power laws

A power law is the form taken by a remarkable number of regularities in economics, and is a relation of the type $Y = kX^{\alpha}$, where Y and X are variables of interest, α is called the power law exponent, and k is a constant. Many economic laws take the form of power laws, in particular macroeconomic scaling laws, the distribution of income, wealth, size of cities and firms, and the distribution of financial variables such as returns and trading volume. This article surveys the empirical evidence and the theoretical explanations for the occurrence of power laws.

A power law (PL), also known as a scaling law, is the form taken by a remarkable number of regularities or 'laws' in economics, and is a relation of the type $Y = kX^{\alpha}$, where Y and X are variables of interest, α is called the power law exponent, and k is a typically unremarkable constant.

A special type is the distributional PL, also called a Pareto law. For instance, the probability that a firm has more than x employees is proportional to $1/x^{\zeta}$, for some positive number $\zeta : P(S > x) = k/x^{\zeta}$, for some k, at least in the upper tail or most of it. The exponent ζ is independent of the units in which the law is expressed. A special case is Zipf's law, which is a Pareto law with $\zeta \simeq 1$.

Understanding what gives rise to the scaling law, and explaining the precise value of the exponent (for example, why it is equal to 1 rather than any other number) is a challenge that has fascinated successive generations. Schumpeter (1949, p. 155) wrote: 'Few if any economists seem to have realized the possibilities that such invariants hold for the future of our science. In particular, nobody seems to have realized that the hunt for, and the interpretation of, invariants of this type might lay the foundations for an entirely novel type of theory.' Champernowne (1953) and Simon (1955) made great strides towards realizing Schumpeter's vision, and the quest continues.

Power laws are also of great interest outside of economics. Understanding PLs is a large part of the theory of critical phenomena, in which many materials behave identically around phase transitions – a phenomenon physicists call 'universality', and which is still only partially understood. Power laws have proven useful for describing and understanding networks. Biology has also many scaling regularities; for example, the daily energy intake of an animal of mass M is proportional to the $M^{3/4}$. This regularity was explained (Brown and West, 2000) via simple physical reasoning, which eschews the need to talk about the feathers and the hair of animals. Simpler and deeper principles underlie the regularities instead. The same holds for economic laws. Power laws give the hope of robust, detail-independent economic laws.

Theory: forces that generate power laws

Proportional random growth

Getting a power law. To explain distributional PLs, a central mechanism is proportional random growth (Sornette, 2001). The process was developed in economics by Champernowne (1953) and Simon (1955). Things are more tractable in continuous time (see Gabaix, 1999).

Take the example of cities in an economy with a constant number of cities and a fixed total population. When the system grows, the same reasoning applies after normalization – S is the normalized size of a city, for example as a multiple of the median city population. Suppose that each city i has a population S_t^i and, between t and $t+1$, increases by a growth rate γ_{t+1}^i:

$$S_{t+1}^i = \gamma_{t+1}^i S_t^i, \tag{1}$$

and suppose that the γ_{t+1}^i are identically and independently distributed, with density $f(\gamma)$, at least in the upper tail. Call $G_t(x) = P(S_t^i > x)$ the counter-cumulative distribution function. The equation of motion of G is:

$$G_{t+1}(x) = P\left(S_{t+1}^i > x\right) = P\left(S_t^i > x/\gamma_{it}\right) = E\left[G_t\left(x/\gamma_{it}\right)\right].$$

Hence:

$$G_{t+1}(S) = \int_0^\infty G_t\left(\frac{S}{\gamma}\right) f(\gamma)\, d\gamma.$$

Its steady state distribution G, if it exists, satisfies

$$G(S) = \int_0^\infty G\left(\frac{S}{\gamma}\right) f(\gamma)\, d\gamma. \tag{2}$$

One can try the functional form $G(S) = a/S^\zeta$, where a is a constant. Plugging it in (2) gives: $1 = \int_0^\infty \gamma^\zeta f(\gamma)\, d\gamma$, that is

$$E\left[\gamma^\zeta\right] = 1. \tag{3}$$

The steady state distribution is (in the upper tail) Pareto, with an exponent ζ that satisfies eq. (3).

To make sure that the steady state distribution exists, one needs some friction, for example a force that prevents small cities from becoming too small.

Getting a Zipf's law. We see that proportional random growth leads to a PL. Why should the exponent $\zeta = 1$ appear in so many economic systems? An answer is the following (see Gabaix, 1999; Luttmer, 2007; Rossi-Hansberg and Wright, 2007). Suppose that the random growth process (1) holds through most of the distribution, and that the system has constant size. Then, $E[S_{t+1}] = E[\gamma]E[S_t]$. As the system has constant size, then we need $E[S_{t+1}] = E[S_t]$, hence $E[\gamma] = 1$. That means that $\zeta = 1$ is a solution of eq. (3). In other words, to get Zipf's law we need a random growth process with small frictions.

In sum, proportional random growth with frictions leads to PLs, and proportional random growth with small frictions leads to a special type of PL- namely Zipf's law.

Inheritance via algebraic transformation

Power laws have excellent inheritance and aggregation properties. The property of being distributed according to a PL is conserved under addition, multiplication, power transformation, min, and max. The general rule is that, when we combine two PL variables, the fatter-tailed (that is, the one with the smaller exponent) dominates. Call ζ_X the PL exponent of X, with $\zeta_X = +\infty$ if X is thinner than any PL, for example is a Gaussian. For X and Y independent random variables, and $\beta > 0$ a constant, we have: $\zeta_{X+Y} = \zeta_{X\cdot Y} = \zeta_{\max(X,Y)} = \min(\zeta_X, \zeta_Y)$, $\zeta_{\min(X,Y)} = \zeta_X + \zeta_Y$, $\zeta_{\alpha X} = \zeta_X$, $\zeta_{X^\alpha} = \zeta_X/\alpha$ (see Jessen and Mikosch, 2006). Those properties generate new PLs from old ones. For instance, if mutual funds are PL distributed, then many of their actions (for example, trading volumes, or the price movements they create) will be PL distributed (Gabaix et al., 2006).

Equilibrium economic mechanisms

Optimization with PL objective function. The early example is the Allais–Baumol–Tobin model of demand for money (see also Mulligan and Shleifer, 2005; Gabaix et al., 2003). Costs and benefits are power functions of the variables of interest, so that maximization also yields a PL – there, money demand is proportional to the interest rate to the power $-1/2$. PL in, PL out.

Matching talents in the upper tail. Another way to generate PLs is in matching the talent of individuals with large firms or audiences. For instance, Gabaix and Landier (2008) study the market for executives. They derive that, in the upper tail of all well-behaved distributions, if $T(x)$ is the talent of an individual in the x upper quantile, then $T'(x)$ is approximately a power function x^{α}. As a result, the competitive matching process generates a PL relation between CEO pay and firm size, and a PL of the pay distribution. Huge differences in pay reward minuscule differences in talent. The PL form of T' is likely to be useful in other superstars markets.

Empirics: the main power laws of economics

Old macroeconomic scaling laws

The first quantitative law of economics is probably the quantity theory of money, which, not coincidentally, is a scaling relation. It states that the price level P is proportional to the mass of money in circulation M, divided by the gross domestic product Y, times a pre-factor V: $P = VM/Y$. If the money supply doubles while GDP remains constant, prices double – a nice scaling law, relevant to policy.

More modern, we have Kaldor's stylized facts on economic growth: with K the capital stock, Y GDP, L population, r the interest rate, K/Y, wL/Y, and r, are roughly constant across time and countries. Explaining these facts led Solow to his growth model.

Reasonably old and well-established laws

Income and wealth. The first PL is the Pareto law of income or wealth, which states that the tail distribution of income (or, respectively, wealth), is PL. The tail exponent of income seems to vary between 1.5 and 3, while the tail exponent of wealth is more stable. While, starting with Champernowne (1953), many models have been proposed to explain it (mainly along the lines of random growth), it is intriguingly unclear why the exponent is rather stable across economies.

Firm sizes. The bulk of the distribution of firm sizes is well described by a Zipf's law (Figure 1). This severely constrains models of firm growth, and means that idiosyncratic shocks of large firms may affect GDP (Gabaix, 2006). Zipf's law holds for different measures of firm sizes and countries (Axtell, 2001; Fujiwara et al., 2004; Gabaix and Landier, 2008).

City sizes. In the upper tail, Zipf's law holds generally well across times and countries (Gabaix and Ioannides, 2004).

Gibrat's law for the growth rate of cities is shown in the United States by Ioannides and Overman (2003).

Roberts' law for executive compensation. Across times and countries, an executive heading a firm of size S earns an amount proportional to S^{κ}, for a

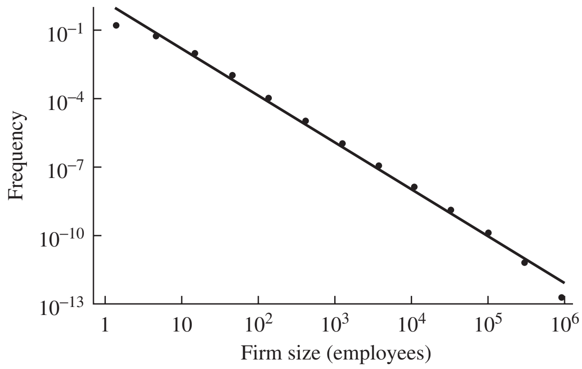

Figure 1 *Note*: Log frequency $\ln f(S)$ vs. log size $\ln S$ of U.S. firm sizes for 1997. OLS fit gives a slope of $1+\zeta = 2.059$(s.e. = 0.054; $R^2 = 0.992$). This corresponds to a frequency $f(S) = kS^{-2.059}$, that is, a power law distribution with exponent $\zeta = 1.059$. Indeed, if $P(\text{Size} > S) = kS^{-\zeta}$ the density is $f(S) = k\zeta S^{-(\zeta+1)}$. This is very close to Zipf's law, which says that $\zeta = 1$. *Source*: Reprinted with permission from Fig. 1 from Robert L. Axtel, *Science* 293, 1818–20 (7 September 2001)

κ around 1/3. Superstars models explain the presence of this scaling (Gabaix and Landier, 2006), but the reason for the 1/3 value remains a mystery.

More recently proposed laws

Power law of stock market activity: returns, trading volume, and trading frequency. Following Mandelbrot, the following regularities have been found. Stock market returns (over one minute to one week) have PL tails, with an exponent around 3 (Gopikrishnan et al., 1999). Individual trades have a PL exponent around 1.5 (Gopikrishnan et al. 2000). The number of trades executed over a short horizon has an exponent close to 3 (Plerou et al., 2000). There is no consensus about the origins for those regularities. The fat tails of the returns might come from GARCH effects. One view (Gabaix et al., 2003; 2006) attributes it to the trades of large institutional investors in relatively illiquid markets, which creates spikes in returns and volume, and generates empirically found exponents.

Supply of regulations. Mulligan and Shleifer (2005) establish another candidate law. In U.S. states, the quantity of regulation is a PL of population.

Estimation of power laws

How does one estimate a distributional PL? We take the example of n cities in the upper tail, ordering them by size, $S_{(1)} \geq \cdots \geq S_{(n)}$. One method is Hill's estimator:

$$\widehat{\zeta}^{Hill} = (n-1) \Big/ \sum_{i=1}^{n-1} \left(\ln S_{(i)} - \ln S_{(n)} \right)$$

which has a standard error $\widehat{\zeta}^{Hill} n^{-1/2}$. The second method is a 'log rank log size regression', where $\widehat{\zeta}$ is the slope in the regression of the log rank i on the log size:

$$\ln(i-s) = \text{constant} - \widehat{\zeta}^{OLS} \ln S_{(i)} + \text{noise}$$

which has a standard error standard error of $\widehat{\zeta}^{OLS} \cdot (n/2)^{-1/2}$. s is a shift,

$s = 0$ is typical, but $s = 1/2$ is optimal (Gabaix and Ibragimov, 2006). Both methods have pitfalls, as true errors are often larger than nominal standard errors (Embrechts, Kluppelberg and Mikosch, 1997; Gabaix and Ioannides, 2004).

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See also

ARCH models;
econophysics;
inequality (measurement);
quantity theory of money, superstar, economics of;
systems of cities;
wealth.

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