A Primer on Tractable Incentive Contracts

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Even in simple settings, the principal-agent problem is typically very difficult to solve: the optimal contract often cannot be derived in closed form. Holmstrom and Milgrom (1987, "HM") showed that simple contracts can be achieved under the joint assumptions of exponential utility, a pecuniary cost of effort, Gaussian noise, and continuous time. However, researchers may desire tractability without imposing the above assumptions – for example, if they wish to model DARA for empirical realism or to obtain wealth effects, or discrete time for clarity. We wished to develop a framework for tractable contracts in broader settings that do not require the above assumptions. This note summarizes this framework in a non-technical manner. We refer the reader to the full paper (Edmans and Gabaix (2011a)) for further detail on the general framework, and to Edmans and Gabaix (2011b) and Edmans, Gabaix, Sadzik, and Sannikov (2011) for two specific applications.

The model has T periods. The agent's reservation utility is $\underline{u}$, and his expected utility is:

$$\mathrm{E}\left[u\left(v\left(c\right)-\sum_{t=1}^{T}g\left(a_t\right)\right)\right]. \qquad (1)$$

- c is the cash paid by the principal to the agent in period T
- v is the "felicity" function over cash, and is increasing and weakly concave
- a_t is an action taken in period t, that benefits output but is costly to the agent (e.g. effort, taking the efficient project rather than private benefits, or choosing not to divert cash flows)
- g is the cost of effort, and is increasing and weakly convex
- u is the utility function, defined over "felicity minus cost of effort", and is increasing and weakly concave

The model contains both a utility function u and a felicity function v to maximize generality:

- Macroeconomic models typically use $\left(ce^{-g(a)}\right)^{1-\Gamma}/\left(1-\Gamma\right)$ (see e.g. the survey of Cooley and Prescott (1995)), which entails $u\left(x\right)=e^{(1-\Gamma)x}/\left(1-\Gamma\right)$ and $v\left(x\right)=\ln x$
- $u(x)=x$ denotes additively separable preferences
- $v(c)=\ln c$ generates multiplicative preferences
- $v(c)=c$ models the cost of effort as a subtraction to cash pay, e.g. if effort requires foregoing an alternative income-generating activity

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The key feature of our model that leads to tractability is our "noise-before-action" timing assumption. In each period t, the agent *first* observes noise η_t, and *then* takes the action a_t. This is similar to theories in which the agent observes total cash flow before deciding how much to divert (e.g., Lacker and Weinberg (1989); Biais et al. (2007); DeMarzo and Fishman (2007)). (Other moral hazard models feature "action-before-noise"). After the action is taken, a verifiable signal:

$$r_t = a_t + \eta_t$$

is publicly observed at the end of each period t. Then, noise η_{t+1} is observed at the start of period $t+1$, a_{t+1} is taken, and so on. We do not require that the noise be Gaussian.[2]

The principal wishes to implement the path of actions $(a_t^*)_{t=1,..,T}$ (exogenous for now), and solves for the cheapest contract $\widetilde{c}\,(r_1, ..., r_T)$ that does so. We take a fully optimal contracting approach that imposes no restrictions on the contract $\widetilde{c}(\cdot)$: for example, it may be nonlinear in the signals r_t. Despite the general setup, the contract takes a surprisingly simple closed form:

Theorem 1 ***(Optimal Contract)*** *The agent is paid*

$$c = v^{-1}\left(\sum_{t=1}^{T} g'\left(a_t^*\right) r_t + K\right), \tag{2}$$

where K is a constant that makes the participation constraint bind ($\mathrm{E}\left[u\left(\begin{array}{c}\sum_t g'\left(a_t^*\right) r_t + \\ K - \sum_t g\left(a_t^*\right)\end{array}\right)\right] = \underline{u}$). The functional form is independent of the utility function u, the reservation utility $\underline{u}$, and the distribution of the noise η; these parameters affect only the scalar K.

To see the intuition for why our timing assumption achieves tractability, consider a one-period model and drop the time subscript for simplicity. With standard "action-before-noise" timing, the agent's IC condition is given by:

$$\mathrm{E}_\eta\left[u'\left(v\left(c\left(r\right)\right) - g\left(a\right)\right)\left(v'\left(c\left(r\right)\right)c'\left(r\right) - g'\left(a\right)\right)\right] = 0, \tag{3}$$

where the expectation is taken over all possible future η. Importantly, the noise η is not known when the agent takes his action a. Thus, the IC condition only pins down his marginal incentives in expectation, i.e. *on average*. Multiple contracts satisfy (3), and so the problem is highly complex as the principal must solve for the cheapest contract out of this continuum.

By contrast, with "noise-before-action" timing, the noise is already known, so the expectations operator can be removed to give:

$$u'\left(v\left(c\left(r\right)\right) - g\left(a\right)\right)\left(v'\left(c\left(r\right)\right)c'\left(r\right) - g'\left(a\right)\right) = 0. \tag{4}$$

This leads to an additional simplification: the $u'\left(\cdot\right)$ term also drops out, which yields

$$v'\left(c\left(r\right)\right)c'\left(r\right) - g'\left(a\right) = 0.$$

[2] We do require that noises $\eta_2, ..., \eta_T$ be log-concave. This property is satisfied by most distribution functions.

This must hold for $a = a^*$, so the contract becomes $c = v^{-1}(g'(a^*)r + K)$.

Intuitively, the first-order condition (4) must hold for every realization of η, i.e., *state-by-state*. This is a tighter restriction than requiring incentives to be correct only on average. This requirement pins down the slope of the contract: for all η, the agent must receive a marginal felicity of $g'(a^*)$ for a one unit increment to the signal r_1. The principal's only degree of freedom is the constant K, which is itself pinned down by the participation constraint. In contrast to the "action-before-noise" case, here there is a single optimal contract and so the contracting problem has a simple solution.

The key advantage of tractability is that it allows the economic forces driving the contract to be transparent. First, (2) shows which parameters do and do not matter for the contract's *functional form*. It depends only on the felicity function v and the cost of effort g, and is independent of the utility function u, the reservation utility $\underline{u}$, and the distribution of noise η: they only affect the scalar K. Second, (2) shows which parameters do and do not matter for the contract's *slope*. Typically, u, $\underline{u}$, and η will affect the slope via their impact on K. However, if $v(c) = c$ (the cost of effort is pecuniary), the contract's slope is linear and independent u, $\underline{u}$, and η; if $v(c) = \ln c$ (multiplicative preferences), the contract is log-linear and independent of u, $\underline{u}$, and η. Third, (2) shows what determines the optimal *curvature* of the contract. If $v(c) = c$, the contract is linear; if $v(\cdot)$ is concave (convex), the contract is convex (concave). More generally, while the HM framework delivers only linear contracts, this framework allows for convex and concave contracts.

We also show that the contract retains the same form in continuous time, where noise and action are simultaneous: the optimal contract becomes

$$c = v^{-1}\left(\int_0^T g'(a_t^*)\, dr_t + K\right).$$

The paper also allows for the action a_t to depend on the noise η_t and shows that a noise-dependent action function $A(\eta)$ can also be implemented with a tractable contract. Finally, we solve for the optimal action function. We prove that, if the output under the agent's control is sufficiently large (e.g., he is a CEO who affects an entire firm), the optimal action is the highest productive effort level $(\overline{a})$ regardless of the noise η (the "high effort principle.") In a cash flow diversion model, this corresponds to zero stealing; in a project selection model, it corresponds to taking all positive-NPV projects while rejecting negative-NPV ones. Intuitively, the optimal action is a trade-off between the benefits and costs of effort. The former are of similar order of magnitude to the output under the agent's control, and the latter (disutility plus the risk imposed by incentives) are of similar order of magnitude to the agent's wage. Thus, if output is sufficiently large, the benefits of effort swamp the costs and so the high effort is efficient regardless of the realized noise.

The model can be extended to allow the principal to choose the highest productive effort level (e.g. plant size or capacity) according to the parameters of the setting. In the first stage, the principal chooses a capacity $\overline{a}$; in the second stage, the above contracting game is played out for this choice of $\overline{a}$. The problem appears complex, since the principal must choose both capacity $\overline{a}$ and the action function $A(\eta)$. However, under certain conditions, the problem can be reduced to optimizing over only capacity $\overline{a}$, which will be the target level of effort $a^* = \overline{a}$:

$$\max_{a^*} B(a^*) - C[a^*], \tag{5}$$

where $B(a^*) = E[b(a^*, \eta)]$ is the principal's expected payoff given effort a^*, and $C[a^*]$ is the expected cost of the contract implementing a^*. $C[a^*]$ obtains in closed forms in a series of cases.

A. Applications

Another advantage of tractability is it allows the model to be easily extended to accommodate other important features of real-life contracting situations. Two examples are below.

A.1. Market Equilibrium

Edmans and Gabaix (2011b) show that the contract can be embedded into a market equilibrium with multiple principals and agents. Typically, it is very difficult to embed a moral hazard model into a market equilibrium unless risk-neutrality is assumed; our setup allows for risk aversion and so we can study how risk affects both the optimal contract and the assignment of managers to firms. As in macroeconomic models, we take $u(x) = e^{(1-\Gamma)x}/(1-\Gamma)$ and $v(x) = \ln x$, so the utility function (1) specializes to:

$$U(c,a) = \frac{\left(ce^{-g(a)}\right)^{1-\Gamma}}{1-\Gamma}. \tag{6}$$

As in Gabaix and Landier (2008, "GL"), there is a continuum of firms of different size and managers with different talent. Firm $n \in [0,N]$ has size $S(n)$ and CEO $m \in [0,N]$ has talent $T(m)$. Low n denotes a larger firm and low m a more talented CEO: $S'(n) < 0$, $T'(m) < 0$. The CEO's talent increases "baseline" firm value according to:

$$s = S + CTS^{\gamma}.$$

Assume a one-period model and drop the time subscript. After taking the action a, the firm's final stock price is given by:

$$P = se^{a-\overline{a}+\eta}/E[e^{\eta}].$$

It can be easily shown that, up to a constant, we have $r = a + \eta$ as before.

GL assume a Pareto firm size distribution $S(n) = An^{-\alpha}$, and the following asymptotic value for the spacings of the talent distribution: $T'(n) = -Bn^{\beta-1}$. The equilibrium expected pay is:

$$w(n) = D(n_*)\, S(n_*)^{\beta/\alpha} S(n)^{\gamma-\beta/\alpha}, \tag{7}$$

where n_* is the index of a reference firm, $S(n_*)$ is the size of that reference firm, and $D(n_*)$ is a constant.

GL do not feature an agency problem and only specify expected pay. We incorporate the above agency model to determine the sensitivity of pay. We denote the marginal cost of effort as $\Lambda_n = g_n'(\overline{a}_n)$, and use σ_n to denote the standard deviation of η_n. The variable

$$\chi_n = g_n(\overline{a}_n) + \frac{\overline{\Gamma}\left(\Lambda_n^2 \sigma_n^2\right)}{2} \tag{8}$$

denotes the "equivalent variation" ("EV") associated with firm n, i.e., the loss suffered by the manager from disutility (the $g_n(\overline{a}_n)$ term) and risk (the $\overline{\Gamma}(\Lambda_n^2\sigma_n^2)/2$ term). Define $\overline{\chi}$ as the average of the firms' EVs:

$$e^{-\overline{\chi}} = E\left[e^{-\chi_n/(\alpha\gamma)}\right]^{\alpha\gamma}. \tag{9}$$

The market equilibrium with heterogeneous moral hazard and risk aversion can be summarized by three simple closed-form equations, one for each of assignment, pay, and incentives:

Theorem 2 ***(CEO Assignment, Pay, and Incentives in Market Equilibrium)*** *Rank managers by their talent T_n and firms by their "effective size" defined by:*

$$\widehat{S}_n = S_n e^{-\chi_n/\gamma}. \tag{10}$$

In equilibrium, the manager of rank n runs a firm whose effective size is ranked n, and receives an expected pay:

$$w_n = D(n_*)\, S(n_*)^{\beta/\alpha} S_n^{\gamma-\beta/\alpha} \exp\left(\frac{\beta}{\alpha\gamma}(\chi_n - \overline{\chi})\right), \tag{11}$$

where χ_n and $\overline{\chi}$ are defined by (8) and (9). The actual pay c_n is given by:

$$\ln c_n = \Lambda_n r_n + \ln w_n - \ln E\left[e^{\Lambda_n r_n}\right]. \tag{12}$$

Again, closed-form solutions allow the economics to be transparent. The first equation is for *assignment.* In a pure assignment model (GL) or an assignment model with homogeneous moral hazard and risk-neutrality (Edmans, Gabaix, and Landier (2009)), managers are matched to firms purely based on their size S_n. The most talented managers are hired by the largest firms. We show that this assignment is distorted with heterogeneous moral hazard and risk aversion. Firms with a high EV, due to either high disutility (high $g_n(\overline{a}_n)$) or high risk (high σ_n) optimally choose to hire less talented managers. Wealth effects (which the framework allows as it does not require exponential utility) are critical for generating this result: more talented (and thus wealthier) managers must be paid a higher compensation for both disutility and risk, and so firms with a high EV choose not to hire such managers.

The second equation is for expected *pay.* In a pure assignment model or an assignment model with homogeneous moral hazard and risk-neutrality, cross-sectional differences in pay stem exclusively from differences in firm size. Here, pay is also greater at firms which exhibit higher disutility or risk. However, what matters is not the absolute level of these parameters (χ_n), but their level compared to other firms in the economy ($\chi_n - \overline{\chi}$). Aggregate, economy-wide increases in disutility and risk have no effect on pay: even though a manager's current firm becomes less attractive to work for, so do the outside options.

The third equation is for *incentives.* Log pay ($\ln c_n$) is linear in the firm's log stock return (r_n), i.e. the optimal measure of incentives is the percentage change in pay for a percentage change in firm value. The model thus provides theoretical justification for this measure of incentives, previously advocated by Murphy (1999) on empirical grounds.

A.2. Dynamic CEO Compensation

Edmans, Gabaix, Sadzik, and Sannikov (2011) add additional dynamics to the core model, by extending it to allow for intermediate consumption, the agent (a CEO) to privately save across periods, and for the CEO to engage in myopic actions which boost the short-term return at the expense of long-run value. For brevity we refer the reader to the paper for the myopia extension.

The CEO now consumes in every period t. He lives in periods 1 through $T \leq \tau$ and retires after period $L \leq T$. We take the utility function (6) from Edmans and Gabaix (2011b) and specialize to $\Gamma = 1$ so his utility is given by:

$$U = \sum_{t=1}^{T} \rho^t (\ln c_t - g(a_t)).$$

We consider two versions of the model: one in which private saving is impossible (so the agent must consume his entire income) and another in which it is possible. In the latter case, we must now distinguish between the CEO's consumption c_t and the income provided by the contract, which we denote y_t. The agent saves $(y_t - c_t)$ at the continuously compounded risk-free rate R. There are now two IC constraints: the principal must ensure that the agent takes action path $(a_t^*)_{t=1,..,T}$, and does not deviate from the contract by privately saving ($y_t = c_t \ \forall \ t$). Again, the optimal contract is tractable, and given as follows:

Theorem 3 ***(Optimal Contract, Full Dynamics)*** *In each period t, the CEO is paid a compensation c_t which satisfies:*

$$\ln c_t = \ln c_0 + \sum_{s=1}^{t} \theta_s r_s + \sum_{s=1}^{t} k_s, \tag{13}$$

where θ_s and k_s are constants. The sensitivity θ_s is given by

$$\theta_s = \begin{cases} \frac{g'(a_s^*)}{1+\rho+\ldots+\rho^{T-s}} \text{ for } s \leq L, \\ 0 \text{ for } s > L. \end{cases} \tag{14}$$

If private saving is impossible, the constant k_s is given by:

$$k_s = R + \ln \rho - \ln E[e^{\theta_s(a_s^* + \eta)}]. \tag{15}$$

If private saving is possible, k_s is given by:

$$k_s = R + \ln \rho + \ln E[e^{-\theta_s(a_s^* + \eta)}]. \tag{16}$$

The initial condition c_0 is chosen to give the agent his reservation utility $\underline{u}$.

Again, tractability allows the economics to be transparent. First, the contract is *history-dependent*: time-t income is linked to the return in the current and all previous periods. The rewards for current performance are spread over all future periods, to achieve intertemporal risk-sharing. Second, regarding the contract's *sensitivity*, with a fixed target action ($a_t^* = a^* \ \forall \ t$) and an infinite horizon ($T = \tau \to \infty$), the sensitivity is constant and given by $\theta_t = \theta = (1 - \rho)\, g'(a^*)$. The sensitivity increases in the marginal cost of effort and decreases in the discount rate: if the CEO is more impatient, he must be rewarded today rather than in the future. With a finite T, the slope θ_t is increasing over time: since there are fewer periods left over which to smooth a given reward for performance, the current reward must increase. Third, it shows the effect of allowing for *private saving* on the contract. Whether the CEO has the option to privately save has no effect on the slope. Instead, it affects the constant k_t and thus the growth rate of consumption. If the CEO cannot privately save, the growth rate is $\ln E\left[c_t/c_{t-1}\right] = R + \ln \rho$. It is positive if and only if the CEO is more patient than the representative agent, as is intuitive. If the CEO can privately save (and so the contract must deter private saving), the growth rate becomes $\ln E\left[c_t/c_{t-1}\right] = R + \ln \rho + \theta_t^2 \sigma_t^2$ and so is faster than in the no-saving case. This is because an incentive contract exposes the CEO to risk, so he wishes to save to insure himself; a rapidly growing contract effectively saves for the CEO, removing the need for him to do so himself. The growth rate of pay depends on the risk to which the CEO is exposed, which is in turn driven by his sensitivity to the firm's returns θ, and firm risk σ.

Due to its simplicity, the contract can be illustrated by a numerical example. We first set $T = 3$, $L = 3$, $\rho = 1$, $a_t^* = a^*$ and $g'\left(a^*\right) = 1$. From (14), the contract is:

$$\begin{aligned}
\ln c_1 &= \frac{r_1}{3} + \kappa_1 \\
\ln c_2 &= \frac{r_1}{3} + \frac{r_2}{2} + \kappa_2 \\
\ln c_3 &= \frac{r_1}{3} + \frac{r_2}{2} + \frac{r_3}{1} + \kappa_3
\end{aligned}$$

where $\kappa_t = \sum_{s=1}^{t} k_s$. An increase in r_1 leads to a permanent increase in log consumption – it rises by $\frac{r_1}{3}$ in all future periods. In addition, the sensitivity $\partial u_t/\partial a_t$ increases over time, from 1/3 to 1/2 to 1/1. The total lifetime reward for effort $\partial U_t/\partial a_t$ is a constant 1 in all periods.

We now consider $T = 5$, so that the CEO lives after retirement. The contract is now:

$$\begin{aligned}
\ln c_1 &= \frac{r_1}{5} + \kappa_1 \\
\ln c_2 &= \frac{r_1}{5} + \frac{r_2}{4} + \kappa_2 \\
\ln c_3 &= \frac{r_1}{5} + \frac{r_2}{4} + \frac{r_3}{3} + \kappa_3 \\
\ln c_4 &= \frac{r_1}{5} + \frac{r_2}{4} + \frac{r_3}{3} + \kappa_4 \\
\ln c_5 &= \frac{r_1}{5} + \frac{r_2}{4} + \frac{r_3}{3} + \kappa_5.
\end{aligned} \tag{17}$$

Since the CEO takes no action from $t = 4$, his pay does not depend on r_4 or r_5. However, it depends on r_1, r_2 and r_3 as his earlier efforts affect his wealth, from which he consumes.

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