_t)\,dt + \left(-\phi x_t - x_t^2\right)dt = -(r - g_* + \phi)\,x_t dt.$$

We note that the $x_t^2$ terms cancel out, which is their raison d'être in (30). So $M_t(1, x_t)$ is a LG process with generator $\omega = \begin{pmatrix} r - g_* & -1 \\ 0 & r - g_* + \phi \end{pmatrix}$. Theorem 4 yields (31).

**Example 3** We define $\widehat{\pi}_t = \pi_t - \pi_*$, and form the LG moments (8)-(9):

$$E_t\left[\frac{M_{t+1}}{M_t}\frac{D_{t+1}}{D_t}\right]=\frac{1+g_*}{1+r}\left(1-\pi_t\right)=\frac{1+g_*}{1+r}\left(1-\pi_*\right)\left(1-\frac{\widehat{\pi}_t}{1-\pi_*}\right)=\alpha\left(1-\frac{\widehat{\pi}_t}{1-\pi_*}\right)$$

$$E_t\left[\frac{M_{t+1}}{M_t}\frac{D_{t+1}}{D_t}\widehat{\pi}_{t+1}\right]=E_t\left[\frac{M_{t+1}}{M_t}\frac{D_{t+1}}{D_t}\right]E_t\left[\widehat{\pi}_{t+1}\right]=\frac{1+g_*}{1+r}\left(1-\pi_t\right)\frac{1-\pi_*}{1-\pi_t}\rho_\pi\widehat{\pi}_t=\alpha\rho_\pi\widehat{\pi}_t.$$

So $M_tD_t\left(1,\widehat{\pi}_t\right)$ is a LG process with generator $\alpha\begin{pmatrix}1 & \frac{-1}{1-\pi_*}\\ 0 & \rho_\pi\end{pmatrix}$. Theorem 2 concludes.

**Example 4** Define $\widehat{\pi}_t=\pi_t-\pi_*,\widehat{g}_t=g_t-g_*$, so that:

$$E_t\left[\frac{M_{t+1}}{M_t}\frac{D_{t+1}}{D_t}\right]=\frac{1}{1+r}\left(1+g_t-\pi_t\right)=\alpha+\frac{\widehat{g}_t-\widehat{\pi}_t}{1+r}$$

$$E_t\left[\frac{M_{t+1}}{M_t}\frac{D_{t+1}}{D_t}\widehat{g}_{t+1}\right]=E_t\left[\frac{M_{t+1}}{M_t}\frac{D_{t+1}}{D_t}\right]E_t\left[\widehat{g}_{t+1}\right]=\frac{1}{1+r}\left(1+g_t-\pi_t\right)\frac{1+g_*-\pi_*}{1+g_t-\pi_t}\rho_g\widehat{g}_t=\alpha\rho_g\widehat{g}_t$$

and likewise for $\widehat{\pi}_t$. Hence $M_tD_t\left(1,\widehat{g},\widehat{\pi}_t\right)$ is LG with generator $\begin{pmatrix}\alpha & 1/\left(1+r\right) & -1/\left(1+r\right)\\ 0 & \alpha\rho_g & 0\\ 0 & 0 & \alpha\rho_\pi\end{pmatrix}$.

In continuous time, the LG moments are:

$$\begin{aligned}
E_t\left[\frac{d\left(M_tD_t\right)}{M_tD_t}\right]/dt &= -r-\pi_t+g_t=-R-\widehat{\pi}_t+\widehat{g}_t\\
E_t\left[\frac{d\left(M_tD_t\widehat{g}_t\right)}{M_tD_t}\right]/dt &= E_t\left[\frac{d\left(M_tD_t\right)/dt}{M_tD_t}\right]\widehat{g}_t+E_t\left[\widehat{g}_t\right]/dt\\
&= \left(-R-\widehat{\pi}_t+\widehat{g}_t\right)\widehat{g}_t-\left(\phi_g-\widehat{\pi}_t+\widehat{g}_t\right)\widehat{g}_t=-\left(R+\phi_g\right)\widehat{g}_t
\end{aligned}$$

and likewise for $\widehat{\pi}_t$. So $M_tD_t\left(1,\widehat{g},\widehat{\pi}_t\right)$ is LG with generator $\begin{pmatrix}R & -1 & 1\\ 0 & R+\phi_g & 0\\ 0 & 0 & R+\phi_\pi\end{pmatrix}$.

**Example 5** $M_tD_t\left(1,x_{1t},\ldots,x_{nt}\right)'$ is a LG with generator $\begin{pmatrix}r_* & -1_n\\ 0 & Diag\left(r_*+\phi_i\right)\end{pmatrix}$.

**Example 6** We calculate the LG moments: $dM_t/M_t = -r_t dt = -(r_* + \widehat{r}_t)\,dt$, and:

$$E_t\left[\frac{d(M_t\widehat{r}_t)}{M_t}\right] = \widehat{r}_t E_t\left[\frac{dM_t}{M_t}\right] + E_t[d\widehat{r}_{tt}] = -\widehat{r}_t(r_* + \widehat{r}_t)\,dt - (\phi - \widehat{r}_t)\,\widehat{r}_t dt = -(r_* + \phi)\,\widehat{r}_t dt$$

Importantly, the $\widehat{r}_t^2$ terms cancel out. Hence we have the LG moments:

$$E_t[dM_t/M_t]/dt = -r_* - \widehat{r}_t, \qquad E_t[d(M_t\widehat{r}_t)/M_t]/dt = -(r_* + \phi)\,\widehat{r}_t$$

So $Y_t = M_t(1, \widehat{r}_t)$ is LG with generator $\begin{pmatrix} r_* & 1 \\ 0 & r_* + \phi \end{pmatrix}$.

**Example 7** We have $E_t\left[\frac{M_{t+1}}{M_t} r_{i,t+1}\right] = \frac{1}{1+r_*}\rho_i r_{i,t}$. So the process $M_t(1, r_{1t}, \ldots, r_{nt})$ has generator: $\frac{1}{1+r_*}\begin{pmatrix} 1 & -1_n \\ 0 & Diag(\rho_i) \end{pmatrix}$. In continuous time, the generator is $\begin{pmatrix} 1 & 1_n \\ 0 & Diag(r_* + \phi_i) \end{pmatrix}$.

**Example 8** With $v \equiv \alpha g_* + \alpha(\alpha - 1)\sigma^2/2$, we calculate $E_t\left[\frac{dC_t^\alpha}{C_t^\alpha}\right]/dt = v + \alpha\widehat{g}_t$ and

$$\begin{aligned} \frac{E_t[d(C_t^\alpha \widehat{g}_t)]}{C_t^\alpha dt} &= (v + \alpha\widehat{g}_t)\,\widehat{g}_t - \phi\widehat{g}_t + \alpha\left\langle d\widehat{g}_t, \frac{dC_t}{C_t}\right\rangle/dt \\ &= (v + \alpha\widehat{g}_t)\,\widehat{g}_t - \phi\widehat{g}_t - \alpha\widehat{g}_t(\widehat{g}_t - A) = (v - \phi + \alpha A)\widehat{g}_t. \end{aligned}$$

Hence $e^{-\rho t}C_t^\alpha(1, \widehat{g}_t)$ is LG with generator $\omega = (\rho - v)\,I_2 + \begin{pmatrix} 0 & -\alpha \\ 0 & \phi - \alpha A \end{pmatrix}$. The statements follow from Theorems 3 and 4.

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