

A Dynamic Theory of Mutual Fund Runs and Liquidity Management*

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Abstract

I model an open-end mutual fund investing in illiquid assets and show that the fund's endogenous cash management can generate shareholder runs even with a flexible NAV. The fund optimally re-builds its cash buffers at time $t + 1$ after outflows at t to prevent future forced sales of illiquid assets. However, cash rebuilding at $t + 1$ implies predictable voluntary sales of illiquid assets, generating a predictable decline in NAV. This generates a first-mover advantage, leading to runs. A time-inconsistency problem aggravates runs: the fund may want to pre-commit not to re-build cash buffers but cannot credibly do so absent a commitment device.

KEYWORDS: open-end mutual fund, illiquid assets, shareholder runs, cash rebuilding, flexible NAV.

JEL: G01, G21, G23, G32, G33, D92

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1 Introduction

There are rising concerns about the financial stability risks posed by open-end mutual funds, which promise daily liquidity to shareholders but have been increasingly holding illiquid assets such as corporate bonds, emerging market assets, bank loans, and even real estates. Given this liquidity mismatch, regulators are worried about the potential for a bank-run-like scenario on mutual funds,¹ and a large number of funds have experienced bank-run-like redemptions in 2015 and 2016.² However, despite the prominence of this issue, the theoretical mechanism of mutual fund runs is not well understood and the existence of runs is still in dispute. First, conventional wisdom suggests that mutual funds with a flexible end-of-day net asset value (NAV) should be immune to bank-run-like crises, which occur only with fixed-NAV claims. Second, observers also argue that careful fund liquidity management can mitigate first-mover advantages and hence prevent runs. With these two points in mind, can there really be runs on mutual funds?

In this paper, I develop a model of an open-end mutual fund that invests in illiquid assets and show that shareholder runs can occur in equilibrium even with a flexible NAV. The main insight is that the combination of a flexible NAV and active fund liquidity management, both of which are viewed as means to mitigate financial stability risks, can make the fund prone to shareholder runs when the underlying illiquid asset prices are not perfectly forward-looking.

The mechanism works as follows. The fund optimally re-builds its cash buffers at time $t+1$ after outflows at t to prevent future forced sales of illiquid assets. However, cash-rebuilding implies predictable voluntary sales of illiquid assets and hence a predictable decline in NAV. This generates a first-mover advantage, leading to shareholder runs. A time-inconsistency problem further aggravates shareholder runs: the fund may want to pre-commit not to re-build its cash buffers but cannot credibly convince the shareholders not to run absent a commitment device. Thus, despite optimal liquidity management, mutual funds are not run-free and runs can lead to higher ex-ante asset sale losses.

My theoretical predictions are consistent with new micro-level evidence. [Chen, Goldstein and Jiang \(2010\)](#), [Feroi, Kashyap, Schoenholtz and Shin \(2014\)](#), [Goldstein, Jiang and Ng \(2015\)](#), [Shek, Shim and Shin \(2015\)](#) and [Wang \(2015\)](#) document that current fund outflows predict a future decline in fund NAV, and the magnitude of the predictable decline in NAV is larger if the fund invests in more illiquid assets or

¹The U.S. SEC implemented new rules on Nov 13, 2016 requiring mutual funds to manage liquidity risks. Also see “Potential Emerging Threats and Vulnerabilities,” Ch. 7 in the *Annual Report*, the U.S. FSOC, May 2015, and “Asset Management Industry and Financial Stability,” Ch. 3 in the *Global Financial Stability Report*, the IMF, April 2015.

²The Third Avenue shutdown on Dec 10, 2015 after severe runs was the first case since the 1940 Act that a U.S. mutual fund shut down redemptions without getting approval from the U.S. SEC. Notably, the Focus Credit Fund was the single largest holder of many high-yield corporate bonds, the fundamental of which were still good. This suggests that liquidity mismatch played an important role in its crisis. Many so-called “liquid-alternative” mutual funds, operated by hedge fund managers such as Whitebox Advisors, J.P. Morgan, and Guggenheim Partners also experienced shareholder runs and were forced to close in 2015. In 2016, many real estate funds in the UK also experienced severe runs after the vote for the Brexit.

has less cash. My model provides one mechanism to generate a first-mover advantage consistent with these patterns and shows that it can indeed lead to runs in equilibrium. Moreover, I show that the potential for runs can in turn distort fund liquidity management.

I formulate the ideas sketched above in a dynamic model of an open-end mutual fund. The fund has both cash and many illiquid assets. It has many shareholders, who may redeem daily at the end-of-day flexible NAV. The fund minimizes total expected asset sale losses by managing its cash buffer over time. Section 2 lays out the model, which is built based on a realistic assumption that the sale prices of illiquid assets are time-varying but not perfectly forward-looking. Specifically, flow-induced sales of illiquid assets can create temporary sale price overshooting at t and partial reversal at $t + 1$, as documented by Coval and Stafford (2007) and Duffie (2010). This price pattern gives rise to a motive for fund liquidity management. After an outflow at t , the fund may actively sell illiquid assets to rebuild its cash buffer at $t + 1$ when the sale price partially rebounds in order to avoid potentially more severe forced sales at $t + 2$ should another outflow shock come, but the prices at t do not fully reflect predictable cash rebuilding and active sales at $t + 1$. Other than this notion of asset illiquidity and the resulting liquidity mismatch, my model does not feature other frictions at the asset level.

I show in Section 3 that under liquidity mismatch, the fund's desire to rebuild its cash buffer can induce shareholder runs, and more rapid cash rebuilding leads to more severe runs. Runs can occur in equilibrium regardless of whether the fund starts with a high cash position or a low one. However, the nature of strategic interactions among shareholders differs between these two cases.

When the fund starts with a high cash position, cash rebuilding induces shareholder runs by endogenously giving rise to a first-mover advantage. When a redemption shock occurs at t , the fund starting with a high cash position can satisfy the projected redemptions at both t and $t + 1$ without incurring sales. This implies that even if the shareholders who initially plan to redeem at $t + 1$ ran at t , the fund would still have enough cash at t , and thus time- t NAV would not adjust. However, since some cash is paid out at t , the fund may want to rebuild its cash buffer by actively selling some illiquid assets at $t + 1$. Thus, the shareholders who initially plan to redeem at $t + 1$ would get a lower NAV if they waited until $t + 1$, and hence may decide to run at t . Fundamentally, cash rebuilding generates a strategic complementarity among shareholders, which ultimately leads to runs.

Alternatively, if the fund starts with a low cash position such that it cannot satisfy the projected redemptions at both t and $t + 1$ without selling illiquid assets, cash rebuilding can still induce shareholder runs despite the flexibly adjusted NAV at t . Interestingly, a shareholder is less likely to run if more of the other shareholders decide to run. This is because runs may force the fund to sell more of its illiquid assets

at an extremely low price at t , and any shareholder who runs at t must share that cost and get a lower NAV. This means that shareholders' run decisions can exhibit strategic substitutability. However, since the fund is already running out of cash and may actively sell more assets at $t + 1$ to rebuild its cash buffer, waiting may only give the shareholders an even lower NAV. Therefore, the fund's desire to rebuild its cash buffer reinforces a strong incentive for shareholders to redeem earlier, despite the strategic substitutability.

Having analyzed the implications of cash rebuilding on shareholder runs for an arbitrary starting level of cash, I endogenize the dynamic cash rebuilding policy of the fund. I show in Section 4 that introducing the potential for runs gives rise to a tension absent in existing liquidity management theories. On the one hand, rebuilding the cash buffer more rapidly at $t + 1$ can trigger shareholder runs at t . As described above, shareholder runs lead to more sales. This run concern makes a more rapid cash rebuilding policy less appealing. On the other hand, adopting a less rapid cash rebuilding policy at $t + 1$ makes the fund more likely to suffer another round of future forced sales at $t + 2$. Moreover, carrying less cash to $t + 2$ also implies that the fund will have to ultimately rebuild its cash buffer more rapidly at time $t + 3$, which can trigger future runs at $t + 2$. With this tension, the fund's optimal dynamic cash rebuilding policy is significantly different from the benchmark case where there are no runs.

Moreover, I show that the potential for shareholder runs introduces a time-inconsistency problem for the fund, which aggravates the tension in choosing between a rapid or slow cash rebuilding policy. When the cost of runs at t is relatively large, ex-ante, the fund may wish to commit itself to rebuilding its cash buffer less rapidly at $t + 1$ to reduce run risks at t . However, ex-post, the fund may instead be tempted to adopt a more rapid cash rebuilding policy at $t + 1$, because the time- t cost is sunk. Anticipating this, shareholders will always have strong incentives to run at t . In other words, in the absence of a commitment device, the fund cannot make credible announcement to convince shareholders not to run. Overall, my paper provides theoretical underpinnings for understanding why open-end mutual funds may not be run-free, in contrast to what the conventional wisdom suggests. The potential for shareholder runs can considerably increase sale losses in expectation despite optimal cash management by the fund.

Fundamentally, shareholder runs in my model are driven by a key property of the NAVs of mutual funds investing in illiquid assets: they are flexible but not perfectly forward-looking. The sale prices of illiquid assets and thus fund NAVs at time t do not fully take into account the predictable asset sales at $t + 1$. Hence, endogenous fund cash rebuilding gives rise to predictable declines in NAV and a first-mover advantage. Other than this insight under liquidity mismatch, my model does neither necessarily predict massive redemptions nor imply that all mutual funds are subject to runs.

Theoretically, my model provides one plausible channel that restores the classic bank run mechanism

(Diamond and Dybvig, 1983) in the mutual fund context, which is viewed by many observers to be run-free because it does not feature a fixed-value claim. In my model, the first-mover advantage does not directly come from an exogenous fixed-NAV claim at t (like the deposit at a bank). Because fund NAVs in my model flexibly and endogenously adjust, a shareholder redeeming at t realizes that more early withdrawals will potentially induce more sales at t and thus lower the proceeds she receives. Hence, if the fund did not rebuild its cash buffer at $t + 1$, the net benefit of running over waiting could be decreasing as more shareholders run. Rather, it is the fund's desire to rebuild its cash buffer at $t + 1$ and the resulting predictable decline in NAV that lead to a strong first-mover advantage. This mechanism highlights a dynamic interaction between the fund and its shareholders. Such an interaction is absent in classic bank run models, which focus on coordination failures among depositors themselves.

The mechanism in my model also differs from that underlying market runs. Bernardo and Welch (2004) and Morris and Shin (2004) argue that if an asset market features a downward-sloping demand curve, investors fearing future liquidity shocks will have an incentive to front-run, fire selling the asset earlier to get a higher price. One might imagine that introducing an intermediary that helps manage liquidity shocks can alleviate such problems. Indeed, in my model fund cash management is beneficial to shareholders because it reduces sale losses. However, the key is that the fund's cash rebuilding also endogenously gives rise to predictable declines in NAV and thus run incentives. In contrast, there is no role for liquidity management in market run models. In this sense, market run models focus on asset markets themselves while my theory focuses on the role of financial intermediaries. This allows me to distinguish between risks that come from active management of financial intermediaries and those that are only a reflection of market-level frictions and would occur in the absence of intermediaries.

My model generates new policy implications, which I explore in Section 5. I consider several fund-level policies, including in-kind redemptions, redemption fees and restrictions, credit lines, and swing pricing, all of which aim to mitigate financial stability risks of mutual funds. Perhaps surprisingly, some of these policies do not necessarily improve shareholder welfare in equilibrium because they may distort fund liquidity management, and thus lead to more asset sale losses. Overall, my model suggests that policies should be designed with the dynamic interdependence of runs and fund liquidity management in mind.

Related Literature. This paper first contributes to the burgeoning literature on financial stability risks posed by open-end mutual funds.^{3,4} Empirically, Feroli, Kashyap, Schoenholtz and Shin (2014)

³Relatedly, Schmidt, Timmermann and Wermers (2016) show the existence of shareholder runs on MMFs.

⁴There is a broader literature on the costs of outflows to non-trading shareholders and to future fund performance; see Christoffersen, Musto and Wermers (2014) for a review. Edelen (1999), Dickson, Shoven and Sialm (2000), Alexander, Cici and Gibson (2007) and Christoffersen, Keim, and Musto (2007) find that flow-induced trades hurt fund performance, and redeeming shareholders impose externalities on non-trading shareholders through trading-related costs (including commissions, bid-ask spreads, and taxes) that are not reflected in current NAVs. Coval and Stafford (2007) further show this by highlighting the

find that fund outflows predict future declines in NAV, suggesting the existence of run incentives for shareholders. At a more micro level, [Chen, Goldstein and Jiang \(2010\)](#) find that the flow-to-performance relationship is stronger for funds investing in less liquid stocks. [Goldstein, Jiang and Ng \(2015\)](#) echo the message by showing that corporate bond funds even exhibit a concave flow-to-performance relationship. [Shek, Shim and Shin \(2015\)](#) explore the underlying channel by showing that outflows are associated with future discretionary bond sales and liquidity rebuilding in an emerging market bond fund context. [Wang \(2015\)](#) finds that outflows predict a stronger decline in future NAVs when the fund has less cash or invests in more illiquid bonds. My model predictions are consistent with these facts above.

Notably, [Chen, Goldstein and Jiang \(2010\)](#) and [Morris and Shin \(2014\)](#) have addressed the potential for mutual fund runs from theoretical perspectives. Both papers deliver new and unique predictions regarding the risks of mutual funds, but their focus and approach is different from this paper. [Chen, Goldstein and Jiang \(2010\)](#) build a static global game model for the purpose of hypothesis development and focus on the relationship between asset illiquidity and the flow-to-performance relationship. In that model, shareholders who run will get a fixed-value claim if the fund is solvent (in the spirit of [Diamond and Dybvig, 1983](#)). They also do not consider fund liquidity management. [Morris and Shin \(2014\)](#) build a model of runs by funds on the asset markets, focusing on managers' relative performance concerns. They do not distinguish between open- and closed-end funds and thus do not consider shareholder runs.

My paper also contributes to the literature of mutual fund liquidity management. This literature suggests that holding cash is costly because funds must give up investment opportunities ([Wermers, 2000](#)), but cash can help them withstand redemption shocks ([Edelen, 1999](#), [Christoffersen, Keim, and Musto, 2007](#), [Coval and Stafford, 2007](#)). [Simutin \(2013\)](#) investigates the determinants of cash management for equity funds. [Chernenko and Sunderam \(2015\)](#) further cover both bond and equity funds and show that even careful liquidity management cannot fully alleviate fire-sale costs. The most relevant theory is [Chordia \(1996\)](#) who shows in a static model that funds hold more cash when there is uncertainty about redemptions. My paper documents a new aspect of fund liquidity management: rebuilding cash buffers by selling illiquid assets can induce shareholder runs, which can in turn distort fund liquidity management.

In addition to the discussion above regarding the difference from bank runs and market runs, my paper contributes to the bank run literature in several aspects.^{5,6} First, [Allen and Gale \(1994, 2005\)](#) show that channel of flow-induced sales. These papers do not examine the potential for shareholder runs.

⁵To review the entire bank run literature is beyond the scope of this paper; I refer interested readers to [Gorton and Winton \(2003\)](#) for a survey. There is also a growing literature about runs on non-bank but leveraged financial institutions, for instance, [Liu and Mello \(2011\)](#) on leveraged hedge fund runs with a focus on how cash buffers help mitigate runs, [Martin, Skeie and von Thadden \(2014\)](#) on repo runs, and [Parlatore \(2015\)](#) on MMF fragility with a focus on sponsor support. These theories resemble classic bank run models in that investors still get a fixed-value claim if they run.

⁶[Green and Lin \(2003\)](#) and [Peck and Shell \(2003\)](#) examine more flexible contracts that allow the bank to condition the payment to each depositor on the number of agents who claimed early withdrawal before her. In other words, the deposit

market liquidity is important in determining the possibility of runs and [Cooper and Ross \(1998\)](#) pioneer to examine how banks manage their liquidity buffers. In that literature, liquidity re-building helps mitigate runs. By combining liquidity management and flexible NAV adjustment, I find the opposite in mutual funds: liquidity re-building can generate unintended shareholder runs. Second, a new literature considers runs in a dynamic framework, for example, [He and Xiong \(2012\)](#) and [Cheng and Milbradt \(2012\)](#). There, investors' run decisions depend on each other inter-temporally but the bank is cashless. My paper differs in that shareholders' run decisions depend on each other through the fund's endogenous liquidity management. Last, [Ennis and Keister \(2010\)](#) is closely related in that they consider the interaction between the bank's endogenous deposit policy intervention and its depositors' endogenous runs. There, depositors anticipate whether the bank will freeze remaining deposits in response to a first wave of runs, and the anticipated policy in turn influences their run behavior. By considering the interaction between shareholder runs and mutual fund liquidity management under a flexible fund NAV, my paper helps reveal the economic conditions under which mutual funds are also subject to run risks.

2 The Model

2.1 Setup

Time is discrete and infinite. Discount rate is normalized to 1. There is a single open-end mutual fund investing in two types of assets: 1) cash, which is liquid and is the only consumption good, and 2) a continuum of many illiquid assets. The illiquid assets have an intrinsic fundamental value $R > 0$, which will be paid off at the end of the game (specified later), but they do not generate any interim cash flows.

At the beginning of any date t , the fund has x_t cash and a_t unit of the basket of illiquid assets. The fund also has n_t existing shareholders, some of whom may exit the fund by redeeming their shares in future. To focus on redemptions, I assume that the fund has no inflows or credit lines.⁷ Redemptions must be met in cash,⁸ so the fund may be forced to sell its illiquid assets if running short of cash, but doing this will generate price impact and thus losses because of the underlying illiquidity problem. Specifically, the unit sale price for any illiquid assets on date t is p_t , which is lower than R and will be specified later.

Flexible Fund Net Asset Value (NAV). The end-of-day flexible NAV will reflect all the asset sale value there is flexible. However, as the authors acknowledge, these bank deposit contracts are not observed in practice.

⁷I will relax this assumption in Section 5.4. As shown there, having credit lines cannot reduce potential financial stability risks of mutual funds but may instead aggravate them in some cases.

⁸In Section 5 I analyze emergency rules such as redemption restrictions and in-kind redemptions.

losses during the given day. Specifically, if the fund does not sell any assets on date t , it will be

$$NAV_t = \frac{a_t R + x_t}{n_t}.$$

However, if the fund sells any assets on date t at the sale price $p_t < R$, the NAV will reflect those losses:

$$NAV_t = \frac{x_t + (a_t - a_{t+1})p_t + a_{t+1}R}{n_t}. \quad (2.1)$$

In (2.1), the *market* prices of those non-traded (and different) assets will not change. This is true for illiquid assets, especially for those traded in OTC markets such as corporate bonds.⁹ This is also consistent with the empirical evidence in Coval and Stafford (2007) that flow-induced sales only have temporary and local price impacts within the assets being sold.

What is crucial in (2.1) is that the end-of-day NAV is flexible in the sense that it considers all the same-day price impact and asset sale losses, which is different from the fixed deposit value in Diamond and Dybvig (1983). However, fund NAV is still not perfectly forward-looking in the sense that it will not perfectly reflect future price impacts induced by future asset sales. These contractual features of fund NAV are robust regardless of the nature of different asset markets and accounting rules.

Fund Management. On any date t , the fund manager's objective is to minimize the total expected asset sale losses that she incurs at t and going forward. The formal objective function will be clear after I describe the asset market.

Since all the shareholders are ex-ante identical, having a fund which minimizes total expected asset sale losses implies that there is no ex-ante agency friction between the fund manager and the collective shareholders. In this sense, the fund's objective parsimoniously captures the outcome of optimal contract design between investors and the asset manager (e.g., Bhattacharya and Pfleiderer, 1985). Minimizing total asset sale losses is also consistent with the fund manager's compensation being tied to the size or equivalently the assets under management (AUM) of the fund, which is common in practice.

Timeline. Each stage consists of two dates, an even date and an odd date.¹⁰ I use $2t$ to denote an even date and $2t + 1$ to denote an odd date. I still use t to denote a stage or a general date when the difference between even and odd dates is not important.

⁹In practice, asset prices may be correlated, and mutual funds may also use matrix pricing for non-traded assets based on the sale price p_t . But as I will discuss in the online appendix B.2, asset price correlations or different accounting rules such as matrix pricing are not crucial for my model mechanism and will not change the insights of this model.

¹⁰I purposefully use this timeline structure to better contrast my model mechanism to classic bank run models; this model can be viewed as a repeated version of Diamond and Dybvig (1983) but with important changes to capture mutual funds. To make distinctions between even and odd dates is also a common modeling tool in the literature to capture time-varying market conditions. See Woodford (1990) and Lagos and Wright (2005) for examples.

At the beginning of each stage (i.e., right before an even date), a shock hits the economy. Specifically, with probability π the game ends; otherwise the game continues. Only if the game continues, will there be projected redemptions on the following two dates within the given stage. The random end-of-game event can be thought of as an upside event in which all the illiquid assets mature at their fundamental value, shareholders get paid off, and there will be no future redemption needs. I call this shock redemption shock in what follows. Figure 1 shows the timeline.

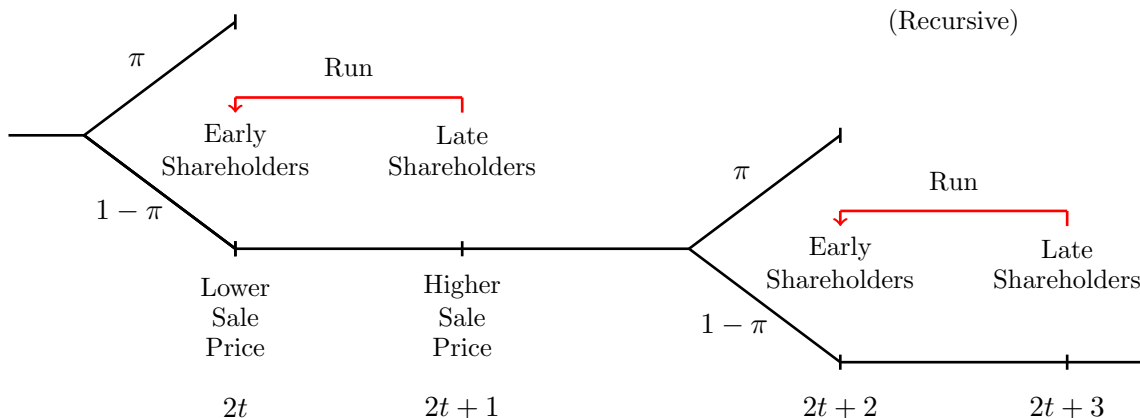


Figure 1: Timeline

Shareholders. There are three groups of shareholders within each stage: early shareholders, late shareholders, and sleepy shareholders, all having no cash in advance. Specifically, if the game continues on an even date $2t$, $\mu_E n_{2t}$ early shareholders and $\mu_L n_{2t}$ late shareholders are hit by unanticipated consumption needs and thus must redeem their shares, where $0 < \mu_E, \mu_L < 1$ and $0 < \mu_E + \mu_L < 1$.¹¹ Since consumption needs are unanticipated, the remaining $(1 - \mu_E - \mu_L) n_{2t}$ sleepy shareholders do nothing but wait until the next stage; they do not plan ahead for future stages although they may randomly become early or late shareholders then.¹² Shareholders get the endogenous and flexible end-of-day NAV when redeeming.

Early shareholders must consume on date $2t$, so they always redeem their shares at the endogenous end-of-day NAV on $2t$. Late shareholders prefer to consume on date $2t + 1$, but can also choose to consume on date $2t$. Formally, late shareholders' utility function is:

¹¹As will become clear soon, the redemptions themselves do not necessarily assume or imply runs. The fractions μ_E and μ_L are assumed to be exogenous and fixed across stages to ensure stationarity and make the model tractable. In the online appendix B.3 I discuss a potential approach to endogenize them to capture the flow-to-performance relationship, which is likely to worsen the trade-off underlying fund liquidity management and make run problems more persistent.

¹²This is consistent with the observation that many mutual fund shareholders are mom-and-pop investors: they do not actively review their portfolios but only do so when subject to unanticipated liquidity shocks (for empirical evidence, see Agnew, Balduzzi and Sunden, 2003, Mitchell, Mottola, Utkus and Yamaguchi, 2006, Brunnermeier and Nagel, 2008). Institutional investors like insurance companies and pension funds also review and update their mutual fund asset portfolio infrequently. From a theoretical point of view, this helps construct a tractable dynamic game with a long-run player (the fund manager) and many generations of short-run players (the shareholders). It also allows me to focus on fund liquidity management as the only channel that links different stages.

$$u_L(c_{2t}, c_{2t+1}) = \theta c_{2t} + c_{2t+1}, \quad (2.2)$$

where $0 \leq \theta \leq 1$. As late shareholders are risk neutral,¹³ their consumption choice boils down to a binary problem: to redeem on date $2t$ or date $2t + 1$. There is no outside storage technology. Thus, if a late shareholder redeems on date $2t$, she gets the endogenous end-of-day NAV on $2t$ and must consume immediately; otherwise she gets the endogenous end-of-day NAV on $2t + 1$ and consume then.

Runs. If a late shareholder chooses to redeem on date $2t$, I define that the late shareholder *runs* the fund in stage t . I allow late shareholders to choose mixed strategies: the run probability of late shareholder i is denoted by $\lambda_{2t}^i \in [0, 1]$. Clearly, a late shareholder's run decision depends on the difference of NAVs between the two dates, which in turn depends on other late shareholders' run decisions and the fund manager's asset re-allocation decision in the given stage. The preference parameter θ in (2.2) parsimoniously captures different types of shareholders with varying propensities to run.^{14,15} Intuitively, when θ is lower, late shareholders are less likely to run even if the NAV is lower on date $2t + 1$.

It is worth noting that the projected redemptions themselves do not necessarily assume or imply runs in my model, and runs do not imply massive redemptions either. Based on the formal definition above, there will be no runs if all the late shareholders redeem on late dates. Rather, runs can happen in any given stage due to an endogenous first-mover advantage within that stage. More interestingly, runs in different stages will become inter-dependent due to the fund's dynamic liquidity management.

The setting described above represents a mutual fund crisis management scenario, during which the fund experiences repeated redemption shocks before a random recovery time.¹⁶ In the setup, π measures how persistent the redemptions shocks are, or in other words how likely the economy is to recover from a bad market condition. When π is lower, the game is more likely to continue, and thus the fund is more likely to experience redemptions in the next stage. As the fund manager never knows when the game will end, liquidity management indeed helps the fund minimize its total expected sale losses, and matters more

¹³In Diamond-Dybvig type bank run models, depositors are usually assumed to be risk averse, and demandable deposit emerges as the optimal contract for risk-sharing between early and late depositors. Instead, I focus on the commonly observed contractual features of open-end mutual funds rather than optimal contract design, so I assume risk-neutrality to help better document the impact of flexible NAVs on shareholders' consumption choices.

¹⁴There are many plausible explanations for different types of shareholders having different values of θ . For example, [Chen, Goldstein and Jiang \(2010\)](#) suggest that institutional investors may have a lower θ because they often have stricter investment targets and are more likely to internalize the market impact posed by own trading activities. Alternatively, [Gennaioli, Shleifer and Vishny \(2015\)](#) argue that mutual funds provide trust to their shareholders. For those shareholders who value such trust, if they choose to leave the fund early, they must give up the trust premium so can also be viewed as having a lower θ .

¹⁵This setting implies that late shareholders' realized marginal utilities can be different on the two dates, a setting also commonly seen in the bank run literature (for example, [Peck and Shell, 2003](#)).

¹⁶Many theories of crisis management in other contexts employ similar structures of shocks followed by a random recovery time, for example, [Lagos, Rocheteau and Weill \(2011\)](#) on crises in OTC asset markets, [He and Xiong \(2012\)](#) on corporate debt runs, and [He and Milbradt \(2015\)](#) on maturity choices in a debt rollover crisis, among others.

when π is lower. Also, I use the end-of-game event to parsimoniously capture a normal-time scenario, in which asset prices are likely to be high and correctly reflecting their fundamentals. In this sense, my model is not intended as a general model of mutual fund management. The crisis management scenarios are also pervasive in reality. They have occurred at various time horizons and for both fundamental and non-fundamental reasons.¹⁷

Illiquid Asset Market and Sale Prices. On any date, the fund manager can sell the illiquid assets to (unmodeled) outside investors. Flow-induced sales of illiquid assets are natural and pervasive (Shleifer and Vishny, 1992, 1997), and they can create temporary price overshooting and reversal, making asset prices not perfectly forward-looking (Coval and Stafford, 2007). Based on this evidence, I assume that:

ASSUMPTION 1. *The sale price of any unit of illiquid assets is $p_E = \delta_E R$ on date $2t$ and $p_L = \delta_L R$ on date $2t + 1$,¹⁸ where $0 < \delta_E, \delta_L < 1$ and $\delta_E + (1 - \delta_E)(\mu_E + \mu_L) < \delta_L$.¹⁹*

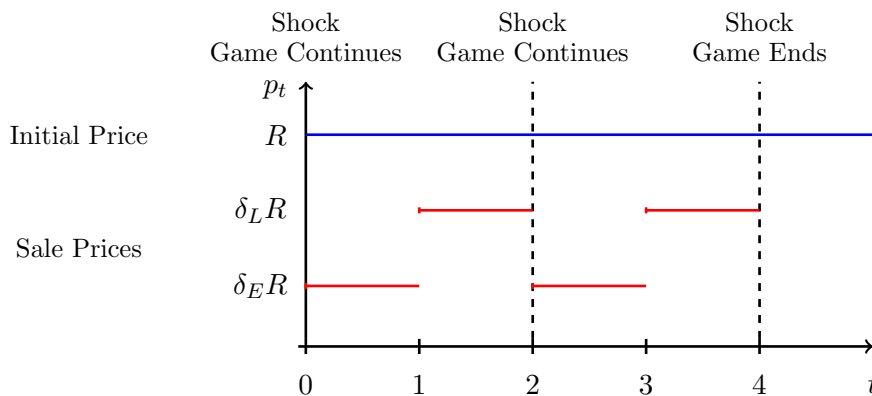


Figure 2: Sample Path of the Selling Prices

Assumption 1 parsimoniously captures two important aspects of flow-induced sales of illiquid assets: 1) the sale price on date $2t$ (i.e., right after a redemption shock) is lower than that on date $2t + 1$ due to the over-shooting and reversal within a stage, but 2) the sale price is only *imperfectly* forward-looking in the sense that p_E does not fully reflect the exact amount of future predictable asset sales on date $2t + 1$. Figure 2 illustrates a sample sale price path when the game lasts for four dates.

This sale price pattern can be micro-founded by the idea of slow-moving capital in illiquid asset markets

¹⁷There are many examples. Between 2013 and 2015, the flagship Total Return fund of the Pacific Investment Management Company (PIMCO), one of the largest fund in the U.S., has experienced net outflows for more than 28 consecutive months. The Prudential M&G's flagship Optimal Income fund, one of the largest bond fund in the Europe, has experienced more than 50 consecutive trading days of net outflows in mid 2015. The largest listed fund manager in Europe, Aberdeen Asset Management, has experienced net outflows for 15 consecutive months as of the end of 2015.

¹⁸The simplifying assumption that the sale prices are constant during a given date and do not depend on the exact amount of asset sales is not crucial at all. What is crucial is that the end-of-day NAV depends on the amount of asset sales according to (2.1), in which the sale prices p_t is not perfectly forward-looking. One can always assume that the asset sale prices follow a downward-sloping demand curve during a day, as shown in Figure 7 in the online appendix B.2. But under the given NAV rule (2.1) the downward-sloping demand curves only make the model less tractable without adding new insights.

¹⁹Besides the fact that the sale price on even dates is sufficiently lower, the specific form of the inequality is not crucial.

(Duffie, 2010) or the idea of liquidity providers' limited attention (Veldkamp, 2011).²⁰ When a redemption shock just hits the economy on date $2t$, there may be only a few liquidity providers available, and thus the fund manager can only get a low sale price. If she waits until the next date $2t+1$, since more (and different) liquidity providers step in, she may find a higher sale price. But since these liquidity providers are different across the two dates, the sale price on date $2t$ would still not fully account for predictable asset sales on $2t+1$, that is, it is not perfectly forward-looking. Such a price pattern has been pervasive and documented in various asset markets in particular when the assets are illiquid (for example, Coval and Stafford, 2007, Mitchell, Pedersen and Pulvino, 2007, Jotikasthira, Lundblad, and Ramadorai, 2012, Hendershott and Menkveld, 2014, Bessembinder, Jacobsen, Maxwell and Venkataraman, 2016, among others). Moreover, if the game continues on date $2t+2$, that is, when another round of redemption shock comes, the sale price drops again and repeats the previous pattern, consistent with the timeline of the model.

Fund Liquidity Management. To meet daily redemption needs in cash, the fund manager manages the cash position of the fund both passively and actively. On the one hand, on any date t , if the fund does not have enough cash to meet date- t projected redemptions at the beginning-of-day NAV (i.e., NAV_{t-1}), the fund will be forced to raise cash until all redemption needs can be met at the end-of-day NAV (i.e., NAV_t). Since there are no inflows and the illiquid assets do not pay interim cash flows, the fund manager can only raise cash by selling illiquid assets passively at the sale price p_t . Denote the amount of illiquid assets that the fund must sell passively by q_t , which will be endogenously determined in equilibrium.

On the other hand, in addition to selling passively for meeting redemptions, the fund can also manage its cash buffer actively. Specifically, the fund manager can actively sell illiquid assets more than contemporary redemption needs to rebuild the cash buffer, also at the sale price p_t . Denote the amount of assets that the fund actively sells on date t by s_t . I call s_t the fund's cash rebuilding policy on date t . Intuitively, a larger s_t means that the fund is rebuilding its cash buffer more rapidly. Notice that selling illiquid assets either passively or actively will always hurt the fund NAV that redeeming shareholders are able to get.

Now I specify the fund's objective function formally. Denote by T the random date on which the game ends. In order to focus on the dynamic interaction between runs and cash rebuilding under *any* initial asset positions, I intentionally omit the fund's portfolio choice problem at the beginning. Specifically, given any initial asset positions (a_t, x_t) , the fund manager chooses a sequence $\{s_\tau\}$ date by date to maximize:

$$- \mathbb{E}_t \sum_{\tau=t}^{T-1} (q_\tau + s_\tau)(R - p_\tau), \quad (2.3)$$

where the expectation is taken over the random variable T . Late shareholders in stage t rely on the

²⁰In the online appendix B.1, I provide a micro-foundation for this price pattern based on the idea of slow-moving capital.

NAVs on date $2t$ and $2t + 1$ to make redemption decisions under rational beliefs about the fund’s cash rebuilding policies $\{s_{2t}, s_{2t+1}\}$ within that stage.²¹ Intuitively, the fund’s cash rebuilding policies will affect shareholders’ run decisions, which will in turn affect the fund’s optimal cash rebuilding policies.

It is important to note that, although feasible, the fund manager will never rebuild the fund’s cash buffer on even dates in any generic equilibrium, that is, $s_{2t} = 0$ for any t . This is intuitive because the fund manages its cash buffer to avoid extremely costly sales on even dates, and hence it never makes sense for the fund manager to actively sell assets then.²² As a result, the fund’s cash rebuilding policy in stage t is solely determined by s_{2t+1} , the amount of illiquid assets the fund manager actively sells on odd date $2t + 1$. In what follows, I consider s_{2t+1} the single choice variable of the fund manager in any stage t .

Ultimately, the above model captures a key friction: liquidity mismatch. On the asset side, because the assets are illiquid, the sale prices are time-varying but not perfectly forward-looking. On the liability side, redemptions must be met daily in cash. This liquidity mismatch eventually leads to the dynamic interaction between the fund’s cash-rebuilding policy and shareholders’ run behavior.

3 Shareholder Runs

I first focus on the stage game, showing that the fund’s desire to rebuild its cash buffer can trigger shareholder runs, and more rapid cash rebuilding leads to more severe runs.

3.1 Stage-Game Equilibrium Definition and Preliminary Analysis

The two-date stage-game equilibrium is a mixed-strategy Nash equilibrium: in any stage t (consisting of dates $2t$ and $2t + 1$), given the fund’s initial portfolio position (a_{2t}, x_{2t}) and the late shareholders’ common beliefs on the fund’s cash rebuilding policy s_{2t+1} , a late shareholder’s run strategy maximizes her utility given other late shareholders’ strategies. Since all the late shareholders are identical, there is no loss of generality to consider symmetric equilibria when mixed strategies are allowed. Formally:

DEFINITION 1. *Given $\mu_E, \mu_L, \delta_E, \delta_L, R, a_{2t}, x_{2t}$, and s_{2t+1} , a symmetric run equilibrium of the stage- t game is defined as a run probability $\lambda_{2t}(a_{2t}, x_{2t}, s_{2t+1}) \in [0, 1]$ such that*

i) given other late shareholders’ run probability λ_{2t} , late investor i ’s optimal run probability $\lambda_{2t}^i = \lambda_{2t}$ maximizes her utility function (2.2),²³ and

²¹In the U.S., mutual funds are required by the SEC to disclose their asset positions quarterly. Theoretically, these requirements allow shareholders to form consistent beliefs about a fund’s cash rebuilding policies.

²²More precisely, rebuilding cash buffers on date $2t$ to punish redeeming shareholders would help only if it helps mitigate runs and the resulting run-induced forced sales on date $2t$. But doing this also means active sales on date $2t$ (at the same low price as forced sales), and it is equally bad or even worse than the fund just letting shareholders run themselves. Because it involves solving the dynamic game, this statement will be proved as Lemma 11 in the online appendix B.4.

²³In a symmetric run equilibrium, the total population of shareholders who redeem on date $2t$ is $(\mu_E + \lambda_{2t}\mu_L)n_{2t}$.

ii) all the late shareholders have a common belief about the fund's cash rebuilding policy s_{2t+1} .²⁴

I first describe three cases of the stage game per the fund's starting cash position, x_{2t} . As will become clear shortly, different x_{2t} implies different nature of strategic interactions among late shareholders.

3.1.1 Cash-to-Assets Ratio Regions

Assuming no cash rebuilding and no shareholder runs as the status quo, I characterize three different cash-to-assets ratio regions of the portfolio position space $\{(a_{2t}, x_{2t})\} \subseteq \mathbb{R}_+^2$. In these different regions, the amounts of illiquid assets that the fund is forced to sell on the two adjacent even and odd dates, that is, q_{2t} and q_{2t+1} , vary. I define the cash-to-assets ratio for any date t :

$$\eta_t \equiv \frac{x_t}{a_x}$$

LEMMA 1. *Suppose the fund does not rebuild its cash buffer and no late shareholder is going to run, that is, $s_{2t+1} = 0$ and $\lambda_{2t} = 0$. Then there are three regions of the cash-to-assets ratio η_{2t} in the stage- t game. In these three regions, the amounts of illiquid assets that the fund must sell passively on dates $2t$ and $2t + 1$ are characterized by:*

$$\left\{ \begin{array}{ll} \text{High Region } G_h: & q_{2t} = 0, q_{2t+1} = 0, \text{ iff } \eta_{2t} \geq \frac{(\mu_E + \mu_L)R}{1 - \mu_E - \mu_L}, \\ \text{Intermediate Region } G_m: & q_{2t} = 0, q_{2t+1} > 0, \text{ iff } \frac{\mu_E R}{1 - \mu_E} \leq \eta_{2t} < \frac{(\mu_E + \mu_L)R}{1 - \mu_E - \mu_L}, \\ \text{Low Region } G_l: & q_{2t} > 0, q_{2t+1} > 0, \text{ iff } \eta_{2t} < \frac{\mu_E R}{1 - \mu_E}. \end{array} \right.$$

The three regions of η_{2t} are intuitive. When $\eta_{2t} \in G_h$, the fund has enough cash to meet all projected redemptions on both date $2t$ and $2t + 1$, and thus no forced sales occur. When $\eta_{2t} \in G_m$, the fund only has enough cash to meet redemptions on date $2t$ but not on date $2t + 1$, so it must passively sell its illiquid asset on $2t + 1$. Finally, when $\eta_{2t} \in G_l$, the fund does not even have enough cash to meet redemption needs on date $2t$, and thus must incur forced sales on both dates.

Lemma 1 implies that the stage game is scale-invariant. The absolute value of (a_{2t}, x_{2t}) plays no role in determining the three regions. This allows me to use a single variable, the cash-to-assets ratio, to characterize shareholder runs in the stage game. Lemma 1 also implies the population of shareholders n_{2t} plays no role, and thus I assume $n_{2t} = 1$ in what follows without loss of generality.

²⁴For simplicity, when analyzing the stage game, I slightly abuse the notation s_{2t+1} to denote both the shareholders' common belief about the fund's cash rebuilding policy and the actual cash rebuilding policy itself.

3.2 High Cash-to-Assets Ratio Region G_h

When the stage game is in the high cash-to-assets ratio region G_h , the next lemma shows that there will always be no forced sales regardless of shareholder runs.

LEMMA 2. *When $\eta_{2t} \in G_h$, $q_{2t}(\lambda_{2t}) = q_{2t+1}(\lambda_{2t}) = 0$ for any given $\lambda_{2t} \in [0, 1]$.*

The intuition of Lemma 2 is clear. When some late shareholders decide to run, there will be effectively more early shareholders and fewer late shareholders, but the total population of redeeming shareholders in the given stage is not changed. Since the fund always has sufficient cash to meet all early and late redemption needs at the initial NAV, it indeed has enough cash on date $2t$ even if all late shareholders are going to run. Thus, NAV_{2t} will never change regardless of whether late shareholders run or not. If the fund does not rebuild its cash buffer on date $2t + 1$, Lemma 2 further implies that $NAV_{2t+1} = NAV_{2t}$ regardless of λ_{2t} , suggesting that there is no strategic interaction among late shareholders absent fund cash rebuilding.

However, given the endogenously fixed NAV_{2t} , late shareholders may decide to run if the fund rebuilds its cash buffer on date $2t + 1$ (i.e., $s_{2t+1} > 0$):

LEMMA 3. *When $\eta_{2t} \in G_h$, late shareholders' run decision λ_{2t} exhibits strategic complementarity if and only if $s_{2t+1} > 0$. Moreover, the strategic complementarity becomes stronger as s_{2t+1} increases. Mathematically:*

$$\frac{\partial \Delta u_L(\lambda_{2t})}{\partial \lambda_{2t}} > 0 \quad \text{and} \quad \frac{\partial^2 \Delta u_L(\lambda_{2t})}{\partial \lambda_{2t} \partial s_{2t+1}} > 0,$$

if and only if $s_{2t+1} > 0$, where $\Delta u_L(\lambda_{2t}) = u_L(\lambda_{2t,i} = 1; \lambda_{2t,-i} = \lambda_{2t}) - u_L(\lambda_{2t,i} = 0; \lambda_{2t,-i} = \lambda_{2t})$, while

$$\frac{\partial \Delta u_L(\lambda_{2t})}{\partial \lambda_{2t}} = 0,$$

when $s_{2t+1} = 0$.²⁵

Lemma 3 suggests the existence of run incentives, which comes from the predictable decline in NAV when the fund rebuilds its cash buffer. To see it better:

$$NAV_{2t+1} = \frac{R \overbrace{(a_{2t} - s_{2t+1})}^{\text{illiquid assets retained}} + \overbrace{x_{2t} - (\mu_E + \lambda_{2t}\mu_L)(Ra_{2t} + x_{2t})}^{\text{cash retained}} + \overbrace{\delta_L R s_{2t+1}}^{\text{cash rebuilt}}}{\underbrace{1 - (\mu_E + \lambda_{2t}\mu_L)}_{\text{shareholders remained on date } 2t+1}} \quad (3.1)$$

$$= NAV_{2t} - \frac{(1 - \delta_L)R s_{2t+1}}{1 - \mu_E - \lambda_{2t}\mu_L}. \quad (3.2)$$

²⁵For brevity, in what follows when I state results about strategic complementarity and substitutability I omit the mathematical definitions because they are standard.

The predictable decline in NAV as shown in (3.2) implies that the fund manager rebuilds its cash buffer at the expense of the late shareholders who initially plan to wait until date $2t + 1$, giving rise to run incentives. In particular, for any given $s_{2t+1} > 0$ and λ_{2t} , the utility gain $\Delta u_L(\lambda_{2t})$ of running over waiting is $\theta NAV_{2t} - NAV_{2t+1}$, which is strictly increasing in λ_{2t} by (3.2).²⁶ This illustrates the underlying strategic complementarity among late shareholders.

Lemma 3 suggests that both cash rebuilding and flexible NAV adjustment play crucial roles in generating the run incentives for late shareholders, which is different from typical bank run models. If $s_{2t+1} = 0$, the stage game features no strategic interaction at all in the high region. If NAV_{2t+1} was fixed, which is the case for MMFs, cash rebuilding would not generate a wedge of value between early and late shareholders. Also notice that although NAV_{2t} is fixed in this case, it is endogenous and flexible by nature.²⁷

I show that the run incentives can indeed lead to shareholder runs in equilibrium:

PROPOSITION 1. *When $\eta_{2t} \in G_h$, late shareholders' run behavior is given by the following three cases:*

i) none of the late shareholders runs, that is, $\lambda_{2t} = 0$, if

$$s_{2t+1} < \underline{s}_h \equiv \frac{(1 - \theta)(1 - \mu_E - \mu_L)(Ra_{2t} + x_{2t})}{(1 - \delta_L)R},$$

ii) all the late shareholders run, that is, $\lambda_{2t} = 1$, if

$$s_{2t+1} > \bar{s}_h \equiv \frac{(1 - \theta)(1 - \mu_E)(Ra_{2t} + x_{2t})}{(1 - \delta_L)R},$$

iii) $\lambda_{2t} \in \{0, \tilde{\lambda}_{2t}, 1\}$, if

$$\underline{s}_h \leq s_{2t+1} \leq \bar{s}_h,$$

where $\tilde{\lambda}_{2t}$ is the solution to

$$s_{2t+1} = \frac{(1 - \theta)(1 - \mu_E - \tilde{\lambda}_{2t}\mu_L)(Ra_{2t} + x_{2t})}{(1 - \delta_L)R}.$$

Moreover, there are $0 \leq \underline{s}_h \leq \bar{s}_h$.

Proposition 1 suggests that the fund's cash rebuilding indeed leads to shareholder runs in equilibrium, and more rapid cash rebuilding can trigger more shareholders to run. The intuitions for the three cases

²⁶More precisely, the utility gain is increasing in λ_{2t} , suggesting that a waiting shareholder is hurt more if more of others are running because she must bear a higher active sale cost per share on date $2t + 1$. It is also increasing in $1 - \delta_L$ and s_{2t+1} , suggesting that a waiting shareholder is hurt more if the price impact is larger or if the fund sells more.

²⁷This is in contrast to typical bank run models in which a fixed-value claim is either exogenously assumed or derived as the optimal contract in an outer risk-sharing problem.

are as follows. In Case i), when the fund sells only a few illiquid assets, NAV_{2t+1} is still high enough. The utility gain of running over waiting would be negative even if all the late shareholders decided to run, so it follows that no one runs. In Case ii), when the fund manager actively sells so many illiquid assets to a point where NAV_{2t+1} is so low and the utility gain of running would be positive even if others did not run, all the late shareholders will run. Both Case i) and Case ii) feature a unique equilibrium. In Case iii), the utility gain of running is negative when no one runs but becomes positive when all the late shareholders are going to run. Strategic complementarity implies that the utility gain of running is increasing when more late shareholders decide to run, so multiple equilibria emerge.

The next question is: what kind of costs do shareholder runs impose on the fund given its objective function (2.3)? I examine this question by examining the law of motions of the fund's portfolio position.

COROLLARY 1. *When $\eta_{2t} \in G_h$, the law of motions of (a_{2t}, x_{2t}) is given by*

$$\begin{aligned} a_{2t+2} &= a_{2t} - s_{2t+1}, \text{ and,} \\ x_{2t+2} &= \underbrace{x_{2t} - (\mu_E + \mu_L)(Ra_{2t} + x_{2t})}_{\text{cash retained if no cash rebuilding}} + \underbrace{\delta_L R s_{2t+1}}_{\text{cash rebuilt}} + \underbrace{\frac{(1 - \lambda_{2t})\mu_L(1 - \delta_L)R s_{2t+1}}{1 - \mu_E - \lambda_{2t}\mu_L}}_{\text{cash saved due to NAV adjustment}}, \end{aligned} \quad (3.3)$$

where λ_{2t} is the run probability induced by (a_{2t}, x_{2t}) and s_{2t+1} , as characterized in Proposition 1.

The law of motions of a_{2t} is straightforward by Lemma 2 because $q_{2t} = q_{2t+1} = 0$ regardless of λ_{2t} . This suggests that even though cash rebuilding can trigger shareholder runs, it will not induce any forced sales in the current stage when $\eta_{2t} \in G_h$.

However, shareholder runs can offset the fund's cash rebuilding efforts and lead to higher risk of future forced sales. This can be seen from the law of motions of x_{2t} in (3.3). To make this clear, I organize the terms in the right-hand side of (3.3) in a way to better reflect the cost of shareholder runs. The first term denotes the amount of cash retained if the fund did not rebuild its cash buffer so that the fund paid the initial NAV to all the early and late shareholders. The second term denotes the actual amount of cash the fund can get by selling s_{2t+1} illiquid assets. Neither of these two terms depends on λ_{2t} . The third term is more interesting; it reflects the fact that the fund can give the late shareholders less cash when it rebuilds its cash buffer on date $2t + 1$. Specifically, when s_{2t+1} is positive, NAV_{2t+1} becomes lower as shown in (3.2). Thus, more cash remains on the fund's balance sheet than that indicated by the first term in (3.3). But the third term is strictly decreasing in λ_{2t} , suggesting that this benefit of cash saving to the fund becomes smaller when more late shareholders are running. Consequently, when more shareholders run in equilibrium, the fund loses more cash in the given stage, carries less cash to future stages under the same

cash rebuilding policy s_{2t+1} , and thus faces higher risk of future forced sales.

With the intuition outlined above, it is convenient to combine the last two terms in (3.3) and define

$$\widehat{p}_L(\lambda_{2t}) \equiv \left[\delta_L + \frac{(1 - \lambda_{2t})\mu_L(1 - \delta_L)}{1 - \mu_E - \lambda_{2t}\mu_L} \right] R \quad (3.4)$$

as the *effective sale price* on the odd dates $2t + 1$. It is decreasing in λ_{2t} , meaning that the effective sale price on odd dates is lower when more shareholders are going to run.

3.3 Low Cash-to-Assets Ratio Region G_l

Now I turn to the low cash-to-assets ratio region G_l . In this region, the fund's starting cash position is so low that it cannot even meet the redemption needs of the early shareholders. Thus, it is forced to sell its illiquid assets on both dates $2t$ and $2t + 1$. If late shareholders run, the fund must sell even more:

LEMMA 4. *When $\eta_{2t} \in G_l$, there are*

$$q_{2t}(\lambda_{2t}) = \frac{\overbrace{(\mu_E + \lambda_{2t}\mu_L)(Ra_{2t} + x_{2t}) - x_{2t}}^{\text{cash gap}}}{\underbrace{[\delta_E + (1 - \delta_E)(\mu_E + \lambda_{2t}\mu_L)] R}_{\text{effective sale price with NAV adjustment}}}, \text{ and,} \quad (3.5)$$

$$q_{2t+1}(\lambda_{2t}) = \frac{\overbrace{(1 - \lambda_{2t})\mu_L \cdot \frac{R(a_{2t} - q_{2t})}{1 - \mu_E - \lambda_{2t}\mu_L}}^{\text{cash gap}}}{\underbrace{\left[\delta_L + \frac{(1 - \lambda_{2t})\mu_L(1 - \delta_L)}{1 - \mu_E - \lambda_{2t}\mu_L} \right] R}_{\text{effective sale price with NAV adjustment}}}, \quad (3.6)$$

where q_{2t} is increasing in λ_{2t} , q_{2t+1} is decreasing in λ_{2t} , and $q_{2t} + q_{2t+1}$ is increasing in λ_{2t} .

I first interpret the intuition behind the expressions. In determining the amounts of forced sales, one must know 1) the amount of cash that the fund is forced to raise (i.e., the ‘‘cash gap’’), and 2) the price at which the fund can sell its assets. Specifically, at the beginning of each date, the cash gap is defined as the difference between the fund's initial cash position and the amount of cash needed to meet projected redemptions at the beginning-of-day NAV, as shown in the numerators of (3.5) and (3.6). However, the fund does not have to raise that much cash in equilibrium, because the NAV goes down as the fund sells its assets, and redeeming shareholders are only entitled to the end-of-day NAV, which reflects those asset sale costs. Hence, it is equivalent to considering a counterfactual in which the fund still sells assets to close the initial cash gap but at a higher effective sale price as the denominators of (3.5) and (3.6) indicate.

Similarly to (3.4), I also formally define the notion of effective sale price on the even dates $2t$:

$$\widehat{p}_E(\lambda_{2t}) \equiv [\delta_E + (1 - \delta_E)(\mu_E + \lambda_{2t}\mu_L)] R. \quad (3.7)$$

Lemma 4 shows that when more late shareholders decide to run, the fund must meet more redemptions on date $2t$ while fewer redemptions on date $2t + 1$. Hence, it is forced to sell more assets on date $2t$ while fewer assets on date $2t + 1$.

More importantly, Lemma 4 also illustrates that runs unambiguously lead to higher total amount of forced sales in the given stage, as shown in the monotonicity of $q_{2t} + q_{2t+1}$ in λ_{2t} . This is because the effective sale price on date $2t$ is always lower than that on date $2t + 1$,²⁸ which means that more early redemptions must be met by selling assets at a lower effective sale price while fewer late redemptions will be met by selling assets at a higher effective sale price. Hence, the increase of q_{2t} will dominate the decrease of q_{2t+1} when more shareholders are going to run.

Lemma 4 implies that both NAV_{2t} and NAV_{2t+1} will be lower when shareholder runs occur. There are

$$\begin{aligned} NAV_{2t}(\lambda_{2t}) &= R \overbrace{(a_{2t} - q_{2t}(\lambda_{2t}))}^{\text{illiquid assets retained}} + \overbrace{x_{2t}}^{\text{initial cash}} + \overbrace{\delta_E R q_{2t}(\lambda_{2t})}^{\text{cash raised}} \\ &= \underbrace{Ra_{2t} + x_{2t}}_{NAV_{2t-1}} - (1 - \delta_E) R q_{2t}(\lambda_{2t}), \end{aligned} \quad (3.8)$$

and

$$NAV_{2t+1}(\lambda_{2t}) = \frac{R \overbrace{(a_{2t} - q_{2t}(\lambda_{2t}) - q_{2t+1}(\lambda_{2t}) - s_{2t+1})}^{\text{illiquid assets retained}} + \overbrace{\delta_L R (q_{2t+1}(\lambda_{2t}) + s_{2t+1})}^{\text{cash raised and rebuilt}}}{\underbrace{1 - \mu_E - \lambda_{2t}\mu_L}_{\text{shareholders remained on date } 2t + 1}}, \quad (3.9)$$

where $q_{2t}(\lambda_{2t})$ and $q_{2t+1}(\lambda_{2t})$ are given in (3.5) and (3.6).

Like in the high region, a predictable decline in NAV_{2t+1} may emerge because of the fund's forced sales and active cash rebuilding on date $2t + 1$. However, differing from the high region, $NAV_{2t}(\lambda_{2t})$ is no longer fixed but decreasing in λ_{2t} , as shown in (3.8). This feature changes the nature of the stage game.

LEMMA 5. *When $\eta_{2t} \in G_I$, late shareholders' run decision λ_{2t} exhibits strategic substitutability for any λ_{2t} satisfying $\theta NAV_{2t}(\lambda_{2t}) \geq NAV_{2t+1}(\lambda_{2t})$ and any feasible s_{2t+1} .²⁹ However, when s_{2t+1} increases, the*

²⁸More precisely, by the monotonicity of the effective sale prices $\widehat{p}_L(\lambda_{2t})$ and $\widehat{p}_E(\lambda_{2t})$, there is

$$\widehat{p}_E(\lambda_{2t}) \leq \widehat{p}_E(1) < \widehat{p}_L(1) \leq \widehat{p}_L(\lambda_{2t})$$

for any $\lambda_{2t} \in [0, 1]$. In other words, the potential for runs may change the effective sale prices on the two adjacent dates in the given stage, but the effective sale price on $2t$ is still lower than that on $2t + 1$ regardless of shareholder runs.

²⁹In equilibrium λ_{2t} will be an endogenous function of s_{2t+1} . But in showing the strategic interaction in the stage game, λ_{2t} should be treated as an independent variable. This also applies to the analysis of the intermediate region.

strategic substitutability becomes weaker and $\theta NAV_{2t}(\lambda_{2t}) - NAV_{2t+1}(\lambda_{2t})$ becomes larger, reinforcing a stronger incentive to redeem earlier.

The intuition behind Lemma 5 is as follows. Different from typical bank run models, a late shareholder who decides to run must bear the higher price impact and thus accept a lower NAV_{2t} when more of other late shareholders also decide to run, implying strategic substitutability among late shareholders. However, when the fund actively sells a sufficiently large amount of assets to rebuild its cash buffer, the resulting large predictable decline in NAV_{2t+1} may reinforce a sufficiently strong run incentive.

PROPOSITION 2. *When $\eta_{2t} \in G_l$, late shareholders' run behavior is given by the following three cases:*

i) none of the late shareholders runs, that is, $\lambda_{2t} = 0$, if

$$s_{2t+1} < \underline{s}_l \equiv \frac{Ra_{2t} - \theta(1 - \mu_E)(Ra_{2t} + x_{2t}) - (1 - \theta(1 - \delta_E)(1 - \mu_E))Rq_{2t}(0)}{(1 - \delta_L)R} - q_{2t+1}(0),$$

ii) all the late shareholders run, that is, $\lambda_{2t} = 1$, if

$$s_{2t+1} > \bar{s}_l \equiv \frac{Ra_{2t} - \theta(1 - \mu_E - \mu_L)(Ra_{2t} + x_{2t}) - (1 - \theta(1 - \delta_E)(1 - \mu_E - \mu_L))Rq_{2t}(1)}{(1 - \delta_L)R},$$

iii) some of the late shareholders run, that is, $\lambda_{2t} = \tilde{\lambda}_{2t}$, if

$$\underline{s}_l \leq s_{2t+1} \leq \bar{s}_l,$$

where $\tilde{\lambda}_{2t}$ is the solution to

$$s_{2t+1} = \frac{Ra_{2t} - \theta(1 - \mu_E - \tilde{\lambda}_{2t}\mu_l)(Ra_{2t} + x_{2t}) - (1 - \theta(1 - \delta_E)(1 - \mu_E - \tilde{\lambda}_{2t}\mu_l))Rq_{2t}(\tilde{\lambda}_{2t})}{(1 - \delta_L)R} - q_{2t+1}(\tilde{\lambda}_{2t}).$$

All the $q_{2t}(\lambda_{2t})$ and $q_{2t+1}(\lambda_{2t})$ are given in Lemma 4. Moreover, there are $\bar{s}_l \geq 0$ and $\bar{s}_l > \underline{s}_l$.

Like Proposition 1, Proposition 2 also suggests that the fund's cash rebuilding leads to shareholder runs in equilibrium, and more rapid cash rebuilding can trigger more shareholders to run, despite the initial strategic substitutability. In Case i), when the fund does not rebuild its cash buffer or only sells a few illiquid assets, NAV_{2t+1} can be still higher regardless of shareholders' redemption decisions, so that late shareholders will not run. In Case ii), when the fund actively sells many illiquid assets, NAV_{2t+1} is so low that the utility gain of running over waiting is positive even if all the late shareholders have already run. Notice that all the late shareholders within the stage do not run unless the fund rebuilds its cash buffer (since $\bar{s}_l \geq 0$), suggesting that only cash rebuilding by the fund can push all the shareholders to run in this

mutual fund context.³⁰ In Case iii), the utility gain of running over waiting is positive when no shareholder runs but becomes negative when all the late shareholders are going to run. In this case, there exists some partial run equilibrium in which the utility gain of running over waiting is zero.

COROLLARY 2. *When $\eta_{2t} \in G_l$, the law of motions of (a_{2t}, x_{2t}) is given by*

$$a_{2t+2} = a_{2t} - \overbrace{(q_{2t}(\lambda_{2t}) + q_{2t+1}(\lambda_{2t}))}^{\text{forced sales}} - s_{2t+1}, \text{ and}, \quad (3.10)$$

$$x_{2t+2} = \underbrace{\delta_L R s_{2t+1}}_{\text{cash rebuilt}} + \underbrace{\frac{(1 - \lambda_{2t})\mu_L(1 - \delta_L)R s_{2t+1}}{1 - \mu_E - \lambda_{2t}\mu_L}}_{\text{cash saved due to NAV adjustment}} = \widehat{p}_L(\lambda_{2t})s_{2t+1}, \quad (3.11)$$

where λ_{2t} is the run probability induced by (a_{2t}, x_{2t}) and s_{2t+1} , as characterized in Proposition 2.

Corollary 2 implies two different costs of shareholder runs. First, shareholder runs force the fund to sell more illiquid assets in the current stage (recall Lemma 4 shows that $q_{2t} + q_{2t+1}$ is increasing in λ_{2t}). Second, like that in the high region, shareholder runs lead to a lower effective sale price on date $2t + 1$ when the fund rebuilds its cash buffer. This means that runs partially offset the fund's efforts of cash rebuilding and thus lead to higher risk of future forced sales.

Compared to the analysis for the high region in Section 3.2, Proposition 2 and Corollary 2 suggest that starting with a low cash position makes a fund financially more fragile. Being in the low cash-to-assets ratio region makes the fund more prone to forced sales initially. Even worse, because the fund is running out of cash, it is likely to rebuild its cash more rapidly (as shown in Section 4), leading to more severe runs despite the initial strategic substitutability.

3.4 Intermediate Cash-to-Assets Ratio Region G_m

In the intermediate region G_m , we still have the universal results of shareholder runs as those in the high and low regions. Also, there are two types of run costs as those in the low region: more current-stage forced sales and higher risk of future sales. However, the underlying strategic interaction among shareholders becomes more involved in the intermediate region. When only a few late shareholders decide to run, the fund will not be forced to sell its illiquid assets on date $2t$, and thus NAV_{2t} will be endogenously fixed. However, when many late shareholders decide to run, the fund will be forced to sell its assets on date $2t$, and thus both NAV_{2t} and NAV_{2t+1} vary. In this sense, the stage game in the intermediate region can be viewed as a hybrid of one game in the high region and another in the low region, which can switch

³⁰This is not true for a comparable bank with fixed-value deposits, in which all shareholders can run in equilibrium even if the bank does not do anything by itself.

from strategic complementarity to substitutability as more shareholders decide to run. However, it is still the fund's cash rebuilding and the resulting predictable decline in NAV_{2t+1} that reinforce a strong run incentive. Given the results in Sections 3.2 and 3.3, I defer the full investigation of the intermediate region to Appendix A.1. The formal results are stated there as Lemma 10, Proposition 13, and Corollary 4.

4 Fund Liquidity Management in the Presence of Runs

In this section, I turn to the dynamic game and endogenize the fund's optimal cash rebuilding policy. I show that the potential for runs gives rise to a new tension: rebuilding the cash buffer more rapidly can trigger runs, while rebuilding it less rapidly puts the fund at higher risk of future forced sales as well as future runs. I then show that a time-inconsistency problem further aggravates this tension, leading to severe sales in expectation despite optimal cash management by the fund.

4.1 Dynamic Equilibrium Definition and Preliminary Analysis

The dynamic equilibrium is Markov perfect: in any stage t (consisting of dates $2t$ and $2t + 1$), if the game continues, both the fund manager and the late shareholders' strategies are functions of the state variables a_{2t} and x_{2t} , the fund's starting portfolio position, and the strategy profile is subgame perfect.

DEFINITION 2. *Given μ_E , μ_L , δ_E , δ_L , and R , a Markov perfect equilibrium is defined as a combination of the fund manager's optimal cash rebuilding policy function $s_{2t+1}^*(a_{2t}, x_{2t})$ and the late shareholders' run decision $\lambda_{2t}(a_{2t}, x_{2t}, s_{2t+1})$ such that*

i) given any state (a_{2t}, x_{2t}) and any generic common belief of the cash rebuilding policy $s_{2t+1}(a_{2t}, x_{2t})$, the late investors' run decision $\lambda_{2t}(a_{2t}, x_{2t}, s_{2t+1}) \in [0, 1]$ constructs a symmetric run equilibrium as defined in Definition 1, which also determines q_{2t} and q_{2t+1} in any stage,

ii) the fund manager's optimal cash rebuilding policy function $s_{2t+1}^(a_{2t}, x_{2t}, \lambda_{2t})$ solves the following Bellman equation:*

$$V(a_{2t}, x_{2t}) = -(1 - \delta_E)Rq_{2t} - (1 - \delta_L)Rq_{2t+1} + \max_{s_{2t+1}^*} [-(1 - \delta_L)Rs_{2t+1} + (1 - \pi)V(a_{2t+2}, x_{2t+2})], \quad (4.1)$$

iii) the state variables (a_{2t}, x_{2t}) are governed by the endogenous laws of motions as described in Corollaries 1, 2 and 4, according to the respective cash-to-assets ratio regions.

Note that the stage game may also admit multiple equilibria in some circumstances, and thus an equilibrium selection mechanism is needed. Since equilibrium selection is not crucial to my main point

concerning the dynamic interdependence between shareholder runs and fund liquidity management, I assume that late shareholders will coordinate to the worst equilibrium whenever multiple equilibria occur.³¹

I first characterize some important properties of the value function $V(a_{2t}, x_{2t})$.

PROPOSITION 3. *A value function $V(a_{2t}, x_{2t})$ exists under the Markov strategies proposed in Definition 2. In particular, $V(a_{2t}, x_{2t})$ is homogeneous of degree one (HD1) in (a_{2t}, x_{2t}) .*

The fact that $V(a_{2t}, x_{2t})$ is HD1 in (a_{2t}, x_{2t}) is important. It implies that the dynamic game is also scale-invariant, and thus the cash-to-assets ratio η_{2t} becomes the single effective state variable.

I also analyze how different values of π , the probability at which the game ends, shape the fund's optimal cash rebuilding policy. Intuitively, when the shock is less persistent (i.e., π is large), the future risk of forced sales is small, and thus cash buffers become less valuable. Therefore, when π is sufficiently large, it makes little sense for the fund to rebuild its cash buffer ex-ante, as doing so only induces current active sales of assets while generating little future benefit. Thus, the model admits a type of equilibria in which the fund finds it optimal not to rebuild its cash buffer at all.

LEMMA 6. *When π is sufficiently large, the equilibrium features $s_{2t+1}^*(a_{2t}, x_{2t}) = 0$ for any starting portfolio position (a_{2t}, x_{2t}) .*

To better illustrate the key trade-off involved in the dynamic model, in the following analysis I will consider an arbitrarily small (but still positive) π . Intuitively, this means that the redemption shocks are sufficiently persistent, consistent with a crisis management scenario. This will introduce significant future risk of sales and thus give rise to a significant trade-off between current runs and future sales.³²

With the help of these preliminary analyses, I solve for the equilibrium for different parameter values of propensity to run, θ , using a guess-and-verify approach as follows.

4.2 The No-Run and Extreme-Run Scenarios: $\theta = 0$ and $\theta = 1$

First, I consider two extreme scenarios of θ , which are sufficient to illustrate the key trade-off underlying the fund's optimal cash management. One is the scenario of $\theta = 0$, in which there are no runs. The other is the scenario of $\theta = 1$, in which late shareholders are indifferent between early and late consumptions so that they have the strongest propensity to run.

³¹This equilibrium selection mechanism can be motivated by that the fund manager may be ambiguity averse to the potential for shareholder runs. Alternative equilibrium selection mechanisms such as selecting the best equilibrium or the static global game approach (Goldstein and Pauzner, 2005) will not qualitatively change my results. Note that some papers in the bank run literature (for example, Allen and Gale, 1998, Cooper and Ross, 1998) assume shareholders to coordinate to the best equilibrium but only to justify the existence of banks, which is irrelevant in our mutual fund context.

³²It should be noted that those equilibria characterized by Lemma 6 are still intuitive and consistent with the model settings. They are just less relevant to the main point of this paper: the dynamic interdependence of shareholder runs and fund liquidity management in a crisis management scenario.

I start by defining some new notations to streamline the presentation. Since the dynamic game is scale-invariant, it is convenient to define

$$\sigma_{2t+1} \equiv \frac{s_{2t+1}}{a_{2t}},$$

the fraction of illiquid assets that the fund actively sell on odd dates $2t + 1$ (relative to the beginning-of-stage asset position a_{2t}), to denote the cash rebuilding policy. Moreover, Corollaries 1, 2, and 4 suggest that η_{2t+2} is uniquely determined by (a_{2t}, x_{2t}) and σ_{2t+1} .³³ Hence, it is also convenient to use η_{2t+2} to denote the fund's cash rebuilding policy when helpful.

I further divide the high region G_h into three different sub-regions: the high-low region G_{hl} , the high-intermediate region G_{hm} , and the high-high region G_{hh} . The three sub-regions of the high region G_h are defined from a dynamic perspective, and they will be useful in describing the optimal dynamic cash rebuilding policy. When the fund starts from G_h and does not rebuild its cash buffer, by definition, after meeting redemptions in the given stage the fund still has a non-negative cash position in the next stage. If the fund ends up into the low region G_l in the next stage, I say that the fund starts from the high-low region G_{hl} . If instead the fund ends up in the intermediate region G_m in the next stage, I say that the fund starts from the high-intermediate region G_{hm} . The high-high region G_{hh} is defined in the same manner.

Region	Cash-to-Assets Ratio
G_l	$\eta_{2t} < \frac{\mu_E R}{1 - \mu_E}$
G_m	$\frac{\mu_E R}{1 - \mu_E} \leq \eta_{2t} < \frac{(\mu_E + \mu_L)R}{1 - \mu_E - \mu_L}$
G_{hl}	$\eta_{2t} \geq \frac{(\mu_E + \mu_L)R}{1 - \mu_E - \mu_L}$ and $\eta_{2t+2} < \frac{\mu_E R}{1 - \mu_E}$ if $\sigma_{2t+1} = 0$
G_{hm}	$\eta_{2t} \geq \frac{(\mu_E + \mu_L)R}{1 - \mu_E - \mu_L}$ and $\frac{\mu_E R}{1 - \mu_E} \leq \eta_{2t+2} < \frac{(\mu_E + \mu_L)R}{1 - \mu_E - \mu_L}$ if $\sigma_{2t+1} = 0$
G_{hh}	$\eta_{2t} \geq \frac{(\mu_E + \mu_L)R}{1 - \mu_E - \mu_L}$ and $\eta_{2t+2} \geq \frac{(\mu_E + \mu_L)R}{1 - \mu_E - \mu_L}$ if $\sigma_{2t+1} = 0$

First, I analyze the behavior of shareholder runs in these two scenarios of $\theta = 0$ and $\theta = 1$, given any generic and feasible cash rebuilding policy of the fund.

LEMMA 7. *When $\theta = 0$, none of the late shareholders run in stage t , that is, $\lambda_{2t}(a_{2t}, x_{2t}) = 0$ for any (a_{2t}, x_{2t}) and any cash rebuilding policy σ_{2t+1} .*

Lemma 7 is straightforward. If a shareholder gets nothing when running, they will never run.

³³Keep in mind that the equilibrium selection mechanism in the stage game is used when needed.

LEMMA 8. When $\theta = 1$, all the late shareholders run in stage t , that is, $\lambda_{2t}(a_{2t}, x_{2t}) = 1$ for any (a_{2t}, x_{2t}) and any positive and feasible cash rebuilding policy $\sigma_{2t+1} > 0$.

Lemma 8 shows that when the shareholders' propensity to run is the highest (i.e., $\theta = 1$), all the late shareholders decide to run if the fund actively sells any illiquid assets to rebuild its cash buffer. This is because when $\theta = 1$ the late shareholders simply compare between NAV_{2t} and NAV_{2t+1} to decide whether to run. If the fund rebuilds its cash buffer, NAV_{2t+1} will be strictly lower than NAV_{2t} regardless of either the fund's initial cash position or other shareholders' run behavior.

Now I turn to the fund's equilibrium cash rebuilding policy. With the help of Lemma 7, the following proposition first characterizes the optimal cash rebuilding policy when $\theta = 0$.

PROPOSITION 4. When $\theta = 0$, the equilibrium cash rebuilding policy of the fund is characterized by:³⁴

i) if $\eta_{2t} \in G_l \cup G_m \cup G_{hl}$, the fund chooses $\sigma_{2t+1}^* > 0$ such that

$$\eta_{2t+2}^* = \frac{\mu_E R}{1 - \mu_E}, \text{ and,}$$

ii) if $\eta_{2t} \in G_{hm} \cup G_{hh}$, the fund does not rebuild its cash buffer, that is, $\sigma_{2t+1}^* = 0$.

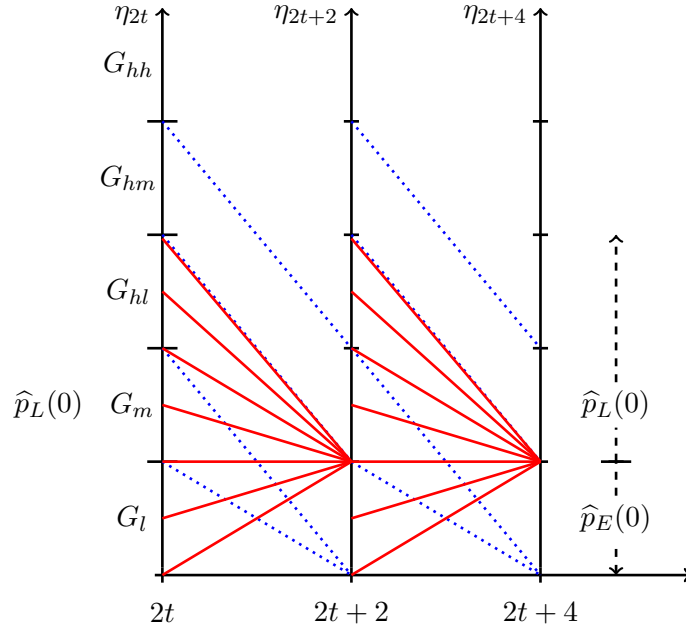


Figure 3: Equilibrium Cash Rebuilding Policy When $\theta = 0$

The fund's optimal dynamic cash rebuilding policy when $\theta = 0$, as characterized in Proposition 4, is illustrated in Figure 3. In this figure, the horizontal axis denotes date, while the vertical axis denotes the

³⁴Keep in mind that the dynamic equilibrium requires sequential optimality. In other words, the fund's cash rebuilding policy is optimal in a stage only when in the next stage the fund also follows its optimal cash rebuilding policy, which is again conditional on the fund's optimal cash rebuilding policy in the following stage, and so on.

cash-to-assets ratio. The blue dotted line depicts the evolution of the cash-to-assets ratio if the fund does not rebuild its cash buffer at all. The red line depicts the evolution of the cash-to-assets ratio when the fund optimally rebuilds its cash buffer. Because of the scale-invariance and the resulting stationarity of the dynamic game, the equilibrium cash rebuilding policy (conditional on the effective state variable, the cash-to-assets ratio η_{2t}) always follows the same pattern in different stages, if the game continues.

Since there are no runs in equilibrium (by Lemma 7), the main insight behind Proposition 4 is a trade-off between current-stage active asset sales (under a policy of more cash rebuilding) and future-stage forced sales (under a policy of no or less cash rebuilding). Intuitively, because the fund manager cares about total expected sale losses, it is worthwhile for her to actively sell more assets at the current stage (on date $2t+1$), if the cash buffer rebuilt can help to avoid more severe sales in the next stage (on date $2t+2$).

Due to flexible NAV adjustment, it will be convenient to use the effective sale prices as defined in (3.4) and (3.7) to illustrate the trade-off. To see this, on the one hand, suppose there is a cash gap $\Delta x_{2t+2} > 0$ on date $2t+2$. As suggested by Lemma 4, the fund manager will be forced to sell at the effective sale price on date $2t+2$ to meet the initial cash gap Δx_{2t+2} . On the other hand, the fund manager can choose to actively sell more assets on date $2t+1$, also at the corresponding effective sale price, to rebuild Δx_{2t+2} unit of cash buffer in advance on date $2t+1$, carry it to date $2t+2$, and thus avoid forced sales on date $2t+2$. This comparison helps pin down the fund's optimal cash rebuilding policy.

Specifically, Proposition 4 says if its initial cash position falls below the high-intermediate region G_{hm} , the fund optimally rebuilds its cash buffer until the next-stage cash-to-assets ratio η_{2t+2} reaches the cutoff between the low region G_l and the intermediate region G_m . The reason is as follows. If the fund did not rebuild its cash buffer, it would end up in the low region in the next stage (i.e., $\eta_{2t+2} \in G_l$). Since the fund will be forced to sell its assets then (as the game continues with a high probability $1 - \pi$), the fund manager may want to rebuild its cash buffer on date $2t+1$ to avoid sales on $2t+2$. Specifically, because late shareholders never run (by Lemma 7), the effective sale price to rebuild cash buffers actively on date $2t+1$ is $\widehat{p}_L(0)$, while the effective sale price to raise cash passively on date $2t+2$ is $\widehat{p}_E(0)$. As $\widehat{p}_L(0) > \widehat{p}_E(0)$, the fund manager always finds it optimal to rebuild its cash buffer on date $2t+1$.

Given that the fund rebuilds its cash buffer, what is the optimal amount of active asset sales? In equilibrium, the fund manager will rebuild the cash buffer up to a point where η_{2t+2} just hits the cut-off between the low and intermediate region. This is because, on the one hand, a lower cash target still implies forced sales on date $2t+2$ at a lower effective sale price $\widehat{p}_E(0)$ and thus is not optimal. On the other hand, any more cash rebuilding on date $2t+1$ means the fund will still have a strictly positive cash buffer on date $2t+3$ after outflows on date $2t+2$. This is also not optimal because that cash buffer is excessive

from the perspective of date $2t + 1$. In other words, even if asset sales occur on date $2t + 3$, the fund manager will be able to sell at the higher effective sale price $\widehat{p}_L(0)$. Since the game only has a less than one probability to continue, selling at the same effective price $\widehat{p}_L(0)$ on date $2t + 1$ to build that excessive cash buffer is not profitable. The same intuition applies to the high-intermediate and high-high regions (i.e., $\eta_{2t} \in G_{hm} \cup G_{hh}$), in which it will not rebuild its cash buffer.

Following the same logic, I then characterize the optimal cash rebuilding policy when $\theta = 1$. This illustrates how the potential for runs interacts with the fund's cash rebuilding policy.

PROPOSITION 5. *When $\theta = 1$, the equilibrium cash rebuilding policy of the fund is characterized by:*

i) if $\eta_{2t} \in G_l \cup G_m \cup G_{hl} \cup G_{hm}$, the fund chooses $\sigma_{2t+1}^* > 0$ such that

$$\eta_{2t+2}^* = \frac{(\mu_E + \mu_L)R}{1 - \mu_E - \mu_L}, \text{ and,}$$

ii) if $\eta_{2t} \in G_{hh}$, the fund does not rebuild its cash buffer, that is, $\sigma_{2t+1}^* = 0$.

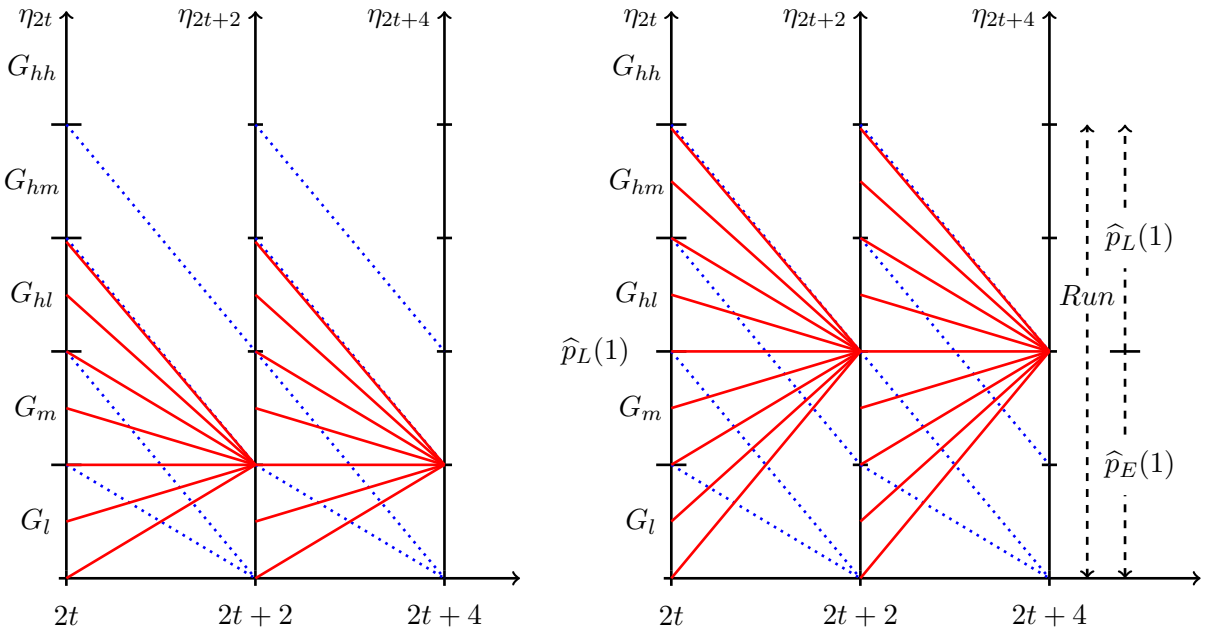


Figure 4: Equilibrium Cash Rebuilding Policies When $\theta = 0$ (Left) and $\theta = 1$ (Right)

The right panel of Figure 4 illustrates the equilibrium cash rebuilding policy when $\theta = 1$. To recap and better show the difference, I illustrate the equilibrium cash rebuilding policy when $\theta = 0$ on the left.

Proposition 5 says that the fund starts to rebuild its cash buffer at a higher starting cash position, and it also rebuilds the cash buffer more rapidly compared to the scenario without runs. Specifically, once the fund's cash position falls below the high-high region G_{hh} , it rebuilds its cash buffer until the next-stage cash-to-assets ratio η_{2t+2} reaches the cutoff between the intermediate region G_m and the high region G_h .

Although Proposition 5 still features the trade-off between current- and future-stage sales, this trade-off becomes subtler in the presence of runs. By Corollaries 1, 2 and 4, runs in equilibrium result in less effective cash rebuilding (i.e., a lower effective sale price) on odd dates and more forced sales on even dates. Thus, when current-stage run risks are relatively high, the fund wants to choose a less rapid cash rebuilding policy. In contrast, when future-stage risk of sales is relatively high, notably when future-stage runs lead to more severe future-stage sales, the fund prefers a more rapid cash rebuilding policy.

To illustrate the intuition, suppose the fund starts from the low or the intermediate region (i.e., $\eta_{2t} \in G_l \cup G_m$). By Lemma 8, all the late shareholders are going to run on date $2t$ if $\sigma_{2t+1} > 0$, which implies a lower effective sale price (for cash rebuilding) $\widehat{p}_L(1)$ on date $2t + 1$. However, if the fund did not rebuild its cash buffer, it would end up in the low region in the next stage, where the fund must sell at an effective price $\widehat{p}_E(1)$. Because $\widehat{p}_L(1) > \widehat{p}_E(1)$, the risk of future forced sales is relatively large. Thus, the fund still finds it optimal to rebuild its cash buffer on date $2t + 1$ to avoid more costly sales on date $2t + 2$, which also justifies shareholders' run behavior on date $2t$.

However, different from the scenario when $\theta = 0$, when $\theta = 1$ the fund does not stop rebuilding its cash buffer even if the next-stage cash-to-assets ratio hits the cutoff between the low and intermediate region. The reasoning is as follows. If the fund ended up in the intermediate region in the next stage (i.e., $\eta_{2t+2} \in G_m$), again by Lemma 8, all the late shareholders in the next stage will run on date $2t + 2$ too. Thus, the fund would be forced to sell its assets on date $2t + 2$ at the effective sale price $\widehat{p}_E(1)$ even if starting in the intermediate region then. Fundamentally, future-stage runs lead to higher risk of future sales. As a result, the fund will keep rebuilding its cash buffer even when $\eta_{2t+2} \in G_m$.

In equilibrium, the fund manager will rebuild the cash buffer up to a point where η_{2t+2} hits the cutoff between the intermediate and high region, which is higher than the counterpart when $\theta = 0$. As analyzed above, a lower cash target implies forced sales on date $2t + 2$ at a lower effective sale price $\widehat{p}_E(1)$ and thus is not optimal. Also, a higher cash target becomes excessive despite runs in the next stage. Specifically, a higher cash target implies that the fund would end up in the high region in the next stage (i.e., $\eta_{2t+2} \in G_h$), where runs only lead to a lower effective sale price $\widehat{p}_L(1)$ on date $2t + 3$. Since the game only has a less than one probability to continue, selling at the same effective price $\widehat{p}_L(1)$ on date $2t + 1$ to build that excessive cash buffer is not profitable. This intuition applies to other regions as well.

Overall, compared to the scenario without runs (i.e., $\theta = 0$), Proposition 5 and the intuition above suggest that the trade-off in fund cash rebuilding becomes more complicated in the presence of runs. When the starting cash position is lower, future risk of sales (notably future-stage forced sales induced by future runs) is relatively high, and thus the fund optimally chooses a more rapid cash rebuilding policy. On the

contrary, when the starting cash position is higher, current-stage costs of runs are relatively high, and thus the fund optimally chooses a less rapid cash rebuilding policy.

4.3 The General Scenarios

I proceed to characterize the dynamic equilibria in the general scenarios when $\theta \in (0, 1)$. In these general scenarios, the shareholders have a moderate propensity to run and thus become less sensitive to the fund's cash rebuilding than they would in the $\theta = 1$ scenario. However, a sufficiently rapid cash rebuilding policy can still push them to run. This in turn shapes the fund's optimal cash rebuilding policy in equilibrium.

Despite the complexity of the general scenarios, all the equilibrium results can be still unified under the same trade-off between current-stage runs and future-stage asset sales as discussed in Section 4.2. The formal result is stated in Proposition 6.

PROPOSITION 6. *When $\theta \in (0, 1)$, there exist two endogenous thresholds $0 < \underline{\theta} < \bar{\theta} < 1$, such that*

i) if $\theta \in (0, \underline{\theta}]$, the equilibrium cash rebuilding policy is characterized by Proposition 4, that is, the cash rebuilding policy follows that in the scenario of $\theta = 0$,

ii) if $\theta \in (\underline{\theta}, \bar{\theta})$, the equilibrium cash policies are characterized by

a) if $\eta_{2t} \in G_l \cup G_m \cup G_{hl} \cup \underline{G_{hm}}$, the fund chooses $\sigma_{2t+1}^ > 0$ such that*

$$\eta_{2t+2}^* = \eta(\check{\lambda}) \equiv \frac{(\mu_E + \check{\lambda}\mu_L)R}{1 - \mu_E - \check{\lambda}\mu_L},$$

where $\check{\lambda}$ is given by

$$\begin{cases} \lambda_{2t}^* > 0 & \text{iff } \eta_{2t} < \eta(\check{\lambda}), \\ \lambda_{2t}^* = 0 & \text{iff } \eta_{2t} \geq \eta(\check{\lambda}), \end{cases}$$

in which λ_{2t}^ denotes the run behavior under the optimal cash rebuilding policy σ_{2t+1}^* , and*

$$\underline{G_{hm}} \equiv \left\{ \eta_{2t} \mid \eta_{2t} \geq \frac{(\mu_E + \mu_L)R}{1 - \mu_E - \mu_L} \text{ and } \frac{\mu_E R}{1 - \mu_E} \leq \eta_{2t+2} < \frac{(\mu_E + \check{\lambda}\mu_L)R}{1 - \mu_E - \check{\lambda}\mu_L} \text{ for } \sigma_{2t+1} = 0 \right\},$$

b) if $\eta_{2t} \in \overline{G_{hm}} \cup G_{hh}$, then $\sigma_{2t+1}^ = 0$, where $\overline{G_{hm}} = G_{hm}/\underline{G_{hm}}$,*

iii) if $\theta \in [\bar{\theta}, 1)$, the equilibrium cash rebuilding policy is characterized by Proposition 5, that is, the cash rebuilding policy follows that in the scenario of $\theta = 1$.

As suggested by Proposition 6, when θ is close to 0, the equilibrium is the same as that when $\theta = 0$, while when θ approaches 1 the equilibrium is the same as that when $\theta = 1$. As θ increases, shareholders become more likely to run, and the fund also chooses a more rapid cash rebuilding policy in equilibrium

cannot happen in any equilibrium path unless the fund can credibly commit to such a policy.

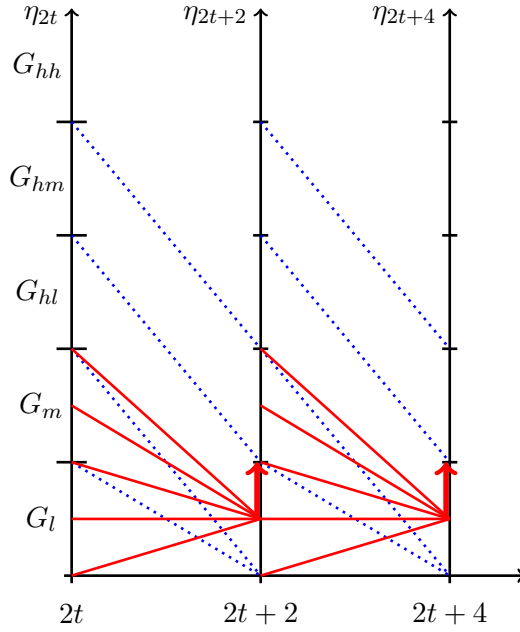


Figure 6: The Time-Inconsistency Problem

The intuition behind Proposition 7 is a time-inconsistency problem, which aggravates the tension in choosing between a rapid or a slow cash rebuilding policy. Starting from the low region or the intermediate region, the fund indeed has a relatively large current-stage cost of shareholder runs because they lead to severe current-stage forced sales. Thus, on date $2t$, the fund may wish to commit itself to rebuilding its cash buffer less rapidly on date $2t + 1$ to reduce such run risks on date $2t$. However, on date $2t + 1$, because all the date- $2t$ costs of runs are sunk, the fund may instead be tempted to adopt a more rapid cash rebuilding policy on date $2t + 1$. Notably, what matters for shareholders' run decisions on date $2t$ are their beliefs about the fund's cash rebuilding policy on date $2t + 1$. In equilibrium, they can always anticipate the fund manager's date- $2t + 1$ temptation to rebuild the cash buffer more rapidly, and thus will always have strong incentives to run on date $2t$. Mathematically, the intuition outlined above can be also seen from the dynamic equilibrium definition (Definition 2) and notably from the Bellman equation (4.1) in the non-commitment benchmark.

Proposition 7 suggests a fundamental difficulty in reducing fund shareholder runs in practice, in which a commitment device can be hard to implement. Consequently, shareholders decide to run not only because they expect other shareholders to run at the same time, but more importantly because they expect the fund to rebuild its cash buffer too rapidly in the future.

4.5 Expected Total Asset Sale Losses

Finally, I show in Proposition 8 that the potential for shareholder runs can lead to unambiguously higher total sale losses ex-ante, regardless of the fund's initial portfolio position. This occurs in a world where both the fund manager and the shareholders are rational, and the fund's cash rebuilding policy is optimal. It suggests that the potential financial stability risks induced by mutual fund shareholder runs can be significant and thus should not be overlooked.

PROPOSITION 8. *When θ increases, the ex-ante total sale losses become higher for any positive starting portfolio position (a_{2t}, x_{2t}) .*

As suggested by Proposition 7, the lack of a commitment device aggravates run problems despite optimal liquidity management by the fund. I show in Proposition 9 that introducing a commitment device can indeed help reduce total sale losses in expectation.

PROPOSITION 9. *When the fund can pre-commit to a cash policy s_{2t+1} on date $2t$, the ex-ante total sale losses become lower for any positive (a_{2t}, x_{2t}) and any $\theta > 0$.*

Intuitively, introducing a commitment device helps reduce total sales in two ways. On the one hand, as suggested by Proposition 7, since the fund can pre-commit to a less rapid cash rebuilding policy, it can directly reduce current-stage forced sales by reducing shareholder runs. On the other hand, from a dynamic perspective, the risk of future-stage sales also becomes lower due to less severe future runs, and thus the fund is also more comfortable in choosing a less rapid cash rebuilding policy by selling assets less aggressively in the current stage.

5 Policy Implications and Extensions

Many regulators and practitioners have proposed fund-level policies, aiming to mitigate potential financial stability risks of open-end mutual funds. In the following, I extend the model to consider several fund-level policies.³⁵ I show that, perhaps surprisingly, some of them are less effective than commonly thought in mitigating potential financial stability risks of mutual funds due to the dynamic interdependence between shareholder runs and fund liquidity management.

5.1 Redemption Fees

The first policy proposal is to increase or eliminate the cap on redemption fees. Open-end mutual funds can charge their shareholders redemption fees when they redeem their shares. Currently, the SEC requires

³⁵Rather than making definitive policy prescriptions, I emphasize how the model adds new insights to policy debates.

mutual fund redemption fees to be lower than 2%.³⁶ Therefore, some observers argue that to increase or eliminate the cap is likely to mitigate potential financial stability risks of mutual funds.

My model suggests that higher redemption fees may help reduce shareholder runs. Suppose $1 - \kappa$ of the redemption proceeds are collected as redemption fees, where $\kappa \in (0, 1)$. Thus, any shareholder who redeems on date t only gets κNAV_t .³⁷ Also, redemption fees are paid back directly to the fund, implying that the fund can save $(1 - \kappa)NAV_t$ cash per share redeemed. To better contrast to the baseline model, I consider $\theta = 1$, that is, when the shareholders' propensity to run is the highest. The following proposition shows that the introduction of redemption fees can lead to less shareholder runs in equilibrium.

PROPOSITION 10. *For any given starting portfolio position (a_{2t}, x_{2t}) , any feasible cash rebuilding policy s_{2t+1} , and any proportional redemption fee $1 - \kappa > 0$, there is $\lambda_{2t}^\kappa \leq \lambda_{2t}$, where λ_{2t}^κ is the equilibrium run probability in the game with the redemption fee while λ_{2t} is that in the game without redemption fees, all other things being equal.*

When the stage game starts from the high cash-to-assets region, redemption fees have a stronger effect. In contrast to the baseline model where any cash rebuilding (i.e., any $s_{2t+1} > 0$) leads to shareholder runs when $\theta = 1$, with the redemption fee there can be completely no runs in equilibrium when s_{2t+1} is small.

COROLLARY 3. *For any given starting portfolio position (a_{2t}, x_{2t}) satisfying $\eta_{2t} \in G_h$ and any proportional redemption fee $1 - \kappa > 0$, there exists a strictly positive $\bar{s} > 0$ such that $\lambda_{2t}^\kappa = 0$ constructs the unique stage-game equilibrium when $s_{2t+1} \leq \bar{s}$.*

Proposition 10 and Corollary 3 suggest that redemption fees can directly reduce shareholders' run incentives. Intuitively, with redemption fees, redeeming shareholders effectively get a value lower than the prevailing NAV, implying a wealth transfer from redeeming shareholders to staying ones. Moreover, in any stage, redemption fees allow the fund to save more cash proportionally, making it easier to meet redemption needs without incurring sales. However, as suggested by Proposition 10, redemption fees do not directly alter the dynamic interdependence between runs and fund liquidity management. They cannot solve the time-inconsistency problem associated with the fund's dynamic cash rebuilding policy either.³⁸

³⁶This is according to Rule 22c-2 of the Investment Company Act of 1940.

³⁷According to Rule 22c-2, the U.S. SEC prohibits discriminative redemption fees solely conditional on shareholder identities as those would effectively create classes of shareholder seniority. This implies that, in my model, the fund cannot intentionally impose different redemption fees on early and late shareholders.

³⁸Redemption fees may also be less effective in practice for the following reasons. First, in my extended model, redemption fees are introduced ex-ante. However, if redemption fees are first introduced on an odd date $2t + 1$ but are expected on the previous date $2t$, the late shareholders will have higher incentives to run to avoid the fees. This represents a real-world concern that imposing higher redemption fees by itself can lead to one-time market turmoil. Other unmodeled but plausible reasons include negative effects on future fund share sales and on the reputation of fund managers.

5.2 In-Kind Redemptions

In practice, open-end mutual funds may satisfy redemption requests by delivering a portion of the underlying basket of assets invested, including cash, which is known as “in-kind redemptions.” Many practitioners argue that the option to elect to in-kind redemptions can largely mitigate any financial stability risks of mutual funds, at least during crisis times. Are in-kind redemptions really a relief?

My model suggests that in-kind redemptions can be very effective in preventing shareholder runs within a fund, but perhaps surprisingly, they do not necessarily help reduce total sale losses or improve total shareholder welfare. The following proposition offers a sufficient condition for such episodes.

PROPOSITION 11. *Electing to in-kind redemptions prevents shareholder runs, that is, $\lambda(a_{2t}, x_{2t}) = 0$ for any (a_{2t}, x_{2t}) . However, when θ , μ_L are sufficiently small and δ_L is sufficiently larger than δ_E , in-kind redemptions lead to higher total sale loss ex-ante than a counterfactual in which the fund sticks to cash redemptions, all other things being equal.*

The intuition behind Proposition 11 relies on three progressive reasons. First, adopting in-kind redemptions completely eliminates any run incentives. This is because late shareholders always get the same basket of assets regardless of the time they redeem, and they must sell the illiquid assets at a lower price p_E for consumptions if they ran, so they prefer not to run. Second, since the fund manager only cares about total sale losses at the fund level, liquidity management becomes irrelevant. The fund will never rebuild its cash buffer, and the initial cash-to-assets ratio η_0 will never change. Third, early shareholders must sell the illiquid assets that they get at the extremely low price p_E for consumptions. These sale losses could have been avoided if the fund manager actively managed its cash buffer. If these sale losses are significant, shareholders will become worse-off than the counterfactual with cash redemptions.

Proposition 11 suggests that in-kind redemptions are not a free lunch, because shareholders who ask their fund to elect to in-kind redemptions effectively give up any benefit they could get from active fund liquidity management. This point highlights the dynamic interdependence between shareholder runs and fund liquidity management. In addition, given that in-kind redemptions are obviously costly during normal times since they discourage the sales of shares, the overall benefit of adopting in-kind redemptions can be even more ambiguous. In reality, in-kind redemptions can also be hard to implement.³⁹

³⁹Rule 18f-1 of the Investment Company Act of 1940 implies that in-kind redemptions will not be effected unless specific approval is first obtained from the SEC. This rule is intended to facilitate mutual fund share sales in jurisdictions where in-cash redemptions are required.

5.3 Redemption Restrictions

A similar emergency rule is redemption restrictions, which give a fund the right to suspend redemptions in given periods as permitted by regulators.⁴⁰ Can redemption restrictions prevent shareholder runs?

I model redemption restrictions by assuming that the fund is able to deny any individual shareholder's redemption request on any date with probability $1 - \zeta$, $\zeta \in (0, 1)$. To better contrast to the baseline model without redemption fees, I also consider $\theta = 1$ when the shareholders' propensity to run is the highest.

PROPOSITION 12. *For any given starting portfolio position (a_{2t}, x_{2t}) and any redemption restriction $1 - \zeta > 0$, there is $\lambda_{2t}^\zeta \leq \lambda_{2t}$, where λ_{2t}^ζ is the equilibrium run probability in the game with the redemption restriction while λ_{2t} is that in the game without redemption restrictions, all other things being equal.*

Proposition 12 suggests that redemption restrictions can help reduce shareholder runs. Introducing redemption restrictions closely resembles the introduction of redemptions fees as analyzed in Proposition 10 and Corollary 3. The intuition for Proposition 12 is clear. By the Law of Large Numbers, only ζ of the redeeming shareholders can get cash out of the fund. Therefore, there will be effectively fewer redemptions. But like the introduction of redemption fees, the introduction of redemption restrictions cannot fully prevent shareholder runs or solve the time-inconsistency problem.

5.4 Credit Lines

In reality, mutual funds may turn to pre-established and usually ultra-short-term credit lines to raise cash. My model suggests that using short-term credit lines may temporarily mitigate the negative effects of current-stage shareholder runs, but can induce more severe sales and runs in the future. Specifically, in stage t , suppose the fund uses pre-established credit lines (rather than selling assets) when it is in the low or the intermediate cash-to-assets region. Thus, the fund does not have to sell any illiquid assets in meeting redemptions on dates $2t$ and $2t + 1$. However, in the next stage (if the game continues) the fund will have no cash to start (i.e., $\eta_{2t+2} = 0$). It thus will face more severe sales unless it can borrow more. What is worse, the fund will be required to pay back its short-term debt first on date $2t + 2$, which may lead to fire sales or default.⁴¹ Intuitively, when using credit lines the fund forgoes the option of cash rebuilding, which is helpful when the redemption shocks are persistent (i.e., π is small). This idea resembles that in Section 5.2: it will be naive to shut down a fund's active liquidity management when attempting to prevent runs.

⁴⁰According to Rule 22e of the Investment Company Act, an open-end mutual fund is generally prohibited from suspending the right or redemption or postponing the payment of redemption proceeds for more than seven days. However, the SEC has the right to deem emergency periods during which a fund is able to suspend redemptions.

⁴¹Moreover, credit lines may expose a fund to debt runs as suggested by He and Xiong (2012). This is in particular relevant when the redemption shocks are persistent so that the fund must repeatedly rollover its credit lines with multiple creditors.

5.5 Swing Pricing

Some observers argue that swing pricing, which allows current NAVs to reflect commissions to asset brokers and dealers, bid-ask spreads, taxes, and other trading-related charges, can reduce the negative externalities imposed by redeeming shareholders on non-trading ones. The new SEC rule passed in October 2016 has allowed U.S. mutual funds to use swing pricing.⁴² Will it prevent shareholder runs?

In fact, swing pricing, in its currently proposed form, has already been incorporated into my baseline model. This is because flow-induced price impacts is the only type of trading-related costs in the model, and current NAVs have already taken them into account. Since they still do not reflect future asset sale costs, they are not able to mitigate the risk of runs induced by active fund liquidity management.

Rather than swing pricing in its current form, my model suggests that forward-looking NAVs may help reduce shareholder runs.⁴³ Theoretically, this can be viewed as the optimal form of swing pricing. To investigate optimal contract design for liquidity provision in a mutual fund context (like [Green and Lin, 2003](#) and [Peck and Shell, 2003](#) in a bank context) is interesting but beyond the scope of this paper.

6 Conclusion

In this paper, I build a model of an open-end mutual fund with a flexible NAV, and show that shareholder runs can occur in equilibrium despite optimal liquidity management by the fund. With a flexible NAV, fund cash rebuilding by selling illiquid assets implies a predictable decline in NAV and thus a first-mover advantage, leading to runs. The presence of shareholder runs further complicates the fund's efforts in liquidity management, leading to higher total sale losses in expectation. Hence, appropriate design of policies aiming for mitigating financial stability risks of mutual funds should take into account the dynamic interdependence of shareholder runs and fund liquidity management.

Fundamentally, shareholder runs are driven by a key contractual property of illiquid mutual funds' NAVs: they are flexible but not perfectly forward-looking. Specifically, the sale prices of illiquid assets and thus fund NAVs at t do not take into account the predictable asset sales at $t + 1$. It implies that cash rebuilding can give rise to predictable declines in NAV and thus the potential for runs.

My model sheds new light on potential systemic risks posed by mutual funds. As mutual fund runs

⁴²The rule is available at <https://www.sec.gov/rules/final/2016/33-10234.pdf>.

⁴³This is equivalent to requiring shareholders to contract on future NAVs directly. However, shareholders cannot contract on future NAVs because mutual funds promise to provide daily liquidity service to their shareholders. In other words, if shareholders instead contracted on future NAVs and they had common and rational beliefs on future NAVs, they would effectively go back to "separate accounts," or equivalently direct holdings of the underlying assets by the shareholders, and there will be no liquidity service provided by the funds. In this sense, there is no point to have a mutual fund in the first place. As a result, forward-looking NAV rules may be hard to implement in reality, and runs can be viewed as a cost that shareholders must bear to have mutual funds engage in liquidity transformation.

lead to more forced asset sales, the underlying asset markets may become even more illiquid. As shown in [He and Milbradt \(2014\)](#), this effect can cause more corporate defaults and impose considerable risks on real economic activities. To be clear, I do not claim that mutual fund runs cause any systemic risks. The systemic implications of mutual fund runs depend not only on the contagion from secondary-market sales to primary-market investment losses, but also on how non-bank financial intermediaries interact with other bank-like financial institutions. A thorough investigation covering all these issues is beyond the scope of this paper, but the results here can naturally serve as a building block for future research on these issues.

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A Appendix

A.1 The Analysis of the Intermediate Cash-to-Assets Ratio Region G_m

I first characterize how shareholder runs affect the fund’s forced sales on dates $2t$ and $2t + 1$, respectively.

I define

$$\widehat{\lambda}_{2t} \equiv \frac{x_{2t} - \mu_E(Ra_{2t} + x_{2t})}{\mu_L(Ra_{2t} + x_{2t})}.$$

By construction, within the intermediate region, there is always $\widehat{\lambda}_{2t} \in [0, 1)$. The economic meaning of $\widehat{\lambda}_{2t}$ will become clear shortly.

LEMMA 9. *When $\eta_{2t} \in G_m$, there are:*

i) if $\lambda_{2t} \in [0, \widehat{\lambda}_{2t}]$, then

$$\begin{aligned} q_{2t}(\lambda_{2t}) &= 0, \\ q_{2t+1}(\lambda_{2t}) &= \frac{(\mu_E + \mu_L)(Ra_{2t} + x_{2t}) - x_{2t}}{\widehat{p}_L(\lambda_{2t})}, \end{aligned} \tag{A.1}$$

where q_{2t+1} is increasing in λ_{2t} , and,

ii) if $\lambda_{2t} \in (\widehat{\lambda}_{2t}, 1]$, then

$$q_{2t}(\lambda_{2t}) = \frac{(\mu_E + \lambda_{2t}\mu_L)(Ra_{2t} + x_{2t}) - x_{2t}}{\widehat{p}_E(\lambda_{2t})}, \quad (\text{A.2})$$

$$q_{2t+1}(\lambda_{2t}) = \frac{(1 - \lambda_{2t})\mu_L \cdot \frac{R(a_{2t} - q_{2t})}{1 - \mu_E - \lambda_{2t}\mu_L}}{\widehat{p}_L(\lambda_{2t})}, \quad (\text{A.3})$$

where q_{2t} is increasing in λ_{2t} but q_{2t+1} is decreasing in λ_{2t} .

Moreover, $q_{2t} + q_{2t+1}$ is increasing in λ_{2t} for all $\lambda_{2t} \in [0, 1]$.

I first discuss the intuition behind the results when $\lambda_{2t} \leq \widehat{\lambda}_{2t}$. In this case, the fund has enough cash to satisfy all the $\mu_E + \lambda_{2t}\mu_L$ redeeming shareholders on date $2t$ at the initial NAV. Thus, no illiquid assets are forced to sell on date $2t$, that is, $q_{2t}(\lambda_{2t}) = 0$. However, in the intermediate region the fund does not have enough cash to satisfy all the late shareholders on the odd date. Specifically, the cash gap at the beginning of date $2t + 1$ is indicated by the numerator of (A.1). Following the same intuition of Lemma 4, the fund manager will close the gap by selling at the effective price $\widehat{p}_L(\lambda_{2t})$. As λ_{2t} increases, the effective sale price $\widehat{p}_L(\lambda_{2t})$ becomes lower, suggesting that the fund will be forced to sell more on date $2t + 1$.

The situation becomes different when $\lambda_{2t} > \widehat{\lambda}_{2t}$. Compared to Lemma 4, the two conditions (A.2) and (A.3) are exactly the same as conditions (3.5) and (3.6) there. This is because when $\lambda_{2t} > \widehat{\lambda}_{2t}$ the cash position x_{2t} becomes inadequate to satisfy the $\mu_E + \lambda_{2t}\mu_L$ redeeming shareholders on date $2t$ at the initial NAV, so that the stage game effectively jumps into the low cash-to-assets ratio region. The monotonicity of q_{2t} , q_{2t+1} , and $(q_{2t} + q_{2t+1})$ all follows the same intuition there.

It is worth noting that, regardless of whether $\lambda_{2t} \leq \widehat{\lambda}_{2t}$ or $\lambda_{2t} > \widehat{\lambda}_{2t}$, more late shareholder runs always lead to unambiguously higher forced sales within the entire stage (including both date $2t$ and $2t + 1$).

Similarly, I can characterize the NAVs. When $\lambda_{2t} \in [0, \widehat{\lambda}_{2t}]$, by Lemma 9 there is

$$NAV_{2t}(\lambda_{2t}) = Ra_{2t} + x_{2t}, \quad (\text{A.4})$$

and

$$\begin{aligned} NAV_{2t+1}(\lambda_{2t}) &= \frac{\overbrace{R(a_{2t} - q_{2t+1}(\lambda_{2t}) - s_{2t+1})}^{\text{illiquid assets retained}} + \overbrace{x_{2t} - (\mu_E + \lambda_{2t}\mu_L)(Ra_{2t} + x_{2t})}^{\text{cash retained}} + \overbrace{\delta_L R(q_{2t+1}(\lambda_{2t}) + s_{2t+1})}^{\text{cash raised and rebuilt}}}{\underbrace{1 - (\mu_E + \lambda_{2t}\mu_L)}_{\text{shareholders remained on date } 2t + 1}} \\ &= NAV_{2t} - \frac{(1 - \delta_L)R(q_{2t+1}(\lambda_{2t}) + s_{2t+1})}{1 - \mu_E - \lambda_{2t}\mu_L}. \end{aligned} \quad (\text{A.5})$$

where $q_{2t+1}(\lambda_{2t})$ is given in (A.1). Clearly, the NAV on date $2t$ as in (A.4) is also constant and the same as that in the high region. The NAV on date $2t + 1$ as in (A.5) also features the same expression as (3.2) in the high region. These suggest that shareholders' strategic interaction in this sub-region is the same as that in the high region.

When $\lambda_{2t} \in (\widehat{\lambda}_{2t}, 1]$, by Lemma 9 there are

$$NAV_{2t}(\lambda_{2t}) = Ra_{2t} + x_{2t} - (1 - \delta_E)Rq_{2t}(\lambda_{2t}), \quad (\text{A.6})$$

and

$$NAV_{2t+1}(\lambda_{2t}) = \frac{R(a_{2t} - q_{2t}(\lambda_{2t}) - q_{2t+1}(\lambda_{2t}) - s_{2t+1}) + \delta_L R(q_{2t+1}(\lambda_{2t}) + s_{2t+1})}{1 - \mu_E - \lambda_{2t}\mu_L}, \quad (\text{A.7})$$

where $q_{2t}(\lambda_{2t})$ and $q_{2t+1}(\lambda_{2t})$ are given in (A.2) and (A.3). Note that, the NAVs as in (A.6) and (A.7) are exactly the same as (3.8) and (3.9) in the low region, suggesting that shareholders' strategic interaction in this sub-region is the same as that in the low region.

The following lemma shows that the stage game in the intermediate region features a switch from strategic complementarity to substitutability when more shareholders are going to run.

LEMMA 10. *When $\eta_{2t} \in G_m$, there are:*

- i) if $\lambda_{2t} \in [0, \widehat{\lambda}_{2t}]$, late shareholders' run decision λ_{2t} exhibits strategic complementarity for any feasible $s_{2t+1} \in [0, a_{2t} - q_{2t+1}(\lambda_{2t})]$, and the strategic complementarity becomes stronger as s_{2t+1} increases, and,*
- ii) if $\lambda_{2t} \in (\widehat{\lambda}_{2t}, 1]$, late shareholders' run decision λ_{2t} exhibits strategic substitutability for any λ_{2t} satisfying $\theta NAV_{2t}(\lambda_{2t}) \geq NAV_{2t+1}(\lambda_{2t})$ and any feasible $s_{2t+1} \in [0, a_{2t} - q_{2t}(\lambda_{2t}) - q_{2t+1}(\lambda_{2t})]$, and the strategic substitutability becomes weaker as s_{2t+1} increases.*

Lemma 10 can be understood in view of Lemma 3 (for the high region) and Lemma 5 (for the low region). In the first sub-region $[0, \widehat{\lambda}_{2t}]$, shareholders who run can get the endogenously fixed NAV on date $2t$ at the expense of shareholders who wait. More running shareholders or a more rapid cash rebuilding policy implies a larger magnitude of predictable decline in the NAV on date $2t + 1$, leading to a stronger strategic complementarity. In the second sub-region, however, running shareholders must accept an endogenously lower NAV themselves because the fund is forced to sell its illiquid assets on date $2t$, when the sale price is extremely low. The resulting higher sale losses suggest that more shareholder runs make other shareholders less likely to run. But again, more rapid cash rebuilding still gives rise to a larger magnitude of predictable decline in the NAV on date $2t + 1$ and thus reinforces the run incentive.

Because of the switch of strategic interaction, shareholders' equilibrium run behavior exhibits a richer pattern. Despite the complicated equilibrium construction in the intermediate region, it still indicates that

fund cash rebuilding leads to runs and more rapid cash rebuilding triggers more severe runs in equilibrium.

PROPOSITION 13. *When $\eta_{2t} \in G_m$, late shareholders' run behavior is given by the following five cases:*

i) *none of the late shareholders runs, that is, $\lambda_{2t} = 0$, if*

$$s_{2t+1} < \underline{s}_m \equiv \frac{Ra_{2t} - \theta(1 - \mu_E - \widehat{\lambda}_{2t}\mu_L)(Ra_{2t} + x_{2t})}{(1 - \delta_L)R} - q_{2t+1}(\widehat{\lambda}_{2t}),$$

ii) *if*

$$s_{2t+1} > \bar{s}_m \equiv \frac{Ra_{2t} - \theta(1 - \mu_E)(Ra_{2t} + x_{2t})}{(1 - \delta_L)R} - q_{2t+1}(0), \text{ then,}$$

a) *all the late shareholders run, that is, $\lambda_{2t} = 1$, if*

$$s_{2t+1} > \bar{s}_m \equiv \frac{Ra_{2t} - \theta(1 - \mu_E - \mu_l)(Ra_{2t} + x_{2t}) - (1 - \theta(1 - \delta_E)(1 - \mu_E - \mu_l))Rq_{2t}(1)}{(1 - \delta_L)R},$$

b) *some of the late shareholder runs, that is, $\lambda_{2t} = \widetilde{\lambda}_{2t} \in [\widehat{\lambda}_{2t}, 1)$, if*

$$s_{2t+1} \leq \bar{s}_m,$$

where $\widetilde{\lambda}_{2t}$ is the solution to

$$s_{2t+1} = \frac{Ra_{2t} - \theta(1 - \mu_E - \widetilde{\lambda}_{2t}\mu_l)(Ra_{2t} + x_{2t}) - (1 - \theta(1 - \delta_E)(1 - \mu_E - \widetilde{\lambda}_{2t}\mu_l))Rq_{2t}(\widetilde{\lambda}_{2t})}{(1 - \delta_L)R} - q_{2t+1}(\widetilde{\lambda}_{2t}),$$

iii) *if $\underline{s}_m \leq s_{2t+1} \leq \bar{s}_m$, then,*

c) *$\lambda_{2t} \in \{0, \widetilde{\lambda}_{2t}, 1\}$, if*

$$s_{2t+1} > \bar{s}_m,$$

where $\widetilde{\lambda}_{2t}$ is the solution to

$$s_{2t+1} = \frac{(1 - \theta)(1 - \mu_E - \widetilde{\lambda}_{2t}\mu_L)(Ra_{2t} + x_{2t})}{(1 - \delta_L)R} - q_{2t+1}(\widetilde{\lambda}_{2t}),$$

d) *$\lambda_{2t} \in \{0, \widetilde{\lambda}_{2t}, \widetilde{\widetilde{\lambda}}_{2t}\}$, if*

$$s_{2t+1} \leq \bar{s}_m,$$

where $\widetilde{\lambda}_{2t}$ is given in Case c) and $\widetilde{\widetilde{\lambda}}_{2t}$ is given in Case b).

The expressions of $q_{2t}(\lambda_{2t})$ and $q_{2t+1}(\lambda_{2t})$ are given in Lemma 9. Moreover, $\bar{s}_m \geq 0$ and $\bar{s}_m > \underline{s}_m$.

The intuition behind Proposition 13 is clear in view of Propositions 1 and 2. By Lemma 10, the stage

game in the intermediate region starts with strategic complementarity when only a small fraction of late shareholders decides to run. Hence, it is the strategic complementarity in the first sub-region $[0, \widehat{\lambda}_{2t}]$ that determines whether any late shareholder will run at all. As in Case i), when $\widehat{\lambda}_{2t}$ of the late shareholders decide to run, if the utility gain of running over waiting is still not positive, none of the late shareholders will ever run. In Case ii), the utility gain of running over waiting is already positive even if no one runs, so that at least $\widehat{\lambda}_{2t}$ of the late shareholders will run due to the strategic complementarity in the sub-region $[0, \widehat{\lambda}_{2t}]$. However, as the stage game switches to the second sub-region $(\widehat{\lambda}_{2t}, 1]$, there can be strategic substitutability. In sub-case a), the fund uses a rapid cash rebuilding policy so that all the late shareholders run despite the strategic substitutability, while in sub-case b) the substitutability is strong so that $\lambda_{2t} = \widetilde{\lambda}_{2t} \in (\widehat{\lambda}_{2t}, 1)$ of the late shareholders are going to run. Finally, in Case iii), the strategic complementarity in the first sub-region $[0, \widehat{\lambda}_{2t}]$ is moderate. When this happens, the worst equilibrium will be determined by the magnitude of strategic substitutability in the sub-region $(\widehat{\lambda}_{2t}, 1]$, as shown in Case c) and Case d).

As usual, I show how shareholder runs increase the risk of forced sales by exploring the laws of motions.

COROLLARY 4. *When $\eta_{2t} \in G_m$, the law of motions of (a_{2t}, x_{2t}) is given by*

$$a_{2t+2} = a_{2t} - (q_{2t}(\lambda_{2t}) + q_{2t+1}(\lambda_{2t})) - s_{2t+1}, \text{ and,} \quad (\text{A.8})$$

$$x_{2t+2} = \widehat{p}_L(\lambda_{2t})s_{2t+1}, \quad (\text{A.9})$$

where λ_{2t} is the run probability induced by (a_{2t}, x_{2t}) and s_{2t+1} , as characterized in Proposition 13.

A.2 Proofs

In this appendix, I provide proofs for the main results in the main text; more technical proofs for Propositions 3 and 6 as well as proofs for the results in Section 5 are provided in the online appendix.

PROOF OF LEMMA 1. First, in the high cash-to-assets ratio region, the fund needs to sell no illiquid assets on either date $2t$ or $2t + 1$. Since no sale losses are incurred in this region, both early and late shareholders are able to get the same NAV as that at the beginning of date $2t$, that is,

$$NAV_{2t} = NAV_{2t+1} = \frac{Ra_{2t} + x_{2t}}{n_{2t}}.$$

Moreover, the initial cash position should be large enough to meet the redemption needs of all shareholders

on dates $2t$ and $2t + 1$ at such a constant NAV:

$$x_{2t} \geq (\mu_E + \mu_L)n_{2t} \cdot \frac{Ra_{2t} + x_{2t}}{n_{2t}},$$

yielding

$$\eta_{2t} \geq \frac{(\mu_E + \mu_L)R}{1 - \mu_E - \mu_L}, \quad (\text{A.10})$$

the criterion for the high region.

Then, in the intermediate region, as no sale is incurred on date $2t$, the initial cash position is high enough to meet the redemption needs of early shareholders at the initial NAV but insufficient to meet late shareholders' redemption needs:

$$\mu_E n_{2t} \cdot \frac{Ra_{2t} + x_{2t}}{n_{2t}} \leq x_{2t} < (\mu_E + \mu_L)n_{2t} \cdot \frac{Ra_{2t} + x_{2t}}{n_{2t}},$$

which leads to

$$\frac{\mu_E R}{1 - \mu_E} \leq \eta_{2t} < \frac{(\mu_E + \mu_L)R}{1 - \mu_E - \mu_L}, \quad (\text{A.11})$$

the criterion for the intermediate region.

Finally, in the low region, the cash position is even inadequate to meet early shareholders' redemption needs at the initial NAV. This means

$$x_{2t} < \mu_E n_{2t} \cdot \frac{Ra_{2t} + x_{2t}}{n_{2t}},$$

which yields

$$\eta_{2t} < \frac{\mu_E R}{1 - \mu_E}, \quad (\text{A.12})$$

the criterion for the intermediate region.

It is straightforward to check that (A.10), (A.11), and (A.12) are also sufficient conditions. \square

PROOF OF LEMMA 2. Suppose $\lambda_{2t}\mu_L$ late shareholders run. This situation is equivalent to a counterfactual in which there are initially $\mu'_E = \mu_E + \lambda_{2t}\mu_L$ early shareholders and $\mu'_L = (1 - \lambda_{2t})\mu_L$ late shareholders but no late shareholder runs. Since $\mu'_E + \mu'_L = \mu_E + \mu_L$, by Lemma 1, $q_{2t} = q_{2t+1} = 0$ is true in the counterfactual situation and so is true in the original situation with $\lambda_{2t}\mu_L$ late shareholders running. \square

PROOF OF LEMMA 3. By Lemma 2 and the definition of $\Delta u_L(\lambda_{2t})$:

$$\Delta u_L(\lambda_{2t}) = \theta NAV_{2t} - NAV_{2t+1} = (\theta - 1)(Ra_{2t} + x_{2t}) + \frac{(1 - \delta_L)Rs_{2t+1}}{1 - \mu_E - \lambda_{2t}\mu_L}.$$

Taking derivatives yields:

$$\frac{\partial \Delta u_L(\lambda_{2t})}{\partial \lambda_{2t}} = \frac{(1 - \delta_L)\mu_L Rs_{2t+1}}{(1 - \mu_E - \lambda_{2t}\mu_L)^2} > 0,$$

which takes value 0 when $s_{2t+1} = 0$, and

$$\frac{\partial^2 \Delta u_L(\lambda_{2t})}{\partial \lambda_{2t} \partial s_{2t+1}} = \frac{(1 - \delta_L)\mu_L R}{(1 - \mu_E - \lambda_{2t}\mu_L)^2} > 0.$$

□

PROOF OF PROPOSITION 1. By Lemma 3, the stage game exhibits strategic complementarity when $s_{2t+1} > 0$. Also notice that any shareholder runs only if $\theta NAV_{2t} \geq NAV_{2t+1}$. Thus, in Case i), none of the late shareholders runs if

$$\theta NAV_{2t} < NAV_{2t+1}(1), \tag{A.13}$$

in which $NAV_{2t+1}(\lambda_{2t})$ is a function of λ_{2t} . Solving inequality (A.13) leads to

$$s_{2t+1} < \frac{(1 - \theta)(1 - \mu_E - \mu_L)(Ra_{2t} + x_{2t})}{(1 - \delta_L)R} \equiv \underline{s}_h.$$

Alternatively, in Case ii), all the late shareholders run if

$$\theta NAV_{2t} > NAV_{2t+1}(0), \tag{A.14}$$

the solution of which is

$$s_{2t+1} > \frac{(1 - \theta)(1 - \mu_E)(Ra_{2t} + x_{2t})}{(1 - \delta_L)R} \equiv \bar{s}_h.$$

Finally, in Case iii), if neither (A.13) nor (A.14) holds, there exists a $\tilde{\lambda}_{2t} \in [0, 1]$ that solves

$$\theta NAV_{2t} = NAV_{2t+1}(\tilde{\lambda}_{2t}).$$

Note that, $\tilde{\lambda}_{2t}$ constructs an equilibrium because by definition $\Delta u_L(\tilde{\lambda}_{2t}) = 0$ and thus no shareholder would have an incentive to deviate from it. In addition, in this case, again by Lemma 3, there are $\theta NAV_{2t} \geq NAV_{2t+1}(1)$ and $\theta NAV_{2t} \leq NAV_{2t+1}(0)$, which means $\lambda_{2t} = 1$ and $\lambda_{2t} = 0$ are also two

equilibria when (A.13) and (A.14) are both violated. \square

PROOF OF COROLLARY 1. By Lemma 2, $q_{2t}(\lambda_{2t}) = q_{2t+1}(\lambda_{2t}) = 0$ for any arbitrary $\lambda_{2t} \in [0, 1]$. Thus, the evolution of the asset position directly follows:

$$a_{2t+2} = a_{2t} - q_{2t} - q_{2t+1} - s_{2t+1} = a_{2t} - s_{2t+1}.$$

For the evolution of the cash position, the fund pays all the redeeming shareholders by cash at the respective end-of-day NAVs on date $2t$ and $2t + 1$, and rebuilds its cash buffer on date $2t + 1$. Note that there will be no cash raised by forced sales. Thus:

$$\begin{aligned} x_{2t+2} &= x_{2t} - (\mu_E + \lambda_{2t}\mu_L)NAV_{2t} - (1 - \lambda_{2t})\mu_L NAV_{2t+1} + p_L s_{2t+1} \\ &= x_{2t} - (\mu_E + \mu_L)(Ra_{2t} + x_{2t}) + \delta_L R s_{2t+1} + \frac{(1 - \lambda_{2t})\mu_L(1 - \delta_L)R s_{2t+1}}{1 - \mu_E - \lambda_{2t}\mu_L}. \end{aligned}$$

\square

PROOF OF LEMMA 4. Recall that, when forced sales occur, the fund sells up to a point at which it can satisfy the redemptions at the end-of-day NAV, which will take into account the losses from forced sales. On the one hand, on date $2t$ the fund starts with a cash position x_{2t} . Hence, on date $2t$, q_{2t} solves

$$x_{2t} + p_E q_{2t} = (\mu_E + \lambda_{2t}\mu_L)[(a_{2t} - q_{2t})R + x_{2t} + p_E q_{2t}],$$

yielding

$$q_{2t}(\lambda_{2t}) = \frac{(\mu_E + \lambda_{2t}\mu_L)(Ra_{2t} + x_{2t}) - x_{2t}}{[\delta_E + (1 - \delta_E)(\mu_E + \lambda_{2t}\mu_L)]R}. \quad (\text{A.15})$$

On the other hand, on date $2t + 1$, the fund has no cash at all at the beginning. Hence, q_{2t+1} solves

$$p_L q_{2t+1} = (1 - \lambda_{2t})\mu_L \frac{(a_{2t} - q_{2t} - q_{2t+1})R + p_L q_{2t+1}}{1 - \mu_E - \lambda_{2t}\mu_L},$$

yielding

$$q_{2t+1} = \frac{(1 - \lambda_{2t})\mu_L(a_{2t} - q_{2t})}{(1 - \mu_E - \lambda_{2t}\mu_L)\delta_L + (1 - \lambda_{2t})\mu_L(1 - \delta_L)}. \quad (\text{A.16})$$

Plugging (A.15) into (A.16) leads to

$$q_{2t+1}(\lambda_{2t}) = \frac{(1 - \lambda_{2t})\mu_L \cdot \frac{R(a_{2t} - q_{2t})}{1 - \mu_E - \lambda_{2t}\mu_L}}{\left[\delta_L + \frac{(1 - \lambda_{2t})\mu_L(1 - \delta_L)}{1 - \mu_E - \lambda_{2t}\mu_L} \right] R}. \quad (\text{A.17})$$

For the monotonicity of $q_{2t}(\lambda_{2t})$, taking derivative of (A.15) leads to

$$\frac{\partial q_{2t}(\lambda_{2t})}{\lambda_{2t}} = \frac{\mu_L(\delta_E R a_{2t} + x_{2t})}{[\mu_E(1 - \delta_E) + (1 - \lambda_{2t}\mu_L)\delta_E + \lambda_{2t}\mu_L]^2} > 0,$$

implying that $q_{2t}(\lambda_{2t})$ is increasing in λ_{2t} . Similar procedures based on (A.15) and (A.17) show that $q_{2t+1}(\lambda_{2t})$ is decreasing in λ_{2t} while $q_{2t}(\lambda_{2t}) + q_{2t+1}(\lambda_{2t})$ is increasing in λ_{2t} . \square

PROOF OF LEMMA 5. By Lemma 4 and the definition of $\Delta u_L(\lambda_{2t})$:

$$\begin{aligned} \Delta u_L(\lambda_{2t}) &= \theta NAV_{2t} - NAV_{2t+1} \\ &= \theta[(Ra_{2t} + x_{2t}) - (1 - \delta_E)Rq_{2t}] - \frac{(Ra_{2t} + x_{2t}) - Rq_{2t} - (1 - \delta_L)R(q_{2t+1} + s_{2t+1})}{1 - \mu_E - \lambda_{2t}\mu_L}, \end{aligned} \quad (\text{A.18})$$

in which q_{2t} and q_{2t+1} are functions of λ_{2t} by Lemma 4. It is straightforward that $\Delta u_L(\lambda_{2t})$ is larger when s_{2t+1} increases. To focus on the value of λ_{2t} that satisfies $\theta NAV_{2t} \geq NAV_{2t+1}$, there is no loss of generality to consider $\theta = 1$, and the analysis for a general θ naturally follows by considering subsets of λ_{2t} . Now plug (A.15) and (A.16) into (A.18) and then take derivative with respect to λ_{2t} . After rearrangement, this yields:

$$\frac{\partial \Delta u_L(\lambda_{2t})}{\partial \lambda_{2t}} = - \frac{(1 - \delta_L)\mu_L \left(x_{2t}C_1 - \frac{R(s_{2t+1}C_2 - a_{2t}\delta_E(1 - \mu_E - \lambda_{2t}\mu_L)^2C_1)}{(1 - \mu_E - \lambda_{2t}\mu_L)^2} \right)}{((1 - \lambda_{2t})\mu_L + \delta_L(1 - \mu_E - \mu_L))^2(\mu_E + \lambda_{2t}\mu_L + \delta_E(1 - \mu_E - \lambda_{2t}\mu_L))^2}, \quad (\text{A.19})$$

where

$$\begin{aligned} C_1 &= (1 - \delta_E)(1 - \lambda_{2t})^2\mu_L^2 + \delta_L(1 - \mu_E - \mu_L)(\mu_E + \mu_L) + \delta_E\delta_L(1 - \mu_E - \mu_L)^2 > 0, \\ C_2 &= ((1 - \lambda_{2t})\mu_L + \delta_L(1 - \mu_E - \mu_L))^2(\mu_E + \lambda_{2t}\mu_L + \delta_E(1 - \mu_E - \lambda_{2t}\mu_L))^2 > 0. \end{aligned}$$

Consider

$$C = x_{2t}C_1 - \frac{R(s_{2t+1}C_2 - a_{2t}\delta_E(1 - \mu_E - \lambda_{2t}\mu_L)^2C_1)}{(1 - \mu_E - \lambda_{2t}\mu_L)^2}$$

for any $0 \leq x_{2t} < \mu_E n_{2t}(Ra_{2t} + x_{2t})$ and any $0 \leq s_{2t+1} \leq a_{2t} - q_{2t}(\lambda_{2t}) - q_{2t+1}(\lambda_{2t})$. Since $C_1 > 0$, there is $x_{2t}C_1 > 0$ and thus

$$C > \frac{R(a_{2t}(\delta_E(1 - \mu_E - \lambda_{2t}\mu_L)^2C_1 - C_2))}{(1 - \mu_E - \lambda_{2t}\mu_L)^2}. \quad (\text{A.20})$$

Notice that C_1 and C_2 are only functions of λ_{2t} , μ_E , μ_L , δ_E , δ_L , and are independent of a_{2t} and x_{2t} . By construction,

$$\delta_E(1 - \mu_E - \lambda_{2t}\mu_L)^2 C_1 - C_2 \geq 0$$

for any $\lambda_{2t} \in [0, 1]$.

As a result, since $a_{2t} > 0$ and $R > 0$, inequality (A.20) implies that $C > 0$. Plugging back to (A.19) finally yields

$$\frac{\partial \Delta u_L(\lambda_{2t})}{\partial \lambda_{2t}} < 0,$$

implying strategic substitutability.

It is straightforward that $\Delta u_L(\lambda_{2t})$ is larger when s_{2t+1} increases. Also, by definition, C is decreasing in s_{2t+1} when $0 \leq s_{2t+1} \leq a_{2t} - q_{2t}(\lambda_{2t}) - q_{2t+1}(\lambda_{2t})$. By (A.19) and the derivation above, the strategic substitutability becomes weaker when s_{2t+1} increases. \square

PROOF OF PROPOSITION 2. Notice that any shareholder runs only if $\theta NAV_{2t} \geq NAV_{2t+1}$. Also by Lemma 5, the stage game exhibits strategic substitutability whenever an incentive to redeem earlier exists. Thus, in Case i), none of the late shareholders runs if

$$\theta NAV_{2t}(0) < NAV_{2t+1}(0), \tag{A.21}$$

which implies that $\theta NAV_{2t}(\lambda_{2t}) < NAV_{2t+1}(\lambda_{2t})$ for any λ_{2t} by using the expressions in Lemma 4. Thus, solving inequality (A.21) leads to

$$s_{2t+1} < \frac{Ra_{2t} - \theta(1 - \mu_E)(Ra_{2t} + x_{2t}) - (1 - \theta(1 - \delta_E)(1 - \mu_E))Rq_{2t}(0)}{(1 - \delta_L)R} - q_{2t+1}(0) \equiv \underline{s}_l.$$

Alternatively, in Case ii), all the late shareholders run if

$$\theta NAV_{2t}(1) > NAV_{2t+1}(1), \tag{A.22}$$

which implies that $\theta NAV_{2t}(\lambda_{2t}) > NAV_{2t+1}(\lambda_{2t})$ for any λ_{2t} despite the underlying strategic substitutability suggested by Lemma 5. Solving inequality (A.22) leads to

$$s_{2t+1} > \frac{Ra_{2t} - \theta(1 - \mu_E - \mu_L)(Ra_{2t} + x_{2t}) - (1 - \theta(1 - \delta_E)(1 - \mu_E - \mu_L))Rq_{2t}(1)}{(1 - \delta_L)R} \equiv \bar{s}_l.$$

Using the expressions in Lemma 4, plugging $q_{2t}(0)$, $q_{2t+1}(0)$ and $q_{2t}(1)$ into the definition of \underline{s}_l and \bar{s}_l

directly yields $\bar{s}_l \geq 0$ and $\bar{s}_l > \underline{s}_l$.

Finally, in Case iii), there exists some $\tilde{\lambda}_{2t} \in [0, 1]$ that solves

$$\theta NAV_{2t}(\tilde{\lambda}_{2t}) = NAV_{2t+1}(\tilde{\lambda}_{2t}),$$

where $\tilde{\lambda}_{2t}$ constructs an equilibrium because by definition $\Delta u_L(\tilde{\lambda}_{2t}) = 0$ and thus no shareholder would have an incentive to deviate from it. This leads to

$$s_{2t+1} = \frac{Ra_{2t} - \theta(1 - \mu_E - \tilde{\lambda}_{2t}\mu_L)(Ra_{2t} + x_{2t}) - (1 - \theta(1 - \delta_E)(1 - \mu_E - \tilde{\lambda}_{2t}\mu_L))Rq_{2t}(\tilde{\lambda}_{2t})}{(1 - \delta_L)R} - q_{2t+1}(\tilde{\lambda}_{2t}).$$

□

PROOF OF COROLLARY 2. By Lemma 4, $q_{2t}(\lambda_{2t}) > 0$ for any arbitrary $\lambda_{2t} \in [0, 1]$ and $q_{2t+1}(\lambda_{2t}) > 0$ for any arbitrary $\lambda_{2t} \in [0, 1)$. Thus, the evolution of the asset position directly follows:

$$a_{2t+2} = a_{2t} - q_{2t}(\lambda_{2t}) - q_{2t+1}(\lambda_{2t}) - s_{2t+1}.$$

For the evolution of the cash position, notice that all the proceeds from forced sales $q_{2t}(\lambda_{2t})$ and $q_{2t}(\lambda_{2t+1})$ go to the redeeming shareholders. Also, by definition of the low cash-to-assets region, the fund starts with no cash on date $2t + 1$. Thus:

$$\begin{aligned} x_{2t+2} &= p_L(q_{2t+1}(\lambda_{2t}) + s_{2t+1}) - (1 - \lambda_{2t})\mu_L NAV_{2t+1} \\ &= \delta_L R s_{2t+1} + \frac{(1 - \lambda_{2t})\mu_L(1 - \delta_L)R s_{2t+1}}{1 - \mu_E - \lambda_{2t}\mu_L}. \end{aligned}$$

□

PROOF OF LEMMA 6. If $a_{2t} = 0$, it is trivial that $s_{2t+1}^*(a_{2t}, x_{2t}) = 0$. So it is only worth considering a strictly positive a_{2t} .

On the one hand, consider a perturbation $\varepsilon > 0$ of cash rebuilding around $s_{2t+1}^*(a_{2t}, x_{2t}) = 0$. On date $2t + 1$ (in stage t), regardless of the starting portfolio position (a_{2t}, x_{2t}) , the effective sale price on $2t + 1$ is at most $\hat{p}_L(0) > 0$. Thus, the sale loss in stage t is at least

$$\frac{\varepsilon(1 - \delta_L)R}{\hat{p}_L(0)} > 0.$$

On other other hand, consider an initial cash gap ε on date $2t + 2$ (in stage $t + 1$). Regardless of the

starting portfolio position (a_{2t+2}, x_{2t+2}) , the effective sale price on $2t + 2$ is at least $\widehat{p}_E(0) > 0$, and the physical sale price in stage $t + 1$ is at least $\delta_E R$. Thus, the expected sale loss in stage $t + 1$ due to this cash gap is at most

$$\frac{\varepsilon(1 - \delta_E)R}{\widehat{p}_E(0)} > 0.$$

Therefore, for any π satisfying

$$\pi > 1 - \frac{(1 - \delta_L)\widehat{p}_E(0)}{(1 - \delta_E)\widehat{p}_L(0)} \in (0, 1),$$

it is optimal to choose $s_{2t+1}^*(a_{2t}, x_{2t}) = 0$. □

PROOF OF LEMMA 7. This directly follows late shareholders' utility function. □

PROOF OF LEMMA 8. First consider the case of $\eta_{2t} \in G_h$. By Proposition 1, $\theta = 1$ implies that $\bar{s}_h = 0$. Again by Proposition 1, there is $\lambda_{2t} = 1$ for any $s_{2t+1} > 0$ regardless of (a_{2t}, x_{2t}) .

Then consider the case of $\eta_{2t} \in G_l$. Similarly, by Proposition 2, $\theta = 1$ implies that $\bar{s}_l = 0$. Again by Proposition 2, there is $\lambda_{2t} = 1$ for any $s_{2t+1} > 0$ regardless of (a_{2t}, x_{2t}) .

Finally, consider the case of $\eta_{2t} \in G_m$. By Proposition 13, $\theta = 1$ implies that $\bar{s}_m = \bar{\bar{s}}_m = 0$. Again by Proposition 13, there is $\lambda_{2t} = 1$ for any $s_{2t+1} > 0$ regardless of (a_{2t}, x_{2t}) . □

PROOF OF PROPOSITION 4. I consider two cases according to the starting cash-to-assets ratio on date $2t$.

CASE 1. $\eta_{2t} \in G_l \cup G_m \cup G_{hl}$. First, consider a perturbation $-\varepsilon < 0$ of cash rebuilding around σ_{2t+1}^* that satisfies $\eta_{2t+2}^* = \mu_E R / (1 - \mu_E)$. On date $2t + 1$ (in stage t), since there are no runs (by Lemma 7), the effective sale price on $2t + 1$ is $\widehat{p}_L(0)$. Thus, the sale loss saved in stage t is

$$\frac{\varepsilon(1 - \delta_L)R}{\widehat{p}_L(0)} > 0.$$

Now consider the same cash gap ε on date $2t + 2$ (under the perturbed cash rebuilding policy $(\sigma_{2t+1}^*, -\varepsilon)$). This implies that $\eta_{2t+2} \in G_l$. Since there are no runs, the fund must sell its assets on date $2t + 2$ at effective sale price $\widehat{p}_E(0) > 0$. Hence, the expected increase of sale loss in stage $t + 1$ due to this cash gap ε is

$$\frac{\varepsilon(1 - \delta_E)R}{\widehat{p}_E(0)} > 0.$$

Since $\delta_E < \delta_L$ and $\widehat{p}_E(0) < \widehat{p}_L(0)$, there is

$$\frac{\varepsilon(1 - \delta_L)R}{\widehat{p}_L(0)} < (1 - \pi) \frac{\varepsilon(1 - \delta_E)R}{\widehat{p}_E(0)}$$

for a sufficiently small but positive π , implying that the perturbation $-\varepsilon$ is not profitable.

Next, consider another perturbation $\varepsilon > 0$ of cash rebuilding around σ_{2t+1}^* that satisfies $\eta_{2t+2}^* = \mu_E R / (1 - \mu_E)$. On date $2t + 1$ (in stage t), similarly, the sale loss increased in stage t is

$$\frac{\varepsilon(1 - \delta_L)R}{\widehat{p}_L(0)} > 0.$$

Under the perturbed cash rebuilding policy $(\sigma_{2t+1}^*, \varepsilon)$ the fund gets ε more cash in stage $t + 1$. This implies that $\eta_{2t+2} \in G_m$. Since there are no runs, the fund does not have to sell its assets on date $2t + 2$. Rather, the marginal cash saves the fund's active asset sales on date $2t + 3$ at the effective sale price $\widehat{p}_L(0) > 0$. Hence, the expected sale saved in stage $t + 1$ due to this marginal cash ε is also

$$\frac{\varepsilon(1 - \delta_L)R}{\widehat{p}_L(0)} > 0.$$

Since $\pi \in (0, 1)$, this perturbation ε is also not profitable. This verifies the optimality of $\eta_{2t+2}^* = \mu_E R / (1 - \mu_E)$ when $\eta_{2t} \in G_l \cup G_m \cup G_{hl}$.

CASE 2. $\eta_{2t} \in G_{hm} \cup G_{hh}$. Consider a perturbation $\varepsilon > 0$ of cash rebuilding around $\sigma_{2t+1}^* = 0$. On date $2t + 1$ (in stage t), similarly, the sale loss increased in stage t is

$$\frac{\varepsilon(1 - \delta_L)R}{\widehat{p}_L(0)} > 0.$$

Similarly, under the perturbed cash rebuilding policy $(\sigma_{2t+1}^*, \varepsilon)$ the fund gets ε more cash in stage $t + 1$. The expected sale loss saved in stage $t + 1$ due to this marginal cash ε is also

$$\frac{\varepsilon(1 - \delta_L)R}{\widehat{p}_L(0)} > 0.$$

Since $\pi \in (0, 1)$, this perturbation ε is again not profitable. This verifies the optimality of $\sigma_{2t+1}^* = 0$ when $\eta_{2t} \in G_{hm} \cup G_{hh}$. This finally concludes the proof. \square

PROOF OF PROPOSITION 5. I consider two cases according to the starting cash-to-assets ratio on date $2t$.

CASE 1. $\eta_{2t} \in G_l \cup G_m \cup G_{hl} \cup G_{hm}$. First, consider a perturbation $-\varepsilon < 0$ of cash rebuilding around σ_{2t+1}^* that satisfies $\eta_{2t+2}^* = (\mu_E + \mu_L)R / (1 - \mu_E - \mu_L)$. On date $2t + 1$ (in stage t), since $\lambda_{2t} = 1$ (by Lemma 8), the effective sale price on $2t + 1$ is $\widehat{p}_L(1)$. Thus, the sale loss saved in stage t is

$$\frac{\varepsilon(1 - \delta_L)R}{\widehat{p}_L(1)} > 0.$$

Now consider the same cash gap ε on date $2t+2$ (under the perturbed cash rebuilding policy $(\sigma_{2t+1}^*, -\varepsilon)$). This implies that $\eta_{2t+2} \in G_l \cup G_m$. Since $\lambda_{2t+2} = 1$, by Lemmas 4 and 9 the fund always must sell its assets on date $2t+2$ at the effective sale price $\widehat{p}_E(1) > 0$, even if $\eta_{2t+2} \in G_m$. Hence, the expected increase of sale loss in stage $t+1$ due to this cash gap ε is

$$\frac{\varepsilon(1 - \delta_E)R}{\widehat{p}_E(1)} > 0.$$

Since $\delta_E < \delta_L$ and $\widehat{p}_E(1) < \widehat{p}_L(1)$, there is

$$\frac{\varepsilon(1 - \delta_L)R}{\widehat{p}_L(1)} < (1 - \pi) \frac{\varepsilon(1 - \delta_E)R}{\widehat{p}_E(1)}$$

for a sufficiently small but positive π . Hence, the perturbation $-\varepsilon$ is not profitable.

Next, consider another perturbation $\varepsilon > 0$ of cash rebuilding around σ_{2t+1}^* that satisfies $\eta_{2t+2}^* = (\mu_E + \mu_L)R/(1 - \mu_E - \mu_L)$. On date $2t+1$ (in stage t), similarly, the sale loss increased in stage t is

$$\frac{\varepsilon(1 - \delta_L)R}{\widehat{p}_L(1)} > 0.$$

Under the perturbed cash rebuilding policy $(\sigma_{2t+1}^*, \varepsilon)$ the fund gets ε more cash in stage $t+1$. This implies that $\eta_{2t+2} \in G_h$. Hence, by Lemma 2, regardless of runs the fund does not have to sell its assets on date $2t+2$. Rather, the marginal cash saves the fund's active asset sales on date $2t+3$ at the effective sale price $\widehat{p}_L(1) > 0$. Hence, the expected sale saved in stage $t+1$ due to this marginal cash ε is also

$$\frac{\varepsilon(1 - \delta_L)R}{\widehat{p}_L(1)} > 0.$$

Since $\pi \in (0, 1)$, this perturbation ε is also not profitable. This verifies the optimality of $\eta_{2t+2}^* = (\mu_E + \mu_L)R/(1 - \mu_E - \mu_L)$ when $\eta_{2t} \in G_l \cup G_m \cup G_{hl} \cup G_{hm}$.

CASE 2. $\eta_{2t} \in G_{hh}$. Consider a perturbation $\varepsilon > 0$ of cash rebuilding around $\sigma_{2t+1}^* = 0$. On date $2t+1$ (in stage t), similarly, the sale loss increased in stage t is

$$\frac{\varepsilon(1 - \delta_L)R}{\widehat{p}_L(1)} > 0.$$

Similarly, under the perturbed cash rebuilding policy $(\sigma_{2t+1}^*, \varepsilon)$ the fund gets ε more cash in stage $t+1$.

The expected sale loss saved in stage $t + 1$ due to this marginal cash ε is also

$$\frac{\varepsilon(1 - \delta_L)R}{\widehat{p}_L(1)} > 0.$$

Since $\pi \in (0, 1)$, this perturbation ε is again not profitable. This verifies the optimality of $\sigma_{2t+1}^* = 0$ when $\eta_{2t} \in G_{hh}$. This finally concludes the proof. \square

PROOF OF PROPOSITION 7. It directly follows Propositions 4, 5, and 6 that for any θ ,

$$\eta_{2t+2}^* < \frac{\mu_E R}{1 - \mu_E}$$

is not optimal when there is no commitment device. Here I provide a sufficient condition that it can be optimal if a commitment device is introduced.

Consider $\underline{\theta}$ as defined in Proposition 6. By definition, when $\theta = \underline{\theta} + \epsilon$, where $\epsilon > 0$ is arbitrarily small, and $\eta_{2t+2} = \mu_E R / (1 - \mu_E)$, there is $\lambda_{2t} > 1$ for any $\eta_{2t} \in G_l$. Consider a perturbation $-\varepsilon < 0$ of cash rebuilding around $\eta_{2t+2} = \mu_E R / (1 - \mu_E)$ when $\eta_{2t} = \mu_E R / (1 - \mu_E)$, where the perturbation is chosen such that there is $\lambda_{2t} = 0$ for any $\eta_{2t} \in G_l$. On the one hand, since $\lambda_{2t+2} = 0$, the cash gap resulted from this perturbation on date $2t + 2$ leads to the following expected increase of sale loss in stage $t + 1$:

$$\frac{\varepsilon(1 - \delta_E)R}{\widehat{p}_E(0)}.$$

On the other hand, when a commitment device is introduced, the determination of η_{2t+2} on $2t$ directly affects q_{2t} and q_{2t+1} through λ_{2t} . Thus, under the proposed perturbation $-\varepsilon < 0$, there are no runs on date $2t$, and thus the sale loss saved in stage t is

$$\Delta q_{2t}(1 - \delta_E)R + \left(\frac{\varepsilon}{\widehat{p}_L(\lambda_{2t})} + \Delta q_{2t+1} \right) (1 - \delta_L)R,$$

where λ_{2t} solves

$$\frac{\varepsilon}{\widehat{p}_L(\lambda_{2t})} = \frac{Ra_{2t} - \theta(1 - \mu_E - \lambda_{2t}\mu_l)(Ra_{2t} + x_{2t}) - (1 - \theta(1 - \delta_E)(1 - \mu_E - \lambda_{2t}\mu_l))Rq_{2t}(\lambda_{2t})}{(1 - \delta_L)R} - q_{2t+1}(\lambda_{2t}),$$

and $\Delta q_{2t} = q_{2t}(\lambda_{2t}) - q_{2t}(0)$ and $\Delta q_{2t+1} = q_{2t+1}(0) - q_{2t+1}(\lambda_{2t})$.

Note that $\Delta q_{2t}(1 - \delta_E)R + \Delta q_{2t+1}(1 - \delta_L)R > 0$. Thus, if

$$\Delta q_{2t}(1 - \delta_E) + \Delta q_{2t+1}(1 - \delta_L) > (1 - \pi) \frac{\varepsilon(1 - \delta_E)}{\widehat{p}_E(0)} - \frac{\varepsilon(1 - \delta_L)}{\widehat{p}_L(\lambda_{2t})}$$

satisfies, it is optimal for the fund to choose a less rapid cash rebuilding policy $\eta_{2t+2} < \mu_E R / (1 - \mu_E)$. \square

PROOF OF PROPOSITION 8. This directly follows Propositions 4, 5, and 6. Under a Markov strategy profile, because the equilibrium is stationary, it suffices to show that a higher θ leads to higher total sale losses within stage t for any given positive (a_{2t}, x_{2t}) . I then consider two cases.

CASE 1. $\eta_{2t} \in G_l \cup G_m$. By Propositions 4, 5, and 6, the optimal cash rebuilding policy is always $\eta_{2t+2}^*(\theta) > 0$ where $\eta_{2t+2}^*(\theta)$ is increasing in θ . By definition, $\eta_{2t+2} = 0$ if $s_{2t+1} = 0$. Thus, the total sale losses in stage t is given by

$$L_t(\theta) = (1 - \delta_E)Rq_{2t} + (1 - \delta_L)Rq_{2t+1} + \frac{\eta_{2t+2}^*}{\hat{p}_L + \eta_{2t+2}^*}(1 - \delta_L)R(a_{2t} - q_{2t} - q_{2t+1}),$$

where q_{2t} , q_{2t+1} , and \hat{p} are both functions of λ_0 and in turn functions of θ . Propositions 4, 5, and 6 also imply that λ_{2t} is increasing in θ for any given positive (a_{2t}, x_{2t}) . Hence, it follows Lemmas 4 and 9 that $L(\theta)$ is increasing in θ .

CASE 2. $\eta_{2t} \in G_h$. By Lemma 2, $q_{2t} = q_{2t+1} = 0$ regardless of λ_{2t} or θ . Thus, the total sale losses in stage t is given by

$$L_t(\theta) = (1 - \delta_L)Rs_{2t+1}^*.$$

Define $\underline{\eta}_{2t+2}$ as the target cash-to-assets ratio if $s_{2t+1} = 0$ and $\eta_{2t} \in G_h$. By Propositions 4, 5, and 6, the difference $\eta_{2t+2}^*(\theta) - \underline{\eta}_{2t+2}$ is increasing in θ . Moreover, \hat{p}_L is decreasing in λ_{2t} and thus decreasing in θ . This implies that s_{2t+1}^* is increasing in θ and so is $L(\theta)$ in this case. This finally concludes the proof. \square

PROOF OF PROPOSITION 9. Recall that, the Bellman equation for the non-commitment case is:

$$V(a_{2t}, x_{2t}) = -(1 - \delta_E)Rq_{2t} - (1 - \delta_L)Rq_{2t+1} + \max_{s_{2t+1}^*} [-(1 - \delta_L)Rs_{2t+1} + (1 - \pi)V(a_{2t+2}, x_{2t+2})]. \quad (\text{A.23})$$

When a commitment device is introduced, the Bellman equation instead becomes:

$$V(a_{2t}, x_{2t}) = \max_{s_{2t+1}^*} [-(1 - \delta_E)Rq_{2t} - (1 - \delta_L)R(q_{2t+1} + s_{2t+1}) + (1 - \pi)V(a_{2t+2}, x_{2t+2})]. \quad (\text{A.24})$$

Also, the fund manager's objective function can be re-written as

$$\max_{\{s_{2\tau+1}^*\}_{\tau=t}^{\infty}} \mathbb{E}_t \sum_{\tau=t}^{T-1} [-(1 - \delta_E)Rq_{2t} - (1 - \delta_L)R(q_{2t+1} + s_{2t+1})],$$

where the expectation is taken over the random variable T ,⁴⁴ which is govern by π . By the Principle of Optimality, the solution to (A.24) maximizes the fund manager's objective function, while the solution to

⁴⁴To be precise, the random variable T here denotes the stage (rather than the date) before which the game ends.

(A.23) is feasible for the sequential problem associated with (A.24).

□

Online Appendix for:

A Dynamic Theory of Mutual Fund Runs and Liquidity Management

Not for Publication

B.1 A Micro-foundation for the Pattern of Asset Sale Prices

In this appendix, I show that the pattern of sale prices in the baseline model can emerge endogenously by modeling slow-moving liquidity providers in the spirit of [Grossman and Miller \(1988\)](#) and [Duffie \(2010\)](#). It shows that the reduced-form assumption can be rationalized as the outcome of a full-fledged equilibrium model with both liquidity demanders and providers. To make the idea more transparent, I set the micro-foundation in continuous time.⁴⁵ I follow the building blocks in [Duffie, Garleanu and Pedersen \(2005, 2007\)](#), [Weill \(2007\)](#) and [Lagos, Rocheteau and Weill \(2011\)](#) to model the gradual entry of liquidity providers and focus on the equilibrium price implications. I also stress that this continuous-time model is designed for proving a theoretical rationale for rather than precisely backing up the sale price pattern as specified in [Assumption 1](#).

Time is continuous and infinite. A probability space (Ω, \mathcal{F}, P) is fixed with an information filtration $\{\mathcal{F}_t, t \geq 0\}$ satisfying the usual measurability conditions. There is a common discount rate $r > 0$. There is a continuum of 1 of risk-neutral, infinitely lived, and competitive investors. There is a centralized market with many different assets. The total supply of all assets is $S \in [0, 1)$. Investors can hold at most one unit of assets and cannot short sell the assets. There is also a riskfree saving account with return r , which can be interpreted as cash equivalents. Under usual non-arbitrage conditions, this implies that the fundamental value of the assets is $1/r$.

There are two types of investors: liquidity providers and liquidity demanders. Liquidity providers enjoy a high utility flow per time by holding one unit of assets, which is normalized to 1, while liquidity

⁴⁵This baseline model is set in discrete time to highlight the discrete nature of daily redemptions and the end-of-day NAV. But in the micro-foundation, the discrete nature is no longer important. As a result, setting a continuous-time model incurs no loss of generality but makes the derivation mathematically more convenient.

demanders enjoy a low utility flow $\delta \in (0, 1)$.

At the beginning, the economy is hit by an unanticipated liquidity shock that makes all investors liquidity demanders. However, as time goes by, they will randomly and pairwise independently switch to liquidity providers.⁴⁶ Specifically, the times at which investors switch to liquidity providers are i.i.d. exponentially distributed with a parameter α . Denote the endogenous population of liquidity providers by $\rho(t)$. By the exact law of larger numbers, there is

$$\rho(t) = 1 - \exp(-\alpha t). \quad (\text{B.1})$$

Intuitively, this implies that there is no liquidity provider available right at the shock time (i.e., $t = 0$), while there will be more and more liquidity providers stepping into the market after the shock.

In this simple framework, the following proposition shows the pattern of asset sale price over time:

PROPOSITION 14. *The asset sale price at time t is characterized by*

$$p(t) = \frac{\delta + (1 - \delta) \exp(-r(t_S - t))}{r},$$

where t_S satisfies $\rho(t_S) = S$ and $\rho(\cdot)$ is given by (B.1).

Intuitively, the sale price drops discontinuously at $t = 0$ from the fundamental value, but rebounds gradually over time (as more liquidity providers become available) until it gets back to the fundamental value at time t_S . When the next shock comes, this process repeats itself, giving rise to the price pattern in the baseline model.

It is instructive to provide the proof here to help build intuition. First of all, I show that there is a time at which the selling positions can be completely absorbed by liquidity providers so that the price goes back to the fundamental. Specifically, condition (B.1) shows that more liquidity providers step into the market as time goes by after the shock. Denote the endogenous time by which liquidity providers can absorb all the asset supply by t_S , which implies that $\rho(t_S) = S$. Since $\rho(t_S)$ is monotone, this uniquely determines t_S . This corresponds to the baseline model that if the game ends (i.e., there are no future shocks), the asset sale price will ultimately reflect the fundamental value.

Then I show that, between the shock time 0 and the full recovery time t_S (before the next possible shock), the asset sale price first drops and then rebounds gradually, as that in the baseline model. Note that, at any time t between 0 and t_S , there are not enough liquidity providers in the market, so that

⁴⁶This dynamic process is in the spirit of [Grossman and Miller \(1988\)](#), in which liquidity providers only enter the market one period after the initial liquidity shock.

the marginal investor is a liquidity demander who has a low valuation of the assets. Since this liquidity demander is infinitely lived, the Hamilton-Jacobi-Bellman equation leads to:

$$rp(t)dt = \delta dt + p(t) \tag{B.2}$$

This condition has an intuitive interpretation. At any time t between 0 and t_S , the left hand side of (B.2) denotes the return of selling the unit of assets at t and investing the proceeds in cash equivalents in the time interval $[t, t + dt)$, while the right hand side denotes the valuation flow by holding one unit of assets in the time interval $[t, t + dt)$ plus the proceeds from selling it after that. In any equilibrium path, the liquidity demander should be indifferent between these two options of selling earlier or later. Therefore, solving the differential equation implied by (B.2) with the boundary conditions yields the equilibrium sale price.

Fundamentally, this micro-foundation follows the spirit of [Grossman and Miller \(1988\)](#) and [Duffie \(2010\)](#), but differs in an important way. Specifically, liquidity providers in their models share risks with liquidity demanders, while in both my baseline model and the micro-foundation, all the investors are risk neutral.⁴⁷ However, similar sale price pattern emerges. This is because liquidity providers in my model have higher valuation of the underlying assets, which resemble the notion of natural buyers in [Shleifer and Vishny \(1992, 1997\)](#) and thus is closer to the interpretation in the baseline model. Like that in [Grossman and Miller \(1988\)](#) and [Duffie \(2010\)](#), liquidity providers step into the market only gradually after the shock, implying that only a few liquidity providers are present in the market right after the shock. Hence, investors who want to sell the assets right after the shock must accept an extremely low sale price. As time goes by (but before the next possible shock comes), more liquidity providers with high valuation of the underlying assets step into the market, implying that it becomes increasingly easier for the liquidity demanders to find a better sale price.

B.2 Asset Sale Price Correlations

In the baseline model, flow-induced sales will not affect the market prices of the non-traded assets. This is realistic given that mutual funds invest in many different illiquid assets, and flow-induced sales only have local and temporary price impacts ([Coval and Stafford, 2007](#)). But asset prices can be potentially correlated with each other, and the fund manager may use alternative accounting rules such as matrix pricing to price these non-traded assets. This appendix proposes an approach to capture this alternative

⁴⁷This assumption of risk neutrality also appears in other search-based models (see [Duffie, Garleanu and Pedersen, 2005](#), [Weill, 2007](#), [Lagos, Rocheteau and Weill, 2011](#), among many others).

setting and suggests that it will not change the main insights of the baseline model.

Specifically, I assume that asset prices are perfectly correlated. This can be effectively viewed as a setting with only one single illiquid asset. In this alternative setting, if the fund sells any assets on date t at the sale price p_t , the end-of-day flexible NAV will be:

$$NAV_t = \frac{x_t + (a_t - a_{t+1})p_t + a_{t+1}p_t}{n_t}. \quad (\text{B.3})$$

The difference between (B.3) and the baseline model's NAV (2.1) is that the market price of the non-traded assets will also be updated to p_t , the temporary sale price. In both (2.1) and (B.3), the NAV is flexible in the sense that it takes into account all the same-day price impact and asset sale losses, while it is not perfectly forward-looking in the sense that it will not reflect future asset sale costs. As a result, this alternative setting would not change my results qualitatively because future fund cash rebuilding would still give rise to a predictable decline in NAV and thus the run incentives.

To model this alternative setting rigorously would require additional parametric assumptions regarding how the sale price p_t depends on the amount of sales $a_t - a_{t+1}$ on date t ; otherwise the NAV would be irrelevant to the amount of asset sales. It would also require additional assumptions regarding how the prices of non-traded assets rebound if there are no subsequent asset sales. One natural and consistent approach is to introduce a downward-sloping demand curve within each trading day, with the slope being larger on even dates than that on odd dates to capture the idea of slow-moving capital provisions. In addition, I also make the realistic assumption that the price impacts induced by asset sales are temporary; the asset price always rebounds to the fundamental value at the beginning of the next trading day. Figure 7 shows a sample path of the selling prices under these new assumptions. However, these additional assumptions will make the model less tractable; it will no longer admit closed-form solutions as those in Sections 3 and 4. Imposing these additional assumptions also makes the model mechanism less general. For these reasons, I choose to follow the simpler NAV rule (2.1) in the baseline model as a benchmark to make the mechanism more transparent.

B.3 Flow-to-Performance Relationship and Endogenous Outflows

The baseline model assumes random redemption shocks, but the realized population of redeeming shareholders in each stage is taken as exogenous. In reality, future fund flows are likely to be positively correlated with past returns, known as the flow-to-performance relationship. Earlier research finds that future flows mostly respond to past good performance (Ippolito, 1992, Sirri and Tufano, 1998), but recent evidence

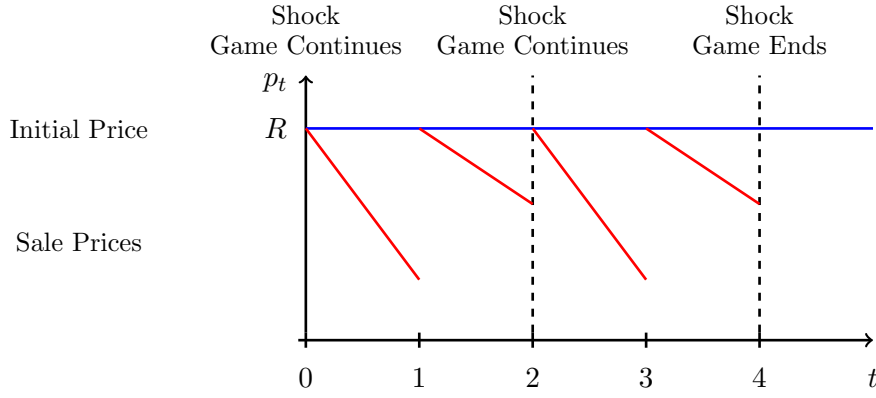


Figure 7: Sample Path of the Selling Prices

suggests that they also respond to bad performance in particular when the underlying assets are illiquid (for example, [Spiegel and Zhang, 2013](#), [Goldstein, Jiang and Ng, 2015](#)). This appendix considers the flow-to-performance relationship and its interaction with fund shareholder runs.

To incorporate the flow-to-performance relationship, I define the fund return in stage t as

$$r_{2t+1} = \frac{NAV_{2t+1}}{NAV_{2t}},$$

which is positive but no greater than one in my baseline model.⁴⁸ I then assume that in any stage t the populations of early and late shareholders are $\gamma_{2t}\lambda_E n_{2t}$ and $\gamma_{2t}\lambda_L n_{2t}$ for the even date $2t$ and the odd date $2t + 1$, respectively, where $\gamma_{2t}(r_{2t-1}) \geq 1$ for $t > 1$ is a decreasing function of r_{2t-1} satisfying $\gamma_{2t}(1) = 1$, and $\gamma_0 = 1$. This implies that, if current fund return is lower, there will be more shareholders redeeming in the next stage if the game continues, capturing the flow-to-performance relationship.

This extended model is no longer stationary and does not admit closed-form solutions, but it suggests that the flow-to-performance relationship will complicate the tension in choosing between a rapid or slow cash rebuilding policy by the fund manager. This can be seen from [Proposition 5](#). Suppose the fund starts from the joint region $G_l \cup G_m \cup G_{hl} \cup G_{hm}$ where it is optimal to sell some illiquid assets to rebuild the cash buffer (i.e., $\sigma_{2t+1}^* > 0$). When the flow-to-performance relationship is introduced, σ_{2t+1}^* suggested by [Proposition 5](#) is no longer optimal. To see this, notice that $\sigma_{2t+1}^* > 0$ implies $r_{2t+1} < 1$ and then $\gamma_{2t+2} > 1$. As a result, the fund either has to increase σ_{2t+1} to prevent more severe future fire sales due to a larger population of redeeming shareholders in the next stage, or to decrease σ_{2t+1} to sustain a higher current fund return but suffer higher risk of future forced sales. Either way, the fund incurs higher risk of shareholder runs and higher total expected sale losses as well.

⁴⁸It can be larger than one in the model with redemption fees or redemption restrictions.

This extended model suggests a new amplification mechanism to explain fund performance persistence in bad times. The flow-to-performance relationship first implies that it is harder for the fund to manage its cash buffer. Due to the interdependence of shareholder runs and fund liquidity management, this further suggests more severe runs and fire sales, leading to worse performance. Only those funds with a sufficiently high cash-to-assets ratio are likely to withstand these hard times without incurring shareholder runs and fire sales.

B.4 Additional Proofs

This appendix provides additional and more technical proofs for several results in the main text.

PROOF OF PROPOSITION 3. The existence of a Markov equilibrium of the dynamic game follows a special case of Theorem 2 and Corollary 6 in [Khan and Sun \(2002\)](#). The key is to find a measurable selection of Nash equilibria in each stage game determined by the state variables (a_{2t}, x_{2t}) . The Arsenin-Kunugui Theorem (see [Kechris, 1995](#) for a textbook treatment) guarantees that any usual equilibrium selection mechanism such as selecting the best, the worst or the one based on the global game approach is measurable.

Under any Markov strategy profile, by definition, the strategies of both the fund manager and all the shareholders are functions of the two state variables (a_{2t}, x_{2t}) , and their strategies are mutually best responses as well. In other words, strategies played in the past stages influence current-stage strategies only through the two state variables. For convenience, in what follows I call a stage game (a_{2t}, x_{2t}) when the fund starts from the portfolio position (a_{2t}, x_{2t}) on date $2t$.

Consider any arbitrary $\phi \in (0, 1)$. Define $a'_{2t} = \phi a_{2t}$, $x'_{2t} = \phi x_{2t}$, and $s'_{2t+1} = \phi a_{2t+1}$. By Lemma 1, game (a_{2t}, x_{2t}) and game (a'_{2t}, x'_{2t}) start from the same cash-to-assets ratio region. By Propositions 1, 2, and 13, if λ_{2t} constructs a run equilibrium in game (a_{2t}, x_{2t}) under the cash rebuilding policy s_{2t+1} , it must also construct a run equilibrium in game (a'_{2t}, x'_{2t}) under the cash rebuilding policy s'_{2t+1} . Hence, by Lemmas 2, 4, and 9, the equilibrium amounts of forced sales in game (a'_{2t}, x'_{2t}) must be $q'_{2t} = \phi q_{2t}$, $q'_{2t+1} = \phi q_{2t+1}$, where q_{2t} and q_{2t+1} are the equilibrium amounts of forced sales in game (a_{2t}, x_{2t}) .

Then consider the dynamics. Fix a consistent equilibrium selection mechanism if multiple equilibria occur. Let (a_{2t+2}, x_{2t+2}) be the next stage game when game (a_{2t}, x_{2t}) is played under the cash rebuilding policy s_{2t+1} . By Corollaries 1, 2 and 4, the next stage game must be (a'_{2t+2}, x'_{2t+2}) , where $a'_{2t+2} = \phi a_{2t+2}$ and $x'_{2t+2} = \phi x_{2t+2}$, if the current stage game (a'_{2t}, x'_{2t}) is played under the cash rebuilding policy s'_{2t+1} . Therefore, if $s_{2t+1}(a_{2t}, x_{2t})$ is the optimal cash rebuilding policy in stage t for game (a_{2t}, x_{2t}) , $s'_{2t+1}(a'_{2t}, x'_{2t}) = \phi s_{2t+1}(a_{2t}, x_{2t})$ must be the optimal cash rebuilding policy in stage t for game (a'_{2t}, x'_{2t}) .

Hence, $V(a'_{2t}, x'_{2t}) = \phi V(a_{2t}, x_{2t})$ is indeed the value function for the dynamic game with a starting position (a'_{2t}, x'_{2t}) .

Finally, it is straightforward to see that $V(0, 0) = 0$. \square

PROOF OF PROPOSITION 6. This proof proceeds in three steps. First, I show that when θ is sufficiently small, the equilibrium is the same as that characterized by Proposition 4. Second, I characterize the equilibrium when θ takes an intermediate value. Lastly, I show that when θ is sufficiently large, the equilibrium is the same as that characterized by Proposition 5.

STEP 1. Recall that when $\theta = 0$, the equilibrium cash rebuilding policy is characterized by Proposition 4. By Propositions 1, 2, and 13, \underline{s}_h , \underline{s}_l , and \underline{s}_m are all continuous in θ . Hence, there exists a $\underline{\theta} > 0$ (explicit expression will be calculated in the next step) such that when $\theta \in (0, \underline{\theta}]$, none of the late shareholders chooses to run in any region if the fund still follows the cash rebuilding policy as described in Proposition 4. In addition, the proof of Proposition 4 only relies on the fact that there are no shareholder runs. This confirms that the cash rebuilding policy as described in Proposition 4 is still optimal when $\theta \in (0, \underline{\theta}]$, which in turn confirms the late shareholders' run decision $\lambda_{2t} = 0$.

STEP 2. By the definition of $\underline{\theta}$, when $\theta > \underline{\theta}$ there exists a non-zero-measure set G_{run} in which at least some of the late shareholders will run given the cash rebuilding policy described in Proposition 4. I first show that G_{run} takes the form of

$$G_{run} = G_l \cup \underline{G}_m, \quad (\text{B.4})$$

where $\underline{G}_m \subseteq G_m$ is connected and

$$\inf \underline{G}_m = \frac{\mu_E R}{1 - \mu_E}.$$

To see this, first recall the definition of \underline{s}_l :

$$\underline{s}_l = \frac{Ra_{2t} - \theta(1 - \mu_E)(Ra_{2t} + x_{2t}) - (1 - \theta(1 - \delta_E)(1 - \mu_E))Rq_{2t}(0)}{(1 - \delta_L)R} - q_{2t+1}(0).$$

Note that, for every pair of (a_{2t}, x_{2t}) and η_{2t+2} , there is an implied s_{2t+1} . Using that as the threshold \underline{s}_l and solving for θ backward yields that, under the cash rebuilding policy $\eta_{2t+2} = \mu_E R / (1 - \mu_E)$, when

$$\theta > \frac{\delta_L}{\mu_E + \mu_L + \delta_L(1 - \mu_E - \mu_L)} \equiv \underline{\theta} \in (0, 1)$$

there must be $\lambda_{2t} > 0$ for $\eta_{2t} \in G_l$.

Similarly, consider the definitions of \underline{s}_m and \underline{s}_h . Also under the cash rebuilding policy $\eta_{2t+2} = \mu_E R / (1 -$

μ_E), solving for the threshold θ backward yields that, when

$$\theta > \frac{\delta_L}{\mu_E + \mu_L + \delta_L(1 - \mu_E - \mu_L)} = \underline{\theta}$$

there must be $\lambda_{2t} > \widehat{\lambda}_{2t}$ for $\eta_{2t} \in G_m$, while when

$$\theta > \frac{\delta_L + \mu_L - \delta_L \mu_L}{\mu_E + \mu_L + \delta_L(1 - \mu_E - \mu_L)} \equiv \bar{\theta} \in (0, 1)$$

there must be $\lambda_{2t} > 0$ for $\eta_{2t} \in G_h$.

Notice that

$$\underline{\theta} < \bar{\theta}.$$

Thus, under the cash rebuilding policy $\eta_{2t+2} = \mu_E R / (1 - \mu_E)$, when $\theta \in (\underline{\theta}, \bar{\theta})$, there is $\lambda_{2t} = 0$ when $\eta_{2t} \in G_h$. This confirms the claim in (B.4).

Now define

$$\eta(\check{\lambda}) \equiv \frac{(\mu_E + \check{\lambda}\mu_L)R}{1 - \mu_E - \check{\lambda}\mu_L} \in G_m.$$

For any $\check{\lambda} \in (0, 1)$, consider the following cash rebuilding policy:

$$\eta_{2t+2} = \eta(\check{\lambda}).$$

Since

$$\frac{\mu_E R}{1 - \mu_E} < \eta(\check{\lambda}) < \frac{(\mu_E + \mu_L)R}{1 - \mu_E - \mu_L},$$

there exists a $\check{\theta} \in (\underline{\theta}, \bar{\theta})$ such that when $\theta = \check{\theta}$, there is

$$\begin{cases} \lambda_{2t} > 0 & \text{iff } \eta_{2t} < \eta(\check{\lambda}), \\ \lambda_{2t} = 0 & \text{iff } \eta_{2t} \geq \eta(\check{\lambda}). \end{cases}$$

Thus, it is natural to define that

$$\underline{G}_m \equiv \left\{ \eta_{2t} \mid \frac{\mu_E R}{1 - \mu_E} \leq \eta_{2t} < \eta(\check{\lambda}) \right\},$$

and

$$\underline{G}_{hm} \equiv \left\{ \eta_{2t} \mid \eta_{2t} \geq \frac{(\mu_E + \mu_L)R}{1 - \mu_E - \mu_L} \text{ and } \frac{\mu_E R}{1 - \mu_E} \leq \eta_{2t+2} < \eta(\check{\lambda}) \text{ for } \sigma_{2t+1} = 0 \right\}.$$

Now I confirm that $\eta_{2t+2}^* = \eta(\check{\lambda})$ is the optimal cash rebuilding policy when $\theta = \check{\theta}$ and $\eta_{2t} \in \underline{G}_m$. First, consider a perturbation $-\varepsilon < 0$ of cash rebuilding around σ_{2t+1}^* that satisfies $\eta_{2t+2}^* = \eta(\check{\lambda})$. On date $2t + 1$ (in stage t), since $\lambda_{2t} = 1$ (by Lemma 8), the effective sale price on $2t + 1$ is $\widehat{p}_L(\check{\lambda})$. Thus, the sale loss saved in stage t is

$$\frac{\varepsilon(1 - \delta_L)R}{\widehat{p}_L(\check{\lambda})} > 0.$$

Now consider the same cash gap ε on date $2t+2$ (under the perturbed cash rebuilding policy $(\sigma_{2t+1}^*, -\varepsilon)$). This implies that $\eta_{2t+2} \in \underline{G}_{hm}$. Since $\lambda_{2t+2} \geq \check{\lambda}$, the fund always must sell its assets on date $2t + 2$ at most at the effective sale price $\widehat{p}_E(\check{\lambda}) > 0$. Hence, the expected increase of sale loss in stage $t + 1$ due to this cash gap ε is at least

$$\frac{\varepsilon(1 - \delta_E)R}{\widehat{p}_E(\check{\lambda})} > 0.$$

Since $\widehat{p}_E(\check{\lambda}) < \widehat{p}_L(\check{\lambda})$, there is

$$\frac{\varepsilon(1 - \delta_L)R}{\widehat{p}_L(\check{\lambda})} < (1 - \pi) \frac{\varepsilon(1 - \delta_E)R}{\widehat{p}_E(\check{\lambda})}$$

for a sufficiently small but positive π . Hence, the perturbation $-\varepsilon$ is not profitable.

Next, consider another perturbation $\varepsilon > 0$ of cash rebuilding around σ_{2t+1}^* that satisfies $\eta_{2t+2}^* = \eta(\check{\lambda})$. On date $2t + 1$ (in stage t), similarly, the sale loss increased in stage t is

$$\frac{\varepsilon(1 - \delta_L)R}{\widehat{p}_L(\check{\lambda})} > 0.$$

Under the perturbed cash rebuilding policy $(\sigma_{2t+1}^*, \varepsilon)$ the fund gets ε more cash in stage $t + 1$. This implies that $\eta_{2t+2} \in G_h$. Since there will be no runs on date $2t + 2$, the marginal cash saves the fund's active asset sales on date $2t + 3$ at the effective sale price $\widehat{p}_L(1) > 0$. Hence, the expected sale saved in stage $t + 1$ due to this marginal cash ε is

$$\frac{\varepsilon(1 - \delta_L)R}{\widehat{p}_L(1)} > 0.$$

Since $\widehat{p}_L(\check{\lambda}) < \widehat{p}_L(1)$ (and also $\pi \in (0, 1)$), this perturbation ε is also not profitable. This verifies the optimality of $\eta_{2t+2}^* = (\mu_E + \mu_L)R / (1 - \mu_E - \mu_L)$ when $\eta_{2t} \in \underline{G}_m$. This analysis can be readily extended to other subset of $G_l \cup G_m \cup G_{hl} \cup \underline{G}_{hm}$ as well as $\overline{G}_{hm} \cup G_{hh}$ following the same argument.

Finally, define

$$\bar{\theta} \equiv \check{\theta}(\check{\lambda} = 1) \in (\underline{\theta}, \bar{\theta}).$$

By construction, when $\theta = \bar{\theta}$, there are $\lambda_{2t} > 0$ for $\eta_{2t} \in G_l \cup G_m$ while $\lambda_{2t} = 0$ for $\eta_{2t} \in G_h$ under the corresponding optimal cash rebuilding policy $\eta_{2t+2}^* = \eta(1)$.

STEP 3. This step shows that when $\theta > \bar{\theta}$ there can not be equilibria other than that described by Proposition 5. In this step, I use Figure 8 to help illustrate the idea. I first show that, when $\theta > \bar{\theta}$, there must be $G_{run} = G_l \cup G_m \cup G_h$. Note that, by Step 2, there must be $G_l \cup G_m \subseteq G_{run}$ when $\theta > \bar{\theta}$, and thus it suffices to show that it cannot be that

$$\sup G_{run} < \sup G_h .$$

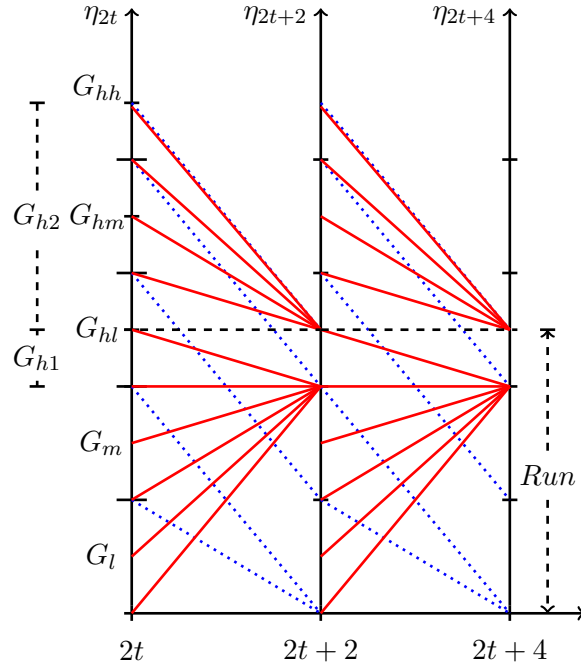


Figure 8: Hypothetical Equilibrium Cash Rebuilding Policy When $\theta \in [\bar{\theta}, 1)$

I prove this by contradiction. First suppose $\sup G_{run} \in G_{hl} \cup G_{hm}$. Define

$$G_{h1} \equiv G_{run} / (G_l \cup G_m) .$$

By the argument in the proof of Proposition 5, when $\eta_{2t} \in G_{run}$ the equilibrium cash rebuilding policy still features

$$\eta_{2t+2}^* = \frac{(\mu_E + \mu_L)R}{1 - \mu_E - \mu_L} . \quad (\text{B.5})$$

However, because $\sup G_{run} \in G_{hl} \cup G_{hm}$, one can now find another non-zero-measure connected set $G_{h2} \subseteq G_{hl} \cup G_{hm}$ that satisfies

$$\inf G_{h2} = \sup G_{h1} ,$$

in which shareholders will not run under the cash rebuilding policy (B.5).

By construction, the optimal cash rebuilding policy when $\eta_{2t} \in G_{h2}$ should be

$$\eta_{2t+2}^* \geq \sup G_{h1} > \frac{(\mu_E + \mu_L)R}{1 - \mu_E - \mu_L}. \quad (\text{B.6})$$

To see this, consider a perturbation $-\varepsilon$ of cash rebuilding around this cash rebuilding policy. On date $2t + 1$ (in stage t), since there are no runs when $\eta_{2t} \in G_{h2}$, the effective sale price on $2t + 1$ is $\widehat{p}_L(0)$. Thus, the sale loss saved in stage t is

$$\frac{\varepsilon(1 - \delta_L)R}{\widehat{p}_L(0)} > 0.$$

Now consider the same cash gap ε on date $2t + 2$ (under the perturbed cash rebuilding policy $(\sigma_{2t+1}^*, -\varepsilon)$). This implies that $\eta_{2t+2} \in G_{h1}$. Because of shareholder runs on date $2t + 2$, the fund will sell its assets at the effective sale price $\widehat{p}_L(1) > 0$. Hence, the expected increase of sale loss in stage $t + 1$ due to this cash gap ε is

$$\frac{\varepsilon(1 - \delta_L)R}{\widehat{p}_L(1)} > 0.$$

Since $\widehat{p}_L(1) < \widehat{p}_L(0)$, there is

$$\frac{\varepsilon(1 - \delta_L)R}{\widehat{p}_L(0)} < (1 - \pi) \frac{\varepsilon(1 - \delta_L)R}{\widehat{p}_L(1)}$$

for a sufficiently small but positive π , implying that the perturbation $-\varepsilon$ is not profitable.

However, under the new, more rapid cash rebuilding policy (B.6), by the definition of \underline{s}_h , there must be a subset of G_{h2} in which late shareholders are going to run. This violates the definition of G_{h2} : a contradiction.

Now instead suppose $\inf G_{hh} \leq \sup G_{run} < \sup G_h$. Again by the monotonicity and continuity of \underline{s}_h in a_{2t} and x_{2t} , there is no loss of generality to assume that $\sup G_{run} = \inf G_{hh} + \varepsilon$, where $\varepsilon > 0$ is arbitrarily small. Similarly, the optimal cash rebuilding policy when $\eta_{2t} = \inf G_{hh} + \varepsilon$ should be

$$\eta_{2t+2}^* = \inf G_{hh} > \frac{(\mu_E + \mu_L)R}{1 - \mu_E - \mu_L}. \quad (\text{B.7})$$

However, when $\eta_{2t} = \sup G_{hm} = \inf G_{hh}$, the optimal cash rebuilding policy already leads to runs. By the definition of \underline{s}_h , there must be shareholder runs when $\eta_{2t} = \inf G_{hh} + \varepsilon$ under the cash rebuilding policy (B.7). This is again a contraction. As a result, there must be $G_{run} = G_l \cup G_m \cup G_h$.

Finally, by Proposition 5, the optimal cash rebuilding policy must be the same as described there

because the pattern of shareholder runs is the same as described by Lemma 8. \square

LEMMA 11. *Under the fund's objective function (2.3), any cash-rebuilding policies that involve $s_{2t} > 0$ in any stage t is not optimal for the fund.*

PROOF OF LEMMA 11. I prove it by contradiction. It suffices to consider a one-stage deviation in which the fund chooses $s'_{2t} > 0$ in a given stage t but conforms to the optimal cash rebuilding policies as characterized by Propositions 4, 5, and 6 thereafter.

I start by showing that $s'_{2t} > 0$ in any stage t is not optimal when $\theta = 0$. Suppose $s'_{2t} > 0$. By Lemma 7, shareholders will never run regardless of the fund's cash rebuilding policies, that is, $\lambda_{2t} = 0$, implying that the effective sale price on date $2t$ is always $\widehat{p}_E(0)$. As a result, by the definition of active cash rebuilding, the fund carries

$$\varepsilon = \widehat{p}_E(0)s'_{2t}$$

more cash into date $2t + 1$. Notice that this ε amount of cash will be either 1) used to meet redemption needs on date $2t + 1$ or 2) carried into the next stage $t + 1$ (i.e., date $2t + 2$).

If this ε amount of cash will be used to meet redemption needs on date $2t + 1$, there is at least a positive cash gap on date $2t + 1$, meaning the fund will be forced to sell at least some illiquid assets on date $2t + 1$. However, since the effective selling price is $\widehat{p}_L(0) > \widehat{p}_E(0)$ on date $2t + 1$, the fund can always sell less than s'_{2t} and at a higher physical sale price on date $2t + 1$ to close that cash cap. This violates the optimality of $s'_{2t} > 0$.

Otherwise, the fund has ε cash buffer on date $2t + 2$. However, since the effective selling price on date $2t + 2$ is also $\widehat{p}_E(0)$, and the game only has a $\pi < 1$ probability to continue to stage $t + 1$, the fund can always wait until stage $t + 1$ to raise that ε amount of cash. This again violates the optimality of $s'_{2t} > 0$.

The analysis above suggests that a positive cash rebuilding $s'_{2t} > 0$ on date $2t$ is not optimal when there are always not shareholder runs. The following considers the scenarios in which shareholder runs are possible. There is no loss of generality to consider $\theta = 1$, in which late shareholders have the greatest propensity to run, and the analysis for a general θ follows the same argument. I consider three different and exclusive cases in this scenario.

CASE 1. The initial stage equilibrium $\{s_{2t}^* = 0, s_{2t+1}^* \geq 0, \lambda_{2t}^* = 0\}$ features no runs. By Proposition 5, this implies that $\eta_{2t} \in G_{hh}$ and $s_{2t+1}^* = 0$. Now suppose $s'_{2t} > 0$. This implies that $NAV'_{2t} \leq NAV_{2t}^*$. Hence, by Proposition 1 for the high cash-to-asset ratio region, there are still no runs in this stage. Since $\lambda'_{2t} = \lambda_{2t}^*$, the effective sale price on date $2t$ is always $\widehat{p}_E(0)$. As a result, by the definition of active cash

rebuilding, the fund carries

$$\varepsilon = \widehat{p}_E(0)s'_{2t}$$

more cash into date $2t + 1$.

By the definition of the high-high region G_{hh} , there is $q'_{2t+1} = q^*_{2t+1} = 0$. Thus, the ε amount of additional cash will then be directly carried into the next stage (regardless of the cash-rebuilding policy on date $2t + 1$, s'_{2t+1}), in particular, to date $2t + 3$. However, since the effective selling price on date $2t + 3$ is also $\widehat{p}_L(0) > \widehat{p}_E(0)$, and the game only has a $\pi < 1$ probability to continue to stage $t + 1$, the fund can always wait until stage $t + 1$ to raise that ε amount of cash. This violates the optimality of $s'_{2t} > 0$.

CASE 2. The stage equilibrium $\{s^*_{2t} = 0, s^*_{2t+1} \geq 0, \lambda^*_{2t} > 0\}$ features runs but no run-induced fire sales on date $2t$, that is, $q^*_{2t} = 0$. By Proposition 5, this implies that $\eta_{2t} \in G_{hm} \cup G_{hl}$, $s^*_{2t+1} > 0$, and $\lambda^*_{2t} = 1$. Now suppose $s'_{2t} > 0$. This implies that $NAV'_{2t} \leq NAV^*_{2t}$ and $\lambda'_{2t} \leq 1$ by Proposition 1 (regardless of the cash-rebuilding policy on date $2t + 1$, s'_{2t+1}). In other words, because of the punishment on date $2t$ by active asset sales, the number of late shareholders who choose to run becomes weakly lower. However, $q^*_{2t} = 0$ implies that there are still no forced asset sales on date $2t$ despite fewer running shareholders, that is, $q'_{2t} = 0$. In other words, no forced asset sales are ever saved in stage t . On the other hand, by the definition of active cash rebuilding, the fund carries

$$\varepsilon = \widehat{p}_E(\lambda'_{2t})s'_{2t}$$

more cash into date $2t + 1$.

By the definition of the high-intermediate region G_{hm} and high-low region G_{hl} , there is still $q'_{2t+1} = q^*_{2t+1} = 0$. Thus, this ε amount of cash will then be directly carried into the next stage $t + 1$. However, in order to rebuild this additional amount of cash on date $2t + 1$, the fund can instead actively sell on date $2t + 1$ and enjoy a higher effective sale price $\widehat{p}_L(\lambda'_{2t}) > \widehat{p}_E(\lambda'_{2t})$ for any λ'_{2t} . This violates the optimality of $s'_{2t} > 0$.

CASE 3. The stage equilibrium $\{s^*_{2t} = 0, s^*_{2t+1} \geq 0, \lambda^*_{2t} > 0\}$ features both runs and run-induced fire sales on date $2t$, that is, $q^*_{2t} > 0$. By Proposition 5, this implies that $\eta_{2t} \in G_m \cup G_l$, $s^*_{2t+1} > 0$, and $\lambda^*_{2t} = 1$. Also notice that in this case there is $q^*_{2t+1} = 0$ because there are effectively no late shareholder left on date $2t + 1$ in the initial stage equilibrium.

There are three exclusive sub-cases here and I consider them separately.

CASE 3.1. Now suppose $s'_{2t} > 0$ and $\lambda'_{2t} = \lambda_{2t} = 1$, which implies $NAV'_{2t} > NAV'_{2t+1}$. In this sub-case, NAV'_{2t} is still high enough relative to NAV'_{2t+1} so that all the late shareholders still run. In other

words, the punishment by active sales on date $2t$ does not help mitigate runs. As a result, we still have $q'_{2t} = q^*_{2t} > 0$ and $q'_{2t+1} = q^*_{2t+1} = 0$. In other words, no forced asset sales are ever saved in stage t . On the other hand, by the definition of active cash rebuilding, the fund carries

$$\varepsilon = \widehat{p}_E(1)s'_{2t}$$

more cash into date $2t + 1$.

Since $q'_{2t+1} = q^*_{2t+1} = 0$, this ε amount of cash will then be directly carried into the next stage $t + 1$. However, in order to rebuild this additional amount of cash on date $2t + 1$, the fund can instead actively sell on date $2t + 1$ and enjoy a higher effective sale price $\widehat{p}_L(1) > \widehat{p}_E(1)$. This violates the optimality of $s'_{2t} > 0$.

CASE 3.2. Now suppose $s'_{2t} > 0$ and $\lambda'_{2t} = 0$, which implies $NAV'_{2t} < NAV'_{2t+1}$. In this sub-case, NAV'_{2t} becomes low enough so that all the late shareholders choose to stay until date $2t + 1$. In addition, there are $q'_{2t} < q^*_{2t}$ and $q'_{2t+1} \geq q^*_{2t+1} = 0$ in this case. However, for any s'_{2t+1} , one can always find another $s''_{2t} \in (0, s'_{2t})$ that satisfies $\lambda''_{2t} = 0$, and $NAV''_{2t} < NAV''_{2t+1}$. Notice that in this case there are $q''_{2t} = q'_{2t}$ and $q''_{2t+1} = q'_{2t+1}$ because $\lambda''_{2t} = \lambda'_{2t}$. This violates the optimality of $s'_{2t} > 0$.

CASE 3.3. Now suppose $s'_{2t} > 0$ and $\lambda'_{2t} \in (0, 1)$, which implies

$$NAV'_{2t}(\lambda'_{2t}) = NAV'_{2t+1}(\lambda'_{2t}). \quad (\text{B.8})$$

Clearly, there are still $q'_{2t} < q^*_{2t}$ and $q'_{2t+1} \geq q^*_{2t+1} = 0$ in this sub-case. Recall that $\eta_{2t} \in G_m \cup G_l$. Because the fund always chooses s'_{2t+1} on date $2t + 1$, Proposition 5 suggests that any optimal s'_{2t+1} on date $2t + 1$ always satisfies that

$$\eta'_{2t+2} = \eta^*_{2t+2} = \frac{(\mu_E + \mu_L)R}{1 - \mu_E - \mu_L}. \quad (\text{B.9})$$

Therefore, under the new cash rebuilding policies (s'_{2t}, s'_{2t+1}) , Lemmas 4 and 9 suggest that

$$x_{2t} - (\mu_E + \mu_L)(Ra_{2t} + x_{2t}) + \widehat{p}_E(\lambda'_{2t})(s'_{2t} + q'_{2t}(\lambda'_{2t})) + \widehat{p}_L(\lambda'_{2t})(s'_{2t+1} + q'_{2t+1}(\lambda'_{2t})) = a'_{2t+2}\eta'_{2t+2}. \quad (\text{B.10})$$

On the other hand, under the initial stage equilibrium, Lemmas 4 and 9 also suggest that

$$x_{2t} - (\mu_E + \mu_L)(Ra_{2t} + x_{2t}) + \widehat{p}_E(1)q^*_{2t}(1) + \widehat{p}_L(1)s^*_{2t+1} = a^*_{2t+2}\eta^*_{2t+2}. \quad (\text{B.11})$$

Notice that

$$\widehat{p}_E(\lambda_{2t}) \leq \widehat{p}_E(1) < \widehat{p}_L(1) \leq \widehat{p}_L(\lambda_{2t}).$$

Plug condition (B.9) into conditions (B.10) and condition (B.11). Then comparing the left hand sides of the two equations under condition (B.8) yields

$$a'_{2t+2} < a^*_{2t+2}.$$

This suggests that although active asset sales on date $2t$ can potentially reduce run-induced forced sales on $2t$, the fund ends up incurring more total asset sale losses in stage t . Since the game only has $\pi < 1$ probability to continue, this is not optimal.

Overall, the key intuition of this proof is that, rebuilding cash buffers on date $2t$ would help only if it helps mitigate runs and the resulting run-induced fire sales on date $2t$. Otherwise, there is no point to actively sell at a lower price. But doing this to prevent runs also means too many active sales on date $2t$ (at the same low price as forced sales), and it is equally bad or even worse than the case in which the fund just let shareholders run themselves. Note that this proof requires Assumption 1 to hold, which guarantees that the price impact on date $2t$ is large enough relative to that on date $2t+1$, i.e., δ_E is sufficiently smaller than δ_L (in other words, the sale price on date $2t$ is low enough). \square

PROOF OF PROPOSITION 10 AND COROLLARY 3. First, according to the starting cash-to-assets ratio η_{2t} , I still divide the stage game into three different regions. Without loss of generality, I consider $n_{2t} = 1$ as in the baseline model. Suppose the fund does not rebuild its cash buffer and no late shareholder is going to run, that is, $s_{2t+1} = 0$ and $\lambda_{2t} = 0$. Then there are three regions of the cash-to-assets ratio η_{2t} in the stage- t game. In these three regions, the amounts of illiquid assets that the fund must sell passively on dates t and $t + 1$ are characterized by:

$$\left\{ \begin{array}{ll} \text{High Region } G_h^\kappa: & q_{2t}^\kappa = 0, q_{2t+1}^\kappa = 0, \\ \text{Intermediate Region } G_m^\kappa: & q_{2t}^\kappa = 0, q_{2t+1}^\kappa > 0, \\ \text{Low Region } G_l^\kappa: & q_{2t}^\kappa > 0, q_{2t+1}^\kappa > 0. \end{array} \right.$$

I use the superscript κ to indicate the existence of the redemption fees. Note that, if a starting position (a_{2t}, x_{2t}) falls into a region G_j , $j \in \{h, m, l\}$, it does not necessarily falls into the same region G_j^κ when

redemption fees are introduced. But by construction, there is

$$G_h^\kappa \cup G_m^\kappa \cup G_l^\kappa = G_h \cup G_m \cup G_l,$$

and

$$G_j^\kappa \cap G_k^\kappa = \emptyset, j \neq k.$$

Thus it suffices to consider the three regions G_h^κ , G_m^κ , and G_l^κ separately. Here I provide a complete analysis of the high region G_h^κ and the derivation for the other two regions directly follows.

In the high region G_h^κ , when $q_{2t}^\kappa = 0$ and $\lambda_{2t} = 0$, there is

$$NAV_{2t}^\kappa = Ra_{2t} + x_{2t},$$

and

$$NAV_{2t+1}^\kappa = \frac{1 - \kappa\mu_E}{1 - \mu_E}(Ra_{2t} + x_{2t}).$$

Thus, $q_{2t}^\kappa = 0$ and $q_{2t+1}^\kappa = 0$ imply

$$\eta_{2t} \geq \frac{\left(\kappa\mu_E + \kappa\mu_L \frac{1 - \kappa\mu_E}{1 - \mu_E} \right) R}{1 - \kappa\mu_E - \kappa\mu_L \frac{1 - \kappa\mu_E}{1 - \mu_E}}.$$

This suggests that $G_h \subseteq G_h^\kappa$. This also suggests that Lemma 2 still holds. That means, for any λ_{2t} :

$$NAV_{2t}^\kappa(\lambda_{2t}) = Ra_{2t} + x_{2t}.$$

Meanwhile, when shareholder runs and cash rebuilding are introduced, there is

$$NAV_{2t+1}^\kappa(\lambda_{2t}) = \frac{R(a_{2t} - s_{2t+1}) + x_{2t} - \kappa(\mu_E + \lambda_{2t}\mu_L)(Ra_{2t} + x_{2t}) + \delta_L Rs_{2t+1}}{1 - \mu_E - \lambda_{2t}\mu_L}.$$

Therefore, when $\theta = 1$, late shareholders' run incentives are governed by

$$\Delta NAV^\kappa(\lambda_{2t}) = \frac{\delta_L Rs_{2t+1}}{1 - \mu_E - \lambda_{2t}\mu_L} - \frac{(1 - \kappa)(\mu_E + \lambda_{2t}\mu_L)(Ra_{2t} + x_{2t})}{1 - \mu_E - \lambda_{2t}\mu_L}. \quad (\text{B.12})$$

When there are no redemption fees, that is, when $\kappa = 1$, this goes back to wedge (3.2) in the baseline model. For any $\kappa \in (0, 1)$ and any $\lambda_{2t} \in [0, 1]$, the second term in (B.12) is strictly positive. This directly

implies that for any feasible s_{2t+1} , there is $\lambda_{2t}^\kappa \leq \lambda_{2t}$, where λ_{2t}^κ is the equilibrium run probability in the game with the redemption fee while λ_{2t} is that in the game without redemption fees, leading to the results in Proposition 10.

Also, for any (a_{2t}, x_{2t}) and any $\kappa \in (0, 1)$, define

$$\bar{s} = \inf_{\lambda_{2t} \in [0,1]} \inf_{s_{2t+1}} \{s_{2t+1} | \Delta NAV^\kappa(s_{2t+1}; \lambda_{2t}) \geq 0\}.$$

By construction, there is $\bar{s} > 0$. Then the result follows because $\Delta NAV^\kappa(\lambda_{2t})$ is strictly increasing in s_{2t+1} . This leads to the results in Corollary 3 and finally concludes the proof. \square

PROOF OF PROPOSITION 11. Under in-kind redemptions, any shareholder who redeems on date t will get a_t/n_t unit of assets and x_t/n_t unit of cash. Since there will be no forced sales at the fund level, the fund will no longer manage its cash buffer. This implies $\eta_t = \eta_0$ for any date t , where η_0 is the initial cash-to-assets ratio.

Consider any late shareholder on any odd date $2t + 1$. If she redeems and consumes on date $2t + 1$, she gets $\delta_L R a_0/n_0 + x_0/n_0$, while if she redeemed and consumed on date $2t$, she would get $\delta_E R a_0/n_0 + x_0/n_0$. Since $\delta_L > \delta_E$, no late shareholder will ever run in an equilibrium.

Now I consider total sale losses when $\theta = 0$. There is no loss of generality to consider $\eta_{2t} = \mu_E R / (1 - \mu_E)$, which is the steady-state cash-to-assets ratio in the baseline model. Again due to the scale-invariance of the dynamic game, it suffices to consider an arbitrary state t . In the baseline model, by Proposition 4, the sale losses in stage t under the optimal cash rebuilding policy are:

$$\begin{aligned} L_t &= (1 - \delta_L) R (q_{2t+1} + s_{2t+1}^*) \\ &= \frac{(1 - \delta_L) R \mu_E \left(a_{2t} - \frac{\mu_L (R a_{2t} + x_{2t})}{\left(\delta_L + \frac{\mu_L (1 - \delta_L)}{1 - \mu_E} \right)} \right)}{(1 - \mu_E) \delta_L + \mu_L (1 - \delta_L) + \mu_E} + \frac{(1 - \delta_L) R \mu_L (R a_{2t} + x_{2t})}{\left(\delta_L + \frac{\mu_L (1 - \delta_L)}{1 - \mu_E} \right)}, \end{aligned} \quad (\text{B.13})$$

while the sale losses in stage t under in-kind redemptions are

$$L_t^{in-kind} = (1 - \delta_E) R \mu_E a_{2t} + (1 - \delta_L) R \mu_L a_{2t}. \quad (\text{B.14})$$

Note that, when $\mu_L = 0$, (B.13) reduces to

$$L_t = \frac{(1 - \delta_L) R \mu_E a_{2t}}{(1 - \mu_E) \delta_L + \mu_E},$$

while (B.14) reduces to

$$L_t^{in-kind} = (1 - \delta_E)R\mu_E a_{2t}.$$

Clearly, when δ_L is sufficiently larger than δ_E such that

$$1 - \delta_E > \frac{1 - \delta_L}{(1 - \mu_E)\delta_L + \mu_E},$$

there is $L_t^{in-kind} > L_t$. □

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