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Précis of Results on Linearity-Generating Processes

This paper uses the Linearity-Generating (LG) processes defined and analyzed in Gabaix (2009); this note offers a brief summary of parts of that paper.[1] LG processes are given by $M_t D_t$, a pricing kernel M_t times a dividend D_t, and X_t, an n-dimensional vector of factors (that can be thought of as stationary). For instance, for bonds, the dividend is $D_t = 1$. Here, I review the discrete-time LG process. By definition, a process $M_t D_t \left(1, X_t\right)$ is LG if and only if there are constant $\alpha \in \mathbb{R}$, γ, $\delta \in \mathbb{R}^n$, and $\Gamma \in \mathbb{R}^{n\times n}$ such that for all $t = 0, 1, ...,$ $M_t D_t > 0$ and

$$\begin{aligned} E_t\left[\frac{M_{t+1}D_{t+1}}{M_t D_t}\right] &= \alpha + \delta' X_t & (1) \\ E_t\left[\frac{M_{t+1}D_{t+1}}{M_t D_t}X_{t+1}\right] &= \gamma + \Gamma X_t. & (2) \end{aligned}$$

Higher moments need not be specified. For instance, the distribution of the noise does not matter, which makes LG processes parsimonious. As a shorthand, $M_t D_t \left(1, X_t\right)$ is an LG process with generator $\Omega = \begin{pmatrix} \alpha & \delta' \\ \gamma & \Gamma \end{pmatrix}$.

Stock and bond prices obtain in closed form. The price of a stock $P_t = E_t\left[\sum_{s\geq t} M_s D_s\right]/M_t$ is, with I_n, the identity matrix of dimension n:

$$P_t = D_t \frac{1+\delta'\left(I_n-\Gamma\right)^{-1}X_t}{1-\alpha-\delta'\left(I_n-\Gamma\right)^{-1}\gamma}. \quad (3)$$

The price-dividend ratio of a "bond," or $Z_t\left(T\right) = E_t\left[M_{t+T}D_{t+T}\right]/\left(M_t D_t\right)$, is:

$$\begin{aligned} Z_t\left(T\right) &= \begin{pmatrix} 1 & 0_n \end{pmatrix}\Omega^T \begin{pmatrix} 1 \\ X_t \end{pmatrix} & (4) \\ &= \alpha^T + \delta' \frac{\alpha^T I_n - \Gamma^T}{\alpha I_n - \Gamma} X_t \text{ when } \gamma = 0. & (5) \end{aligned}$$

[1] It is in particular the way it is used in Gabaix (forth.)

Hence, the "recipe" to solve a model using LG processes is very simple: First, calculate the LG moments (1)-(2), to obtain the values of α, δ, γ, and Γ. Second, use (3) and (4)-(5) to solve for stock and bond prices.

Conversely, the "recipe" to construct a model using LG processes is to force the model's primitives (e.g., twists in the AR(1) processes) to satisfy (1)-(2). Then, the model is very easy to solve by the above procedure.

To ensure that the process is well-behaved (and, hence, will prevent prices from being negative), the volatility of the process has to go to zero near some boundary. Gabaix (2009) details these conditions.

Reference

Gabaix, Xavier, "Linearity-Generating Processes: A Modelling Tool Yielding Closed Forms for Asset Prices," Working Paper, New York Uuniversity, 2009.

Gabaix, Xavier "Variable Rare Disasters: An Exactly Solved Framework for Ten Puzzles in Macro-Finance," *Quarterly Journal of Economics*, forthcoming